TECHNIQUE IN ALGEBRA: THE TASK OF RENDERING IT CONCEPTUAL

CAROLYN KIERAN
Université du Québec à Montréal, CANADA
Département de mathématiques
kieran.carolyn@uqam.ca

Research presented at the Fields Institute, Toronto, January 2009

I wish to express my appreciation to the colleagues, post-doctoral fellows, and collaborators who have contributed to the research that I will be presenting today:

- André Boileau, Caroline Damboise, Paul Drijvers, José Guzmán, Fernando Hitt, Ana Isabel Sacristán, Luis Saldanha, and Denis Tanguay -- as well as the teachers and students of the participating schools, and our project consultant, Michèle Artigue.
- Our gratitude also to the Social Sciences and Humanities Research Council of Canada, and the Ministère de Relations Internationales, who have funded, and continue to fund, this research.

OUTLINE OF THE PRESENTATION

- The first part: The technical-conceptual interface in algebra and what I mean by conceptual (theoretical) understanding of algebraic technique.

- The second part: The ways in which students learn to draw such conceptual aspects from their work with algebraic techniques in a technology environment.

1. INTRODUCTION

- WHAT IS COMPUTER ALGEBRA SYSTEM (CAS) TECHNOLOGY?
  - A **computer algebra system** (**CAS**) is a software program that facilitates symbolic mathematics. The core functionality of a CAS is manipulation of mathematical expressions in symbolic form (Wikipedia, Sept. 5, 2007)

1. INTRODUCTION

SOME FINDINGS FROM CAS RESEARCH

- Since the mid-1990s, in France, when CAS started to make their appearance in secondary school mathematics classes, researchers (Artigue et al., 1998) noticed that teachers were emphasizing the conceptual dimensions while neglecting the role of the technical work in algebra learning.

1. INTRODUCTION

- However, this emphasis on conceptual work was producing neither a clear lightening of the technical aspects of the work nor a definite enhancement of students' conceptual reflection (Lagrange, 1996).
- From their observations, the research team of Artigue and her colleagues came to think of techniques as a link between tasks and conceptual reflection, in other words, that the learning of techniques was vital to related conceptual thinking.

1. INTRODUCTION

Our research group was intrigued by the theoretical notion that algebra learning at the high school level might be conceptualized in terms of a dynamic among Task-Technique-Theory (T-T-T) within technological environments.

- And so it came to be that we began a series of studies in 2002, which continue to this day, that explored the relations among task, technique, and theory in the algebra learning (and teaching) of grade 9, 10, 11, and 12 students in CAS environments.

In brief, we have found that:

- As reported in Kieran & Drijvers, 2006:
  - Technique and theory emerged in mutual interaction: Techniques gave rise to theoretical thinking; and the other way around, theoretical reflections led students to develop and use techniques.
- As reported in Kieran & Damboise, 2007:
  - A comparative study of a CAS class and non-CAS class revealed that the CAS class improved much more than the non-CAS class in both technique and theory, but especially in theory -- and the sequence of lessons was one where the technical component was clearly in the forefront.

1. INTRODUCTION

This brings us to the main question to be addressed in this talk:

- How does the learning of algebraic technique in a CAS environment lead to the emergence of students' theoretical/conceptual growth?
- In other words, how is technique rendered conceptual? What does it mean to have a conceptual understanding of algebraic technique?

2. The interface between technique and theory in algebra

- Note that I will be using the terms *conceptual* and *theoretical* interchangeably.
- Note also that the context of this presentation is related to the letter-symbolic aspects of algebra. Why?
  - A great deal of research exists already with respect to the benefits of multi-representational approaches (e.g., graphical representations) in making algebraic objects more meaningful to students.
  - However, algebra involves more than representational activity; symbolic transformational activity lies at its core.

2. The interface between technique and theory in algebra

- What is meant by a *CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE*?

  We propose that it includes:

  - Being able to see a certain **form** in algebraic expressions and equations, such as a linear or quadratic form;
  - Being able to see **relationships**, such as the equivalence between factored and expanded expressions;
  - Being able to see through algebraic transformations (the technical aspect) to the underlying changes in form of the algebraic object and being able to explain/justify these changes.

Some classic examples of conceptual understandings in algebra include:

- The distinctions
  - between variables and parameters,
  - between identities and equations,
  - between mathematical variables and programming variables, ...
- Both the knowledge of the objects to which the algebraic language refers (generally numbers and the operations on them) and the need, at times, to include certain semantic aspects of the mathematical context so as to be able to interpret the objects being treated. ...

