Nice Moment Swaps

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Moment Swaps

A characteristic is a moment-related property of an asset price – or log return – distribution

Moment swaps are bets on a characteristic

Most common is the variance swap:

$$\text{pay-off} = [\text{rv} - \text{vsr}] \times \text{pv}$$

- rv = realised variance – floating, computed at expiry under $\mathbb{P}$
- vsr = variance swap rate – fixed, agreed at inception
- pv = point value

Moment Swaps

More generally, two parties exchange:

- realised characteristic – floating, computed at expiry under $\mathbb{P}$
- swap rate – fixed, agreed at inception

An indicative, or 'fair-value' swap rate should be determined by setting

$$\mathbb{E}^{\mathbb{Q}}\left[\text{pay-off}\right] = 0$$

This is problematic for standard variance swaps, but not for 'nice' moment swaps

Neuberger (2012) *Realised Skewness*, RFS

'Nice' moment swaps also have the aggregation property:

$$\mathbb{E}^{\mathbb{M}}\left[\text{realised characteristic}\right] \text{ is independent of frequency}$$

for any martingale measure $\mathbb{M}$

(But $\mathbb{V}^{\mathbb{M}}\left[\text{realised characteristic}\right]$ is dependent on frequency)

For instance, if the underlying is a martingale under $\mathbb{P}$:

$$\mathbb{E}^{\mathbb{P}}\left[RS(\text{annual returns})\right] = \mathbb{E}^{\mathbb{P}}\left[RS(\text{daily returns})\right]$$

Model-Free Fair-Value Swap Rates

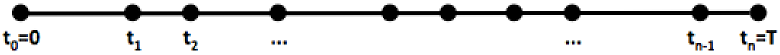
The property $\mathbb{E}^{\mathbb{Q}}$ [pay-off] $= 0$ for nice moment swaps holds under minimal conditions – the underlying process must follow a martingale under $\mathbb{Q}$

For instance, following Harrison and Kreps (1979) it could be the discounted price process of a non-dividend paying asset in an arbitrage-free market

This 'model-free' property follows because the derivation of the fair-value swap rate is very simple

Aggregation property $\Rightarrow$ replication of the realised characteristic using actual sum over the partition defined in the T&C

Basic Notation and Assumptions



- Denote by $\mathbf{t}_n$ a partition of the interval $\mathbf{t} = [0, T]$ where T is the maturity of the variance swap
- Let $s = \{s_t\}_{t \in \mathbf{t}}$ follow a $\mathbb{Q}$-martingale process
- So $\mathbb{E}_0^{\mathbb{Q}}[s_{t_i}] = s_0$
- And increments $\hat{s}_i = s_{t_i} - s_{t_{i-1}}$ are uncorrelated

Conventional Variance Swap

For a standard variance swap it is common practice to set:

- Realised variance $:= T^{-1} \sum_{\mathbf{t}_n} \ln \left(s_{t_i} / s_{t_{i-1}} \right)^2$
- Fair-value variance swap rate $:= 2T^{-1} \int_{\mathbb{R}^+} k^{-2} q\left(k, T\right) dk$

where $q(k, T)$ denotes the current price of a vanilla option with maturity T and strike k

In the following we shall ignore the normalization T^{-1} to ease notation

Discrete Monitoring Error

The fair-value swap rate for a standard variance swap is derived by assuming the realised variance is monitored continuously

This assumption induces an error:

$$\mathbb{E}^{\mathbb{Q}}\left[\langle \ln s\rangle_{\mathbf{t}} - \sum_{\mathbf{t}_n} \ln\left(s_{t_i}/s_{t_{i-1}}\right)^2\right] =: \varepsilon_{\mathbf{t}_n} \qquad (1)$$

where

$$\langle \ln s\rangle_{\mathbf{t}} =: \lim_{\mathbf{t}_n\to\mathbf{t}} \sum_{\mathbf{t}_n} \ln\left(s_{t_i}/s_{t_{i-1}}\right)^2$$

