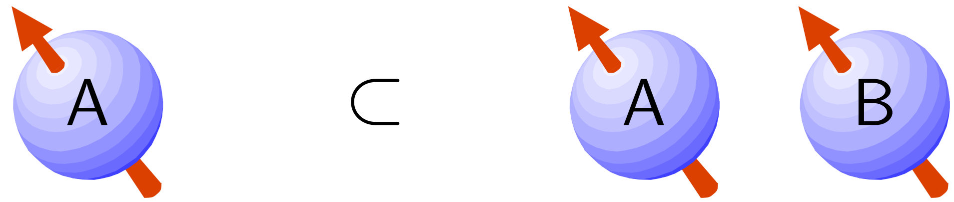
Quantum Error Correction I

Protecting Quantum Information

Manny

- Qubits are subsystems.
- Error control methods.
- Algebraic error models.
- Error detection.
- Error correction.
- Stabilizer codes.

One of Two Qubits



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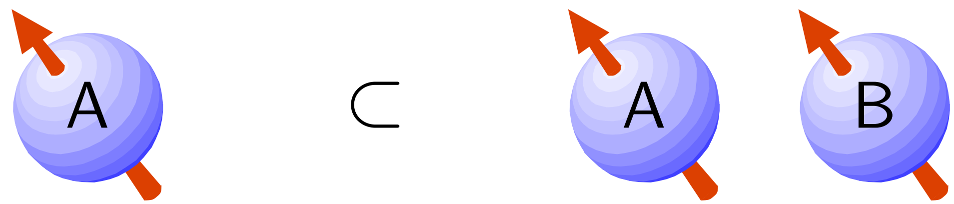

- State spaces:

$$\alpha|0\rangle_A + \beta|1\rangle_B \qquad \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$$

$$\mathcal{Q} \quad \overset{?}{\subset} \quad \mathcal{Q} \otimes \mathcal{Q}$$

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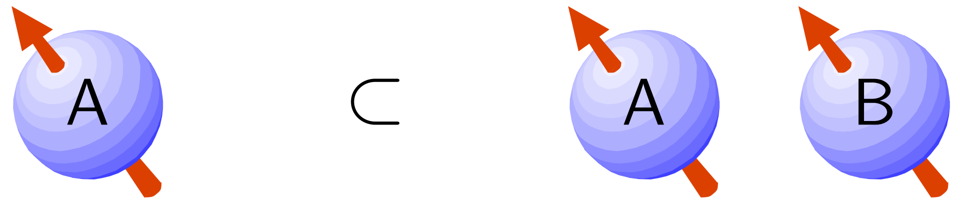

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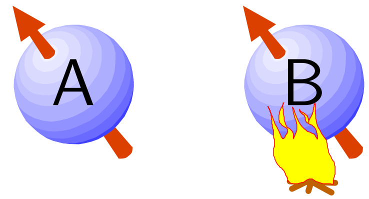
- Observable algebras:

$$\sigma_x{}^{(\text{A})}, \sigma_y{}^{(\text{A})}, \sigma_z{}^{(\text{A})}, \ldots \quad \subset \quad \sigma_x{}^{(\text{A})}, \ldots, \sigma_x{}^{(\text{B})}, \ldots, \sigma_x{}^{(\text{A})}\sigma_x{}^{(\text{B})}, \ldots$$

Passive Error Control



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$$\mathbb{I}, \quad {\sigma_x}^{(B)}, {\sigma_y}^{(B)}, {\sigma_z}^{(B)}.$$

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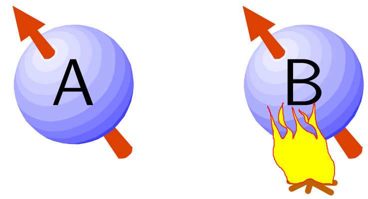

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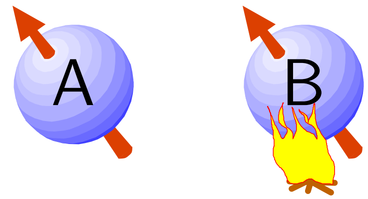

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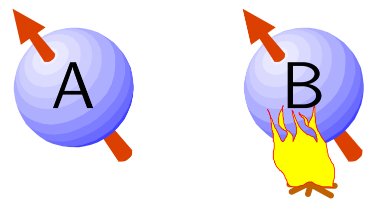