But what might be some examples of that which is intended by '*CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE*'

- 1. Seeing through symbols to the underlying forms, e.g.,

  (a) seeing $x^6 - 1$ as $((x^3)^2 - 1)$
  and as $((x^2)^3 - 1)$,
  and so being able to factor it in 2 ways.

  (b) seeing that x^2+5x+6 and x^4+7x^2+10
  are both of the form ax^2+bx+c.

Examples of what is intended by a *CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE ...*

- 2. Conceptualizing the equivalence of the factored and expanded forms of algebraic expressions,

  e.g., awareness that the same numerical substitution (not a restricted value) in each step of the transformation process of expanding will yield the same value:

  $(x+1)(x+2)$ -- factored form --

  $= x(x+2) + 1(x+2)$

  $= x^2 + 2x + x + 2$

  $= x^2 + 3x + 2$ -- expanded form --

  and so substituting, say 3, into all four expressions is seen to yield 20 for each exp.

Examples of what is intended by a *CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE ...*

- 3. Coordinating the "nature" of equation solution(s) with the equivalence relation between the two expressions that comprise the original equation, e.g., for the following task,

  Given the 3 expressions

  $x(x^2-9)$, $(x+3)(x^2-3x)-3x-3$, $(x^2-3x)(x+3)$,

  (a) determine which of these three expressions are equivalent;

  (b) construct an equation using one pair of the given expressions, which are not equivalent, and find its solution;

  (c) construct an equation from another pair of the given expressions, which are not equivalent, and by logical reasoning only (i.e., without actually solving the equation), determine its solution.

Exp1: $x(x^2-9)$
Exp2: $(x+3)(x^2-3x)-3x-3$
Exp3: $(x^2-3x)(x+3)$

- Which are equivalent?

  Only Exp1 and Exp3 are equivalent.
- An equation using a pair of non-equivalent expressions? And its solution?

  say, Exp1=Exp2

  solution: $x=-1$ (with CAS)
- An equation from another pair of non-equivalent expressions? And its solution?

  Exp3=Exp2; the solution has to be the same as above. Why?

  (a conceptual understanding allows one to answer this last question)

Is it important to foster a *CONCEPTUAL UNDERSTANDING OF ALGEBRAIC TECHNIQUE?*

- National and international mathematics assessments during the 1980s and 1990s reported that secondary school students, in order to cover their lack of understanding, resorted to memorizing rules and procedures and that students eventually came to believe that this activity represented the essence of algebra (e.g., Brown et al., 1988).
- While more recent reform movements have led to infusing “real-world” problem-solving activities into algebra curricula, the traditional dichotomy of skills/procedures and concepts has tended to remain in algebraic discourse.
- Although Skemp (1976) described “relational understanding” as knowing both the rules and why they work, there has never been much movement in the direction of describing what this might mean for algebra.

The role of tasks in the TTT triad …

- At a recent PME Research Forum on "The Significance of Task Design in Mathematics Education", Ainley and Pratt (2005) -- the organizers of the Forum -- argued that,
- "We see task design as a crucial element of the learning environment … [and contend that] the nature of the task influences the activity of students."

Also, with respect to tasks:

- Lagrange (1999) suggested that task situations ought to be created in such a way as to “bring about a better comprehension of mathematical content” (p. 63) via the progressive acquisition of techniques in the achievement of a solution to the task.
- Guin and Trouche (1999) added that tasks should aim at fostering experimental work (investigation and anticipation).

2. The interface between technique and theory in algebra

- So, to sum up, before moving on:
  With recent advances in

  a) the development of theoretical frameworks, such as that of **Task-Technique-Theory**,

  b) the increasing use of technology in schools, for example, **CAS** at the secondary school level, and

  c) the attention being paid to the role that the nature of the task/situation plays in student learning,

  we are well poised to make headway in reflecting upon the ways in which technique can be viewed from a conceptual angle in the teaching and learning of algebra and, in fact, how technology can enhance such conceptualizing of technique.

3. How Year 10 students in our project drew conceptual aspects from their work with algebraic techniques in a CAS environment

- Concerning the tasks:
  - The tasks went beyond merely asking technique-oriented questions;
  - The tasks also called upon general mathematical processes that included:

    observing/focusing, predicting, reflecting, verifying, explaining, conjecturing, justifying.
- Concerning the technologies:
  - Both CAS and paper-and-pencil were used, often with requests to coordinate the two;
  - The CAS provided the data upon which students formulated conjectures and arrived at provisional conclusions.