For instance, with stochastic volatility (log returns are not NID)

$\varepsilon_{\mathbf{t}_n} << 0$ during volatile periods

Jump Error

Also under continuous monitoring, with a generic process having both jump and diffusion components (Carr and Wu, 2009):

$$\mathbb{E}^{\mathbb{Q}}\left[\langle \ln s\rangle_{\mathbf{t}}\right] = 2\int_{\mathbb{R}^+} k^{-2} q\left(k, T\right) dk + \iota_{\mathbf{t}}$$

Therefore, for a standard discretely monitored variance swap with

$$\text{rv} := \sum_{\mathbf{t}_n} \ln\left(s_{t_i}/s_{t_{i-1}}\right)^2$$

the fair-value swap rate is actually

$$\text{vsr}^* = 2\int_{\mathbb{R}^+} k^{-2} q\left(k, T\right) dk + \iota_{\mathbf{t}} - \varepsilon_{\mathbf{t}_n}$$

General f-Swaps

Generalised characteristics relate to some k-dimensional stochastic process $\mathbf{z} = \{\mathbf{z}_t\}_{t \in \mathbf{t}} \in \mathbb{R}^k$ with increments $\hat{\mathbf{z}}_i = \mathbf{z}_{t_i} - \mathbf{z}_{t_{i-1}}$

Given a continuous function $f : \mathbb{R}^k \to \mathbb{R}$ we define an f-swap:

- realised f-characteristic $:= \sum_{\mathbf{t}_n} f\left(\hat{\mathbf{z}}_i\right)$
- implied f-characteristic $= \mathbb{E}^{\mathbb{Q}}\left[\sum_{\mathbf{t}_n} f\left(\hat{\mathbf{z}}_i\right)\right]$
- 'true' f-characteristic $= \mathbb{E}^{\mathbb{P}}\left[\sum_{\mathbf{t}_n} f\left(\hat{\mathbf{z}}_i\right)\right]$

Aggregation Property (Neuberger, 2012)

Assumptions for 'nice' f-swaps:

- No arbitrage in $\mathbf{z}$
- The characteristic f is chosen so that $(f, \mathbf{z})$ satisfies AP:

$$\mathbb{E}\left[\sum_{\mathbf{t}_n} f\left(\hat{\mathbf{z}}_i\right)\right] = \mathbb{E}\left[f\left(\sum_{\mathbf{t}_n} \hat{\mathbf{z}}_i\right)\right] \tag{2}$$

Here and henceforth $\mathbb{E}$ is under a martingale measure

$\Rightarrow$ fair-value f-swap rate depends only on f and T, not on $\mathbf{t}_n$

$\varepsilon_{\mathbf{t}_n} = 0 \Rightarrow$ Aggregation Property (AP)

The AP follows from the absence of a discrete monitoring error. Suppose:

$$\mathbb{E}\left[\langle \mathbf{z} \rangle_{\mathbf{t}}^{f} - \sum_{\mathbf{t}_n} f\left(\hat{\mathbf{z}}_i\right)\right] = 0$$

where

$$\langle \mathbf{z} \rangle_{\mathbf{t}}^{f} := \lim_{\mathbf{t}_n \to \mathbf{t}} \sum_{\mathbf{t}_n} f\left(\hat{\mathbf{z}}_i\right)$$

if the limit exists. Then the implied characteristic must be independent of the partition $\mathbf{t}_n$ and in particular

$$\mathbb{E}\left[\sum_{\mathbf{t}_n} f\left(\hat{\mathbf{z}}_i\right)\right] = \mathbb{E}\left[f\left(\mathbf{z}_T - \mathbf{z}_0\right)\right] = \mathbb{E}\left[f\left(\sum_{\mathbf{t}_n} \hat{\mathbf{z}}_i\right)\right]$$