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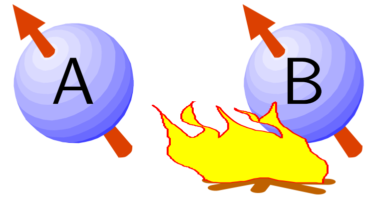
$$|00\rangle_{\mathrm{AB}} \overset{\sigma_x{}^{(\mathrm{B})}}{\longrightarrow} |01\rangle_{\mathrm{AB}}$$

  - Observables for A *commute* with errors.

$$\sigma_u{}^{(\mathrm{A})}\sigma_v{}^{(\mathrm{B})} = \sigma_v{}^{(\mathrm{B})}\sigma_u{}^{(\mathrm{A})}.$$

Active Error Control



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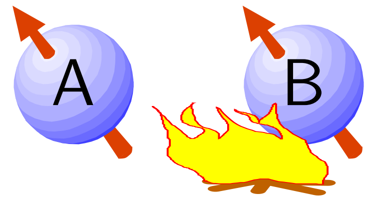

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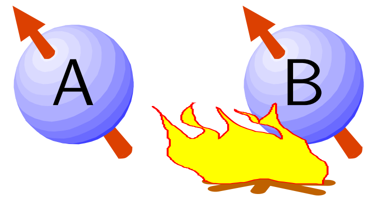

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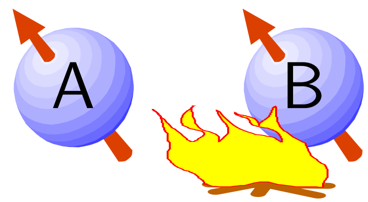

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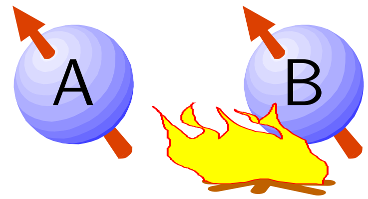

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  - Solution: Reset B before errors.

Back to: Error Correction I

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- Active error control.
  - Periodic error correction.

References

- Prehistory:
  - Quantum Zeno effect: Misra&Sudarshan 1977 [14].
  - Deutsch 1993, Barenco&*al.* 1996 [3].
- Discovery and Theory:
  - Shor 1995 [17], Steane 1995 [19].
  - Bennett&DiVincenzo&Smolin&Wootters 1996 [4], Knill&Laflamme 1996 [10].
  - Calderbank&Shor 1996 [6], Gottesman 1996 [7], Calderbank&Rains&Shor&Sloane 1997 [5].
- Fault tolerance and threshold accuracies:
  - Shor 1996 [18], Kitaev 1997 [9].
  - Aharonov&Ben-Or 1996 [1, 2], Knill&Laflamme&Zurek 1996 [12], Gottesman&Preskill 1997 [8, 16].
- Toward subsystems:
  - Quasi-particles . . .
  - Zanardi&Rasetti 1997 [23], Lidar&Chuang&Whaley 1998 [13].
  - Viola&Knill&Lloyd 1998 [21, 20, 22].
  - Knill&Laflamme 1996 [10], Knill&Laflamme&Viola 2000 [11].

General reference: (M)ike, Ch. 10. Nielsen&Chuang 2001 [15]

The Pauli Error Model

- Error operators:

$$\mathcal{E}_1 \quad = \quad \{\, \mathbb{I},\ {\sigma_x}^{(1)},\ {\sigma_y}^{(1)},\ {\sigma_z}^{(1)},\ {\sigma_x}^{(2)},\ {\sigma_y}^{(2)},\ \ldots \,\}$$

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- Weight 2 error operators:

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  - The linear span of $\mathcal{E}$ contains all operators.