CONCEPTUALIZING THAT EMERGED WHILE LEARNING NEW TECHNIQUES WITH THE AID OF CAS TECHNOLOGY:

(an example from Kieran & Drijvers, 2006)

- The task involved factoring polynomials (adapted from Mounier & Aldon, 1996).
- The family of expressions: $x^n - 1$
- Aim: to arrive at a general form of factorization for $x^n - 1$ and then to relate this to the complete factorization of particular cases for integer values of n from 2 to 13. Proving one of these cases was part of the two-lesson sequence, but is not included today.

One of the initial tasks of the activity

1. Perform the indicated operations: $(x-1)(x+1)$; $(x-1)(x^2+x+1)$.
2. Without doing any algebraic manipulation, anticipate the result of the following product
   $$(x-1)\left(x^3+x^2+x+1\right)=$$
3. Verify the above result using paper and pencil, and then using the calculator.
4. What do the following three expressions have in common? And, also, how do they differ?
   $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)\left(x^3+x^2+x+1\right)$.
5. How do you explain the fact that when you multiply: i) the two binomials above, ii) the binomial with the trinomial above, and iii) the binomial with the quadrinomial above, you always obtain a binomial as the product?
6. Is your explanation valid for the following equality:
   $(x-1)(x^{134}+x^{133}+x^{132}+\ldots+x^2+x+1)=x^{135}-1$? Explain.

After students had worked on these questions, either in groups or individually, the teacher opened up a whole-class discussion and asked students to state their responses to one particular question.

- What do the following three expressions have in common? And also, how do they differ?

  $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, $(x-1)(x^3+x^2+x+1)$

- The teacher's aim in having the whole-class discussion was to encourage students to learn from what some of their peers had noticed.

Different pupils noticed different things in the expression patterns.

$(x-1)(x+1)$, $(x-1)(x^2+x+1)$, $(x-1)(x^3+x^2+x+1)$

- One student noticed that the exponents were different in the second brackets.
- One pair of students focused on the "$x+1$" that was present at the end of each of the second brackets.
- However, one student's contribution to the whole-class discussion, presented just below, helped others to "refine their noticing":
  - "They are all multiplied by $(x-1)$, but each of them adds on an x with a higher exponent in the second expression: $((x+1) \rightarrow (x^2 + x + 1) \rightarrow (x^3 + x^2 + x + 1)$"

After arriving at a general form of factorization for x^n-1 based on a few examples, $x^n - 1 = (x-1)(x^{n-1}+x^{n-2} + \dots x+1)$, the students worked on the following task for *n* being the integers from 2 to 6, where they were confronted with the completely factored forms produced by the CAS.

In this activity each line of the table below must be filled in completely (all three cells), one row at a time. Start from the top row (the cells of the three columns) and work your way down. If, for a given row, the results in the left and middle columns differ, reconcile the two by using algebraic manipulations in the right hand column.

Factorization using <u>paper and pencil</u>	Result produced by the <u>FACTOR command</u>	Calculation to reconcile the two, if necessary
$x^2 - 1 =$		
$x^3 - 1 =$		
$x^4 - 1 =$		
$x^5 - 1 =$		
$x^6 - 1 =$		

An example of a student's work -- first with p/p (in 1st column), then with CAS (in 2nd column), and then involving a reconciliation of the two (in 3rd column) for x^4-1.

This example shows reconciliation by multiplying the 2nd and 3rd CAS factors:

Factorization using paper and pencil	Result produced by FACTOR command	Calculation to reconcile the two, if necessary
$x^2-1=(x-1)(x+1)$	$(x-1)(x+1)$	N/A
$x^3-1=(x-1)(x^2+x+1)$	$(x-1)(x^2+x+1)$	N/A
$x^4-1=(x-1)(x^3+x^2+x+1)$	$(x-1)(x+1)(x^2+1)$	$(x-1)(x+1)(x^2+1)$ $(x-1)(x^3+x^2+x+1)$

After completing the factorization task for n = 2 to 6 in $x^n - 1$, students were presented with the following Conjecture task.

Conjecture, in general, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?

ii) contain more than two factors?

iii) include $(x+1)$ as a factor?

Please explain.

At first, many students incorrectly conjectured that, for all odd n's, the complete factorization of x^n-1 would contain exactly two factors.