Example in One Dimension

The AP does not hold for (f, x) where $x = \ln s$ and $f(x) = \hat{x}^2$:

$$\mathbb{E}\left[\sum_{\mathbf{t}_n} \ln\left(s_{t_i}/s_{t_{i-1}}\right)^2\right] \neq \mathbb{E}\left[\ln\left(s_T/s_0\right)^2\right]$$

However, Neuberger (2012) finds two alternatives:

- the log variance

$$\ell(\hat{x}) := 2\left(\mathrm{e}^{\hat{x}} - 1 - \hat{x}\right) \tag{3}$$

- and the entropy variance

$$h(\hat{x}) := 2\left(\hat{x}\mathrm{e}^{\hat{x}} - \mathrm{e}^{\hat{x}} + 1\right) \tag{4}$$

AP for the Log and Entropy Variances

The AP holds for (ℓ, x) – and it also holds for (h, x) but only under the additional assumption of independent increments

$$\begin{aligned}
\mathbb{E}\left[\sum_{\mathbf{t}_n} \ell\left(\hat{x}_i\right)\right] &= \mathbb{E}\left[\sum_{\mathbf{t}_n} 2\left(\mathrm{e}^{\hat{x}_i} - 1 - \hat{x}_i\right)\right] \\
&= \mathbb{E}\left[-2\sum_{\mathbf{t}_n} \hat{x}_i\right] \quad \text{since} \quad \mathbb{E}\left[\mathrm{e}^{\hat{x}_i}\right] = 1 \\
&= \mathbb{E}\left[2\left(\exp\left(\textstyle\sum_{\mathbf{t}_n} \hat{x}_i\right) - 1 - \displaystyle\sum_{\mathbf{t}_n} \hat{x}_i\right)\right] \\
&= \mathbb{E}\left[\ell\left(\sum_{\mathbf{t}_n} \hat{x}_i\right)\right]
\end{aligned}$$

A Nice Variance Swap

Neuberger (2012) replaces the sum of squared log returns by the realised log variance, i.e.

$$\text{rv} := \sum_{\mathbf{t}_n} \ell\left(\hat{x}_i\right) = \sum_{i=1}^{n} 2\left(e^{\hat{x}_i} - 1 - \hat{x}_i\right)$$

Then the fair-value swap can be derived using the standard gamma-weighted replication theorem (Bakshi and Madan, 2000):

$$\text{vsr}^* = \mathbb{E}\left[\ell\left(x_T - x_0\right)\right] = 2\int_{\mathbb{R}^+} k^{-2} q\left(k, T\right) dk$$

It is exact for all partitions and all processes $x = \ln s$, provided only that s follows a $\mathbb{Q}$-martingale

Aggregating Log Return Characteristics

Now let $\mathbf{z} = (x, v_g)'$ include a generalised variance process

$$v_{g,t} = \mathbb{E}_t \left[g \left(x_T - x_t \right) \right], \quad \lim_{\hat{x} \to 0} g \left(\hat{x} \right) / \hat{x}^2 = 1 \tag{5}$$

and denote by $\hat{x}$ and $\hat{v}_g$ increments in x and v_g respectively

Neuberger (2012) derives the set G of characteristics that satisfy the AP w.r.t. $\mathbf{z}$:

$$\left\{ \phi_1 \hat{x} + \phi_2 \hat{v}_g + \phi_3 \left(\mathrm{e}^{\hat{x}} - 1 \right) + \phi_4 \left(2\hat{x} - \hat{v}_g \right)^2 + \phi_5 \left(2\hat{x} + \hat{v}_g \right) \mathrm{e}^{\hat{x}} \right\}$$