Algebraic Error Models

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|0010\rangle &\overset{{\sigma_x}^{(2)}}{\longrightarrow} |0110\rangle \\
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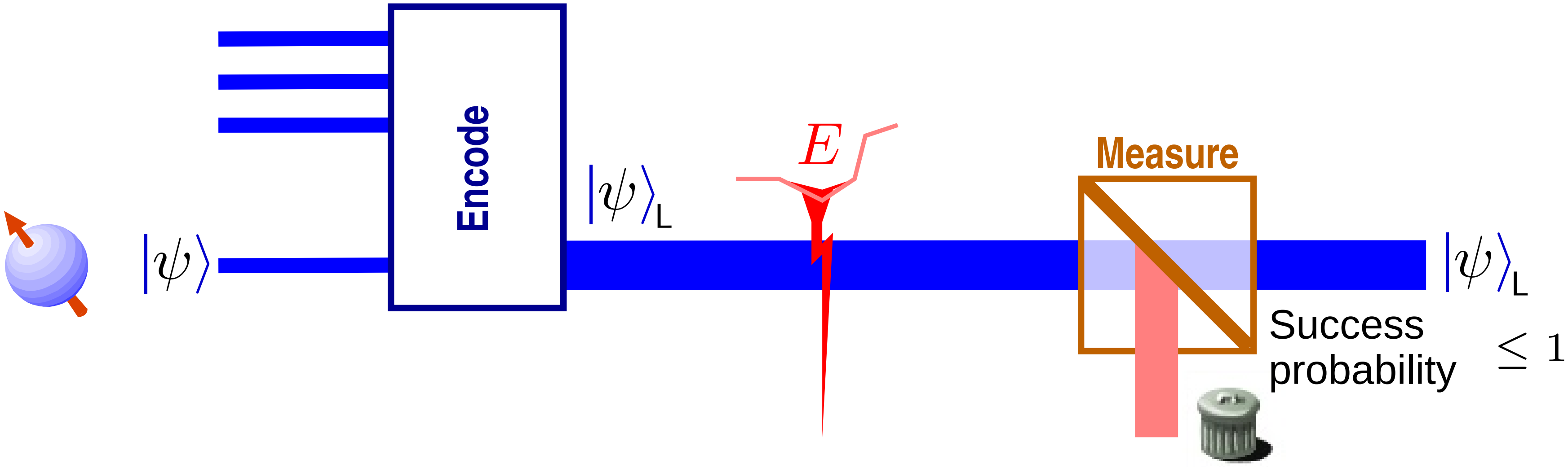
- Size of common eigenspaces of ${\sigma_x}^{(i)}$?

Error Detection I



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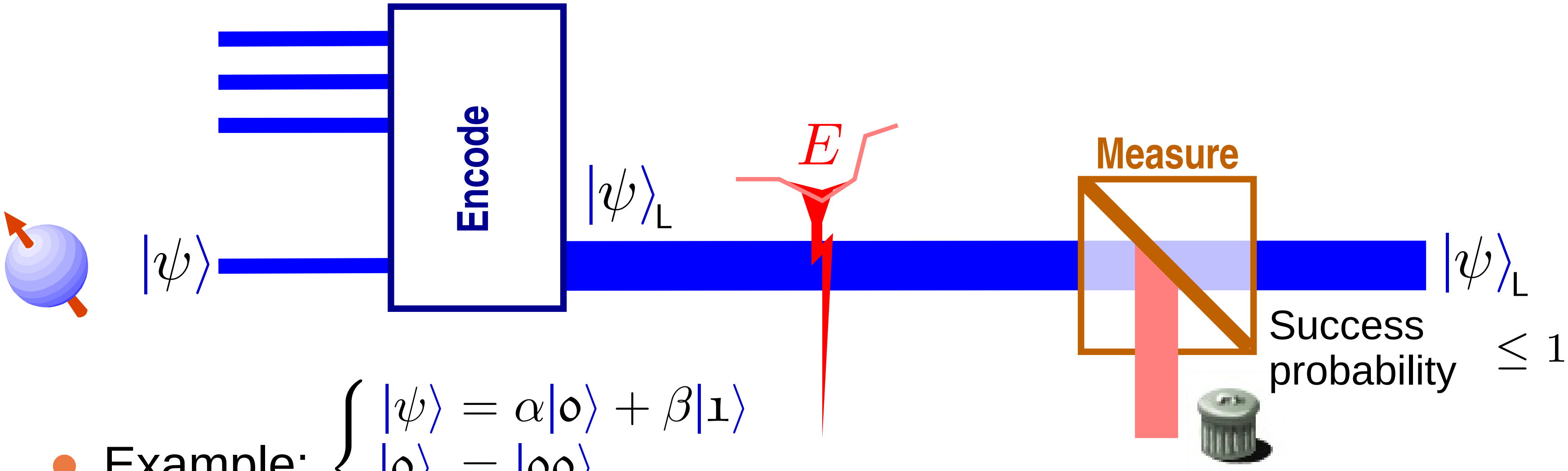