- The CAS played a pivotal role in allowing them not only to test their conjecture, but also to successively refine it.
  - One group of two pupils, whom we videotaped, stated -- after trying a few examples of their own: "There seem to be some exceptions to our rule." They then tested with the CAS for $n = 15, 21, 27, 99$ to arrive at the conclusion that n could not be a multiple of 3. They followed a similar pathway to eliminate, in turn, both the multiples of 5 and 7. Finally the Eureka moment: x^n-1 has exactly two factors when n is a prime number.

Here, we see the final revision of their conjecture regarding the numbers *n* (i.e., prime numbers) that yield exactly two factors for the factorization of x^n-1:

II.(B).2. On the basis of patterns you observe in the table II.B above, revise (if necessary) your conjecture from Part A. That is, for what numbers n will the factorization of $x^n - 1$:

i) contain exactly two factors?
ii) contain more than two factors?
ii) include $(x+1)$ as a factor?

Please explain:

i.) ~~odd numbers (for the~~ exponent) that is ~~not divisible by three and five, seven~~ all prime numbers

ii.). composite numbers.

iii). even numbers.

- With the aid of the CAS technology -- and different sorts of questions within the task set -- the students were able to focus their trials on certain multiples of the exponent, to try out extreme cases, ... in short, to arrive at a new conceptualization of the factors for expressions from a certain family of polynomials.

Further evidence for the emergence of theoretical/conceptual ideas arising from work with CAS techniques was gathered from a study we carried out with two classes of weak algebra students.
(Kieran & Damboise, 2007)

- TASK AND TEST DESIGN:
  - A set of parallel activities was developed -- on factoring and expanding.
  - Tasks were identical except that where one class was to use p/p only, the other class was to use CAS or a combination of CAS and p/p.
  - Some tasks were technique-oriented; others were theory-oriented.
  - A pretest and posttest were also created with some questions being technical and others theoretical.

SOME OF THE TASKS:
from Activity 3 (CAS version)

Activity 3 (CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Use the calculator in completing the table below.

Given trinomial (in "dissected" form)	Factored form using FACTOR	Expanded form using EXPAND
(a) $x^2 + (3+4)x + 3 \bullet 4$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why did the calculator not factor the trinomial expressions of 1(d) and 1(e) above?

2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?

2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.

2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

And the non-CAS version of the same task:

Activity 3 (non-CAS): Trinomials with positive coefficients and $a = 1$ ($ax^2 + bx + c$)

1. Complete the table below by following the example at the beginning of the table.

Given trinomial (in "dissected" form)	Factored form	Expanded form
Example: $x^2 + (3+4)x + 3 \bullet 4$	$x^2 + (3+4)x + 3 \bullet 4$ $= x^2 + 3x + 4x + 3 \bullet 4$ $= x(x+3) + 4(x+3)$ $= (x+3)(x+4)$	$x^2 + 7x + 12$
(a) $x^2 + (5+6)x + 5 \bullet 6$		
(b) $x^2 + (3+5)x + 3 \bullet 5$		
(c) $x^2 + (4+6)x + 4 \bullet 6$		
(d) $x^2 + (3+5)x + 3 \bullet 3$		
(e) $x^2 + (3+4)x + 3 \bullet 6$		

2(a) Why could you not factor the trinomial expressions in 1(d) and 1(e) above?

2(b) How can you tell by looking at the "dissected" form (left-hand column) if a trinomial is factorable?

2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.

2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

IN THIS STUDY, THE TECHNOLOGY WAS FOUND TO PLAY SEVERAL ROLES IN THE CAS CLASS:

- it provoked discussion;
- it generated exact answers that could be scrutinized for structure and form;
- it helped students to verify their conjectures, as well as their paper-and-pencil responses;
- it motivated the checking of answers; and
- it created a sense of confidence and thus led to increased interest in algebraic activity.

THE FINDING THAT:
CAS generated exact answers that could be scrutinized for structure and form

- Of all the roles that the CAS played in this study, this was found to be the most crucial to the success of these weak algebra students.
- It proved to be the main mechanism underlying the evolution in the CAS students' algebraic thinking.
- Ironically, the crucial nature of this role was first made apparent to us by the voicing of frustration by one of the students in the non-CAS class:

One of the students of the non-CAS class remarked when faced with these two questions of the task just seen:

2(c) If a trinomial is not in its "dissected" form but is in its expanded form, how can you tell if it is factorable? Explain and give an example.

2(d) How would you describe the relation between the factored form and the expanded form of the above trinomials in 1(a) – 1(c)?

"How can we describe the relation between the factored form and the expanded form of these trinomials? – we don't even know if our paper-and-pencil factorizations and expansions from Question 1 are right."