Limitations

- G is subject to the following parameter constraints:

$$\begin{aligned} &\text{if } \phi_4 \neq 0,\ \phi_5 = 0 \text{ and } g \equiv \ell \text{ as defined in (3)} \\ &\text{if } \phi_5 \neq 0,\ \phi_4 = 0 \text{ and } g \equiv h \text{ as defined in (4)} \\ &\text{if } \phi_4 = \phi_5 = 0,\ v_g \text{ is any generalised variance} \end{aligned}$$

- The set G contains a unique third moment characteristic and no higher moment characteristics
- Kozhan et al. (2011) demonstrate empirically that the skewness and variance risk premiums derived from G are highly correlated

Theorem

Let $\mathbf{z} \in \mathbb{R}^k$ be a k-dimensional martingale process derived from one (or more) tradable assets in an arbitrage-free market. Let $\hat{\mathbf{z}}$ denote an increment in $\mathbf{z}$. Then

$$F_{\mathbf{z}} := \left\{ f : \mathbb{R}^k \to \mathbb{R} \,|\, f(\hat{\mathbf{z}}) = \mathbf{a}'\hat{\mathbf{z}} + \hat{\mathbf{z}}'\mathbf{A}\hat{\mathbf{z}}, \mathbf{a} \in \mathbb{R}^k, \mathbf{A} \in \mathbb{M}_k \right\}$$

forms a vector space over $\mathbb{R}$ that contains all twice continuously differentiable functions $f : \mathbb{R}^k \to \mathbb{R}$ such that the aggregation property (2) holds w.r.t. $(f, \mathbf{z})$

We term the elements of $F_{\mathbf{z}}$ martingale aggregating characteristics

A Simple Example

Let $\mathbf{z} = s$ and $\mathbf{t}_n = \{0, t, T\}$

$$
\begin{aligned}
& \mathbb{E}_0\left[(s_T - s_t)^2 + (s_t - s_0)^2\right] \\
= \;& \mathbb{E}_0\left[\mathbb{E}_t\left[s_T^2 - 2s_T s_t + s_t^2 + s_t^2 - 2s_t s_0 + s_0^2\right]\right] \\
= \;& \mathbb{E}_0\left[s_T^2 - s_0^2\right] \\
= \;& \mathbb{E}_0\left[s_T^2 - 2s_T s_0 + s_0^2\right] \\
= \;& \mathbb{E}_0\left[(s_T - s_0)^2\right]
\end{aligned}
$$

Generalization

$$
\begin{aligned}
\mathbb{E}\left[\sum_{\mathbf{t}_n} f\left(\hat{\mathbf{z}}_i\right)\right] &= \mathbb{E}\left[\textstyle\sum_{\mathbf{t}_n} \hat{\mathbf{z}}_i' \mathbf{A} \hat{\mathbf{z}}_i\right] = \mathbb{E}\left[\textstyle\sum_{\mathbf{t}_n} \operatorname{tr}\left(\mathbf{A} \hat{\mathbf{z}}_i \hat{\mathbf{z}}_i'\right)\right] \\
&= \operatorname{tr}\,\mathbb{E}\left[\mathbf{A} \textstyle\sum_{\mathbf{t}_n} \left(\mathbf{z}_{t_i} - \mathbf{z}_{t_{i-1}}\right)\left(\mathbf{z}_{t_i} - \mathbf{z}_{t_{i-1}}\right)'\right] \\
&= \operatorname{tr}\,\mathbb{E}\left[\mathbf{A} \textstyle\sum_{\mathbf{t}_n} \left(\mathbf{z}_{t_i} \mathbf{z}_{t_i}' - \mathbf{z}_{t_{i-1}} \mathbf{z}_{t_{i-1}}'\right)\right] \\
&= \operatorname{tr}\,\mathbb{E}\left[\mathbf{A}\left(\mathbf{z}_T \mathbf{z}_T' - \mathbf{z}_0 \mathbf{z}_0'\right)\right] \\
&= \operatorname{tr}\,\mathbb{E}\left[\mathbf{A}\left(\mathbf{z}_T - \mathbf{z}_0\right)\left(\mathbf{z}_T - \mathbf{z}_0\right)'\right] \\
&= \mathbb{E}\left[\left(\mathbf{z}_T - \mathbf{z}_0\right)' \mathbf{A} \left(\mathbf{z}_T - \mathbf{z}_0\right)\right] = \mathbb{E}\left[f\left(\sum_{\mathbf{t}_n} \hat{\mathbf{z}}_i\right)\right]
\end{aligned}
$$