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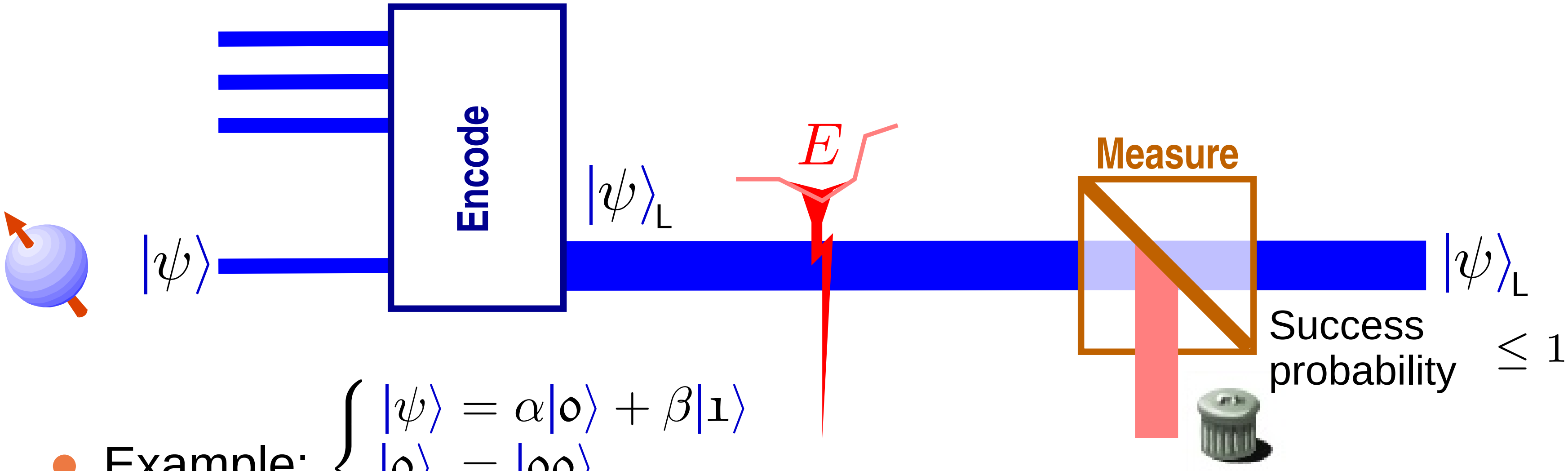

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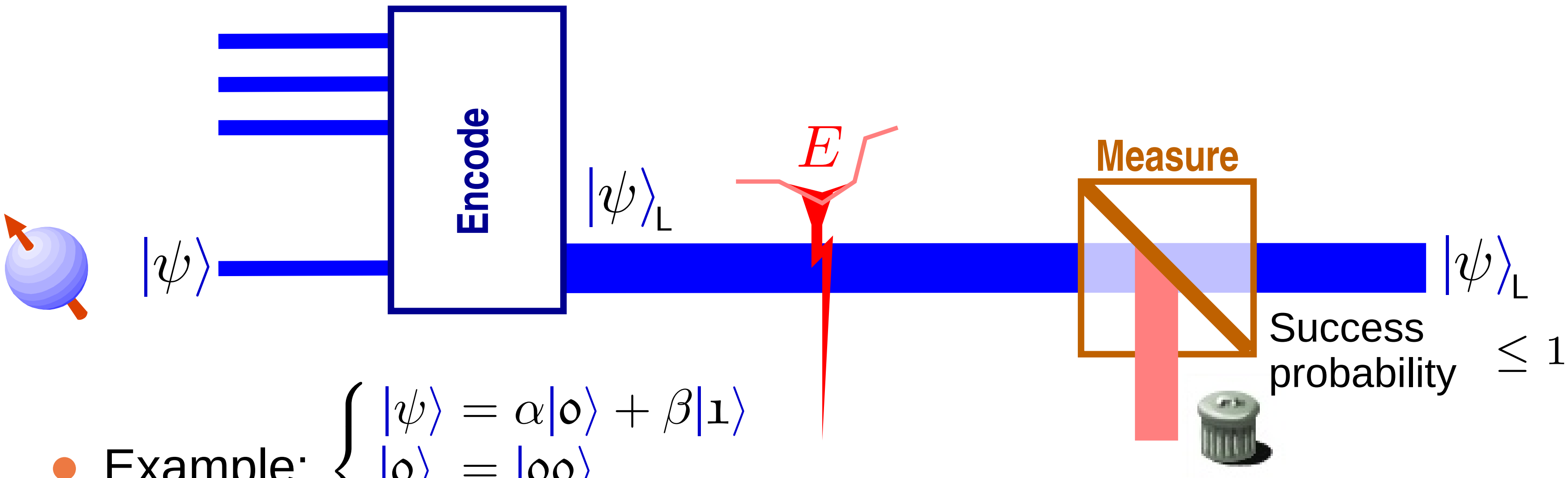