- This study analyzed the improvements of two classes of weak algebra students in both *technique* (being able to do) and *theory* (i.e., being able to explain why and to note some structural aspects), in the context of tasks that invited technical and theoretical development.
- At the outset, both the CAS class and the non-CAS class scored at the same levels in a pretest that included technical and theoretical components.
- However, the CAS class improved more than the non-CAS class on both components, but especially on the theoretical component.

We see this finding as being of some interest

- Being able to generate exact answers with the CAS allowed students to examine their CAS work and to see patterns among answers that they were sure were correct. This kind of assurance, which led the CAS students to theorize, was found to be lacking in the uniquely paper-and-pencil environment where students made few theoretical observations. The theoretical observations made by CAS students worked hand-in-hand with improving their technical ability.
- In other words, their **technique had become theorized,** which in turn led to further improvement in technique.

Thank you

References

Ainley, J., & Pratt, D. (2005). The significance of task design in mathematics education: Examples from proportional reasoning. In H.L. Chick & J.L. Vincent (Eds,), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 93-122). Melbourne, Australia: PME.

Artigue, M., Defouad, B., Duperier, M., Juge, G., & Lagrange, J.-B. (1998). *Intégration de calculatrices complexes dans l'enseignement des mathématiques au lycée* [Integration of complex calculators in the teaching of mathematics at the lycée], Paris: Université Denis Diderot Paris 7, Équipe DIDIREM.

Ball, L., Pierce, R., & Stacey, K. (2003). Recognising equivalent algebraic expressions: An important component of algebraic expectation for working with CAS. In N.A. Pateman, B.J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 15-22). Honolulu, USA: PME.

Brown, C.A., Carpenter, T.P., Kouba, V.L., Lindquist, M.M., Silver, E.A., & Swafford, J.O. (1988). Secondary school results for the fourth NAEP mathematics assessment: Algebra, geometry, mathematical methods, and attitudes. *Mathematics Teacher, 81*, 337-347.

Drijvers, P.H.M. (2003). *Learning algebra in a computer algebra environment* (doctoral dissertation). Utrecht, The Netherlands: Freudenthal Institute.

Guin, D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3,* 195-227.

Heid, M.K. (1988). Resequencing skills and concepts in applied calculus using the computer as tool. *Journal for Research in Mathematics Education, 19*, 3-25.

Heid, M.K., Blume, G.W., Hollebrands, K., & Piez. C. (2002) Computer algebra systems in mathematics instruction: Implications from research. *Mathematics Teacher, 95*(8), 586-591.

Kieran, C., & Damboise, C. (2007). "How can we describe the relation between the factored form and the expanded form of these trinomials? – we don't even know if our paper-and-pencil factorizations are right": The case for Computer Algebra Systems (CAS) with weaker algebra students. In J.-H. Woo, H.-C. Lew, K.-S Park, & D.-Y Seo (Eds.), *Proceedings of 31st PME Conference* (Vol. 3, pp. 105-112). Seoul, Korea: PME.

Kieran, C., & Drijvers, P., with Boileau, A., Hitt, F., Tanguay, D., Saldanha, L., & Guzmán, J. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection: A study of CAS use in secondary school algebra. *International Journal of Computers for Mathematical Learning* 11(2), 205-263. Also available through: http://www.springerlink.com/content/u7t3580294652u37/

Lagrange, J.-B. (1996). Analyzing actual use of a computer algebra system in the teaching and learning of mathematics. *International DERIVE Journal, 3*, 91-108.

Lagrange, J.-B. (1999). Complex calculators in the classroom: Theoretical and practical reflections on teaching pre-calculus. *International Journal of Computers for Mathematical Learning, 4,* 51-81.

Mounier, G., & Aldon, G. (1996). A problem story: Factorisations of x^n-1. *International DERIVE Journal, 3*, 51-61.

National Council of Teachers of Mathematics. (1999). *Dialogues: Calculators – What is their place in mathematics classrooms?* May/June, pp. 1-16.

Shaw, N., Jean, B., & Peck, R. (1997). A statistical analysis on the effectiveness of using a computer algebra system in a developmental algebra course. *Journal of Mathematical Behavior, 16*, 175-180.

Skemp, R.R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20–26.

Zbiek, R.M. (2003). Using research to influence teaching and learning with computer algebra systems. In J.T. Fey, A. Cuoco, C. Kieran, L. McMullin, & R.M. Zbiek (Eds.), *Computer algebra systems in secondary school mathematics education* (pp. 197-216). Reston, VA: National Council of Teachers of Mathematics .