A Nice Skewness Swap

Let $\mathbf{z} = (s, v)'$ with $v_t = \mathbb{E}_t\left[s_T^2\right]$

By construction, $\mathbf{z}$ is a martingale

From the theorem we know that

$$f\left(\hat{\mathbf{z}}\right) = \hat{s}\hat{v} - 2s_0\hat{s}^2$$

is a martingale aggregating characteristic

A Nice Skewness Swap

Furthermore, the implied characteristic of the swap equals the third moment:

$$f\left(\mathbf{z}_T - \mathbf{z}_0\right) = (s_T - s_0)(v_T - v_0) - 2s_0(s_T - s_0)^2$$

Hence

$$\begin{aligned}
\mathbb{E}_0\left[f\left(\mathbf{z}_T - \mathbf{z}_0\right)\right] &= \mathbb{E}_0\left[\left(s_T - s_0\right)\left(s_T^2 - \mathbb{E}_0\left[s_T^2\right]\right) - 2s_0\left(s_T - s_0\right)^2\right] \\
&= \mathbb{E}_0\left[s_T^3 - s_0 s_T^2 - 2s_0\left(s_T^2 - s_0^2\right)\right] \\
&= \mathbb{E}_0\left[s_T^3 - 3s_T^2 s_0 + 3s_T s_0^2 - s_0^3\right] \\
&= \mathbb{E}_0\left[\left(s_T - s_0\right)^3\right]
\end{aligned}$$

A Nice Skewness Swap

The exact fair-value of a swap that pays

$$\text{realised skewness} := \sum_{\mathbf{t}_n} \left(\hat{s}_i \hat{v}_i - 2 s_0 \hat{s}_i^2 \right)$$

can be evaluated according to the replication theorem of Bakshi and Madan (2000):

$$\text{implied skewness} = \mathbb{E}^{\mathbb{Q}} \left[\left(s_T - s_0 \right)^3 \right] = 6 \int_{\mathbb{R}^+} \left(k - s_0 \right) q \left(k, T \right) dk$$

Conjecture

Let $\mathbf{z} \in \mathbb{R}^k$ be a k-dimensional martingale process derived from one (or more) tradable assets in an arbitrage-free market and set $\mathbf{y} = (\mathbf{z}, \ln \mathbf{z}) \in \mathbb{R}^{2k}$ with $\hat{\mathbf{y}} = \left(\hat{\mathbf{z}}, \widehat{\ln \mathbf{z}}\right) \in \mathbb{R}^{2k}$. Then

$$F_{\mathbf{y}} := \left\{ f : \mathbb{R}^{2k} \to \mathbb{R} \middle| f(\hat{\mathbf{y}}) = \mathbf{a}'\hat{\mathbf{y}} + \mathbf{b}'\left(\mathrm{e}^{\widehat{\ln \mathbf{z}}} - \mathbf{1}\right) + \hat{\mathbf{z}}'\mathbf{A}\hat{\mathbf{z}} + \hat{\mathbf{z}}'\mathbf{B}\mathrm{e}^{\gamma\widehat{\ln \mathbf{z}}}, \right.$$
$$\left. \mathbf{a} \in \mathbb{R}^{2k}, \mathbf{b} \in \mathbb{R}^k, \mathbf{A}, \mathbf{B} \in \mathbb{M}_k \right\}$$

forms a vector space over $\mathbb{R}$ that contains all twice continuously differentiable functions $f : \mathbb{R}^k \to \mathbb{R}$ such that the aggregation property (2) holds w.r.t. $(f, \mathbf{y})$