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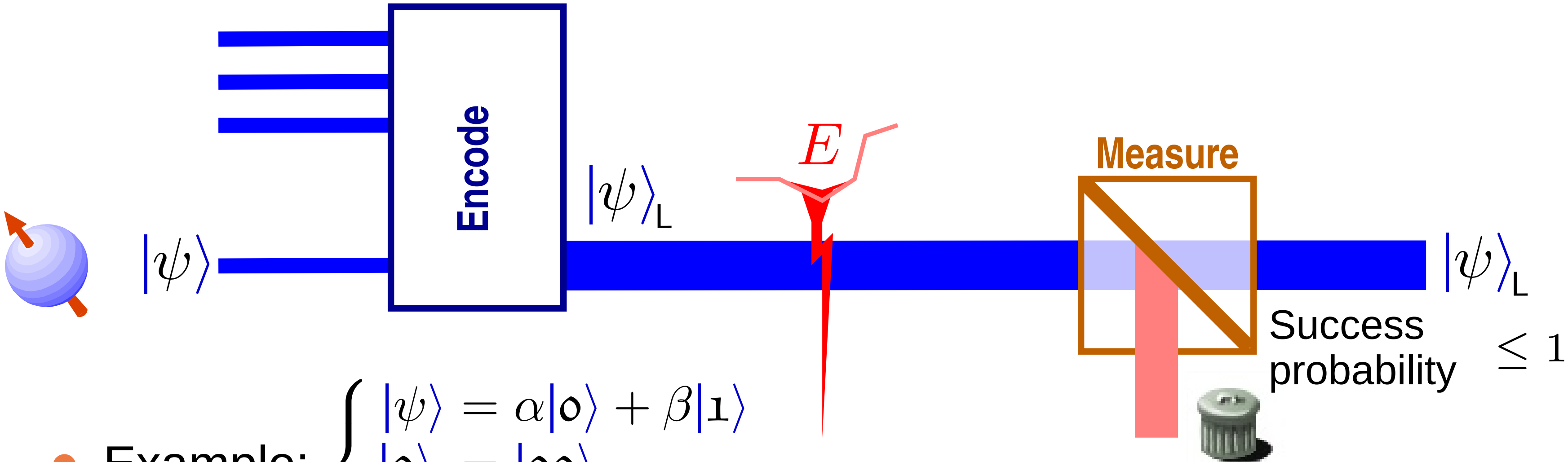

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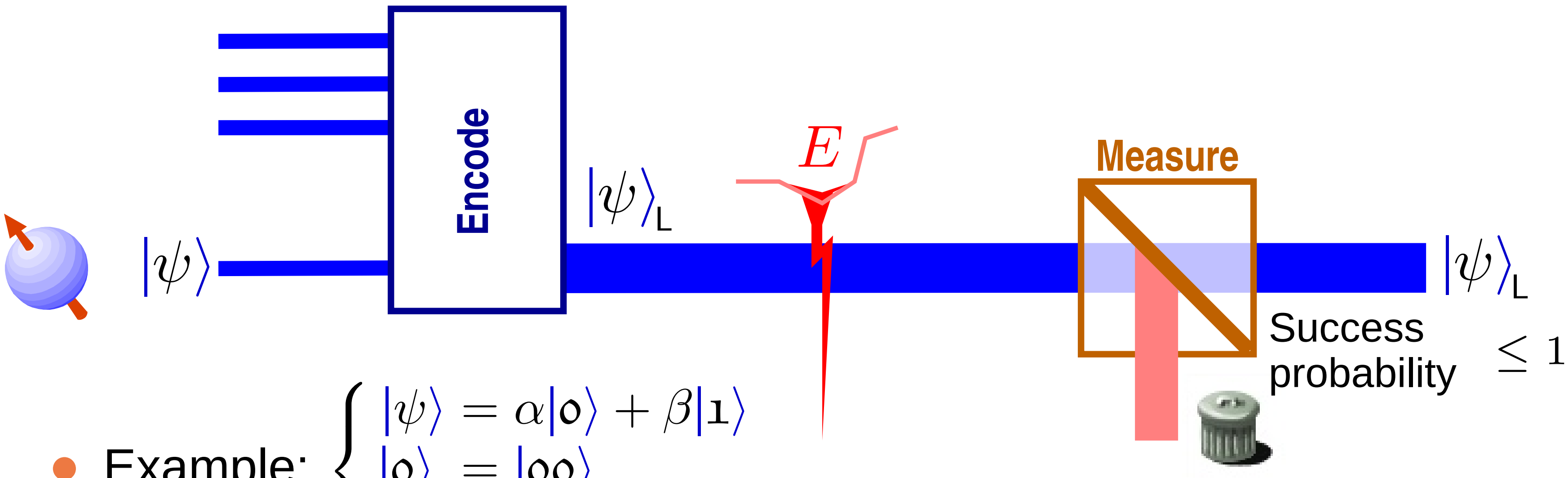

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