We term the elements of $F_{\mathbf{y}}$ aggregating characteristics

Application to Generalized Moment Swaps

Our vector space of aggregating characteristics contains many m-th moment characteristics for $m = 2, 3$ and even $4, 5, \ldots$

Thus we can design 'nice' moment swaps, which have the following properties:

- A model-free fair-value swap rate may be determined exactly (model-free in the sense that all we need is the no-arbitrage assumption)
- The realised characteristic may be defined according to any partition of $[0, T]$

Fundamental Contracts

The fair-value swap rate for these 'nice' moment swaps can be related to a few fundamental contracts, each derived from the market prices of vanilla options using the standard replication theorem

Some of these contracts (e.g. the log, entropy and squared log contracts) are already familiar from previous research

Log Contract

The log contract pays $x_T = \ln s_T$ at maturity

Fair value:

$$L_T := \mathbb{E}^{\mathbb{Q}}\left[x_T\right] = x_0 - \int_{\mathbb{R}^+} k^{-2} q\left(k, T\right) dk$$

Under a GBM with constant volatility σ

$$L_T = x_0 - \frac{1}{2}\sigma^2 T$$

Intuitively, the implied total variance of the log contract is

$$v_\ell = 2\left(x_0 - L_T\right) = 2\int_{\mathbb{R}^+} k^{-2} q\left(k, T\right) dk$$

Entropy Contract

The entropy contract pays $s_T x_T = s_T \ln s_T$ at maturity

Fair value:

$$H_T := \mathbb{E}^{\mathbb{Q}}[s_T x_T] = s_0 x_0 + \int_{\mathbb{R}^+} k^{-1} q(k, T)\, dk$$

Under a GBM with constant volatility σ

$$H_T = s_0 x_0 + \frac{s_0}{2}\sigma^2 T$$

Intuitively, the implied total variance of the entropy contract is

$$v_h = 2\left(s_0^{-1} H_T - x_0\right) = 2 s_0^{-1} \int_{\mathbb{R}^+} k^{-1} q(k, T)\, dk$$

Squared Log Contract

The squared log contract pays $x_T^2 = (\ln s_T)^2$ at maturity

Fair value:

$$SQ_T := \mathbb{E}^{\mathbb{Q}}\left[x_T^2\right] = x_0^2 + 2\int_{\mathbb{R}^+} (1 - \ln k)\, k^{-2} q\,(k, T)\, dk$$

Under a GBM with constant volatility σ

$$SQ_T = x_0^2 + \sigma^2 T \left(1 - x_0 + \frac{1}{4}\sigma^2 T\right)$$

Intuitively, the implied total variance of the squared log contract is the positive root of $v_{sq}^2 + 4\,(1 - x_0)\, v_{sq} + 4\left(x_0^2 - SQ_T\right) = 0$

Summary

We have generalised the set of aggregating characteristics in Neuberger (2012) to a vector space that contains many higher moment characteristics

A ‘model-free’ fair-value ‘nice’ moment swap rate can be derived exactly, in terms of a few fundamental contracts

This rate applies to all partitions, e.g. realised moments computed from daily, weekly or monthly log returns all have the same fair-value swap rate

Outlook

- Empirical investigation of aggregating variance and skewness characteristics and comparison with standard definitions
- Empirical investigation of aggregating variance and skewness risk premia over long horizons (with applications)
- For some simple $\mathbf{y}$, derive basis for $F_{\mathbf{y}} \Rightarrow$ nice moment swaps trading uncorrelated risks

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Outline of Proof

- It is easy to show the AP for all linear and quadratic terms
- If the AP holds for $(f, \mathbf{z})$ and a general martingale $\mathbf{z}$, it has to hold for $(f, \mathbf{c})$ and a continuous martingale $\mathbf{c}$ in particular
- We can therefore derive necessary conditions for f from $\mathbf{c}$ and find a set of potential solutions
- If the AP holds for these candidates and the general martingale $\mathbf{z}$, the derived condition (Hessian constant) is also sufficient

Proof of Theorem

We start from the AP for $(f, \mathbf{c})$, i.e.

$$\mathbb{E}\left[\sum_{\mathbf{t}_n} f\left(\hat{\mathbf{c}}_i\right)\right] = \mathbb{E}\left[f\left(\sum_{\mathbf{t}_n} \hat{\mathbf{c}}_i\right)\right]$$

and, following Itô's Lemma, write

$$f\left(\mathbf{c}_T - \mathbf{c}_0\right) = \int_{\mathbf{t}} \mathbf{J}'_{t,0} d\mathbf{c}_t + \frac{1}{2}\mathrm{tr} \int_{\mathbf{t}} \mathbf{H}_{t,0} d\langle \mathbf{c} \rangle_t \tag{6}$$

and similarly for the partial increments where $\langle \mathbf{c} \rangle_t$ denotes the quadratic covariation and $\mathbf{J}_{t,s} := \nabla f\left(\hat{\mathbf{c}}_{t,s}\right)$, $\mathbf{H}_{t,s} := \nabla\nabla' f\left(\hat{\mathbf{c}}_{t,s}\right)$ with ∇ being the vector operator of first partial derivatives

Proof of Theorem (cont'd)

Using that $\mathbf{c}$ follows a martingale we can write the AP as

$$\mathbb{E}\left[\operatorname{tr}\int_{\mathbf{t}}\left\{\mathbf{H}_{t,0}-\mathbf{H}_{t,m(t)}\right\}d\langle\mathbf{c}\rangle_t\right]=0 \qquad (7)$$

where $m(t)=\max\{t_i\in\mathbf{t}_n, t_i\leq t\}$. Now write

$$\mathbf{H}_{t,0}-\mathbf{H}_{t,m(t)} =: \mathbf{E}_t\boldsymbol{\Lambda}_t\mathbf{E}_t' \qquad (8)$$

where $\boldsymbol{\Lambda}_t=\operatorname{diag}\left\{\lambda_t^1,\ldots,\lambda_t^k\right\}$ is a diagonal matrix of eigenvalues and $\mathbf{E}_t$ is an orthogonal matrix of eigenvectors

Proof of Theorem (cont'd)

W.l.o.g. we assume the dynamics

$$d\mathbf{c}_t = \exp\left(\frac{1}{2}\xi\left\{\mathbf{H}_{t,0} - \mathbf{H}_{t,m(t)}\right\}\right) d\mathbf{w}_t$$

where $\mathbf{w}$ is a multivariate Wiener process with $T^{-1}\langle\mathbf{w}\rangle_{\mathbf{t}} = \mathbf{I}_k$, the identity matrix and $\xi \in \mathbb{R}$ is an arbitrary constant.

Then

$$d\langle\mathbf{c}\rangle_t = \mathbf{E}_t \exp\left\{\xi\mathbf{\Lambda}_t\right\} \mathbf{E}'_t dt \tag{9}$$

Proof of Theorem (cont'd)

Inserting (8) and (9) into (7) and using the cyclic property of the trace yields

$$\mathbb{E}\left[\operatorname{tr}\int_{\mathbf{t}} \mathbf{\Lambda}_t \exp\left\{\xi\mathbf{\Lambda}_t\right\} dt\right] = 0$$

Differentiating w.r.t. T and ξ and evaluating the function at $\xi = 0$ yields the condition

$$\mathbb{E}\left[\operatorname{tr}\mathbf{\Lambda}_t^2\right] = \sum_{i=1}^{k} \mathbb{E}\left[\left(\lambda_t^i\right)^2\right] = 0,$$

which implies that all eigenvalues must be equal to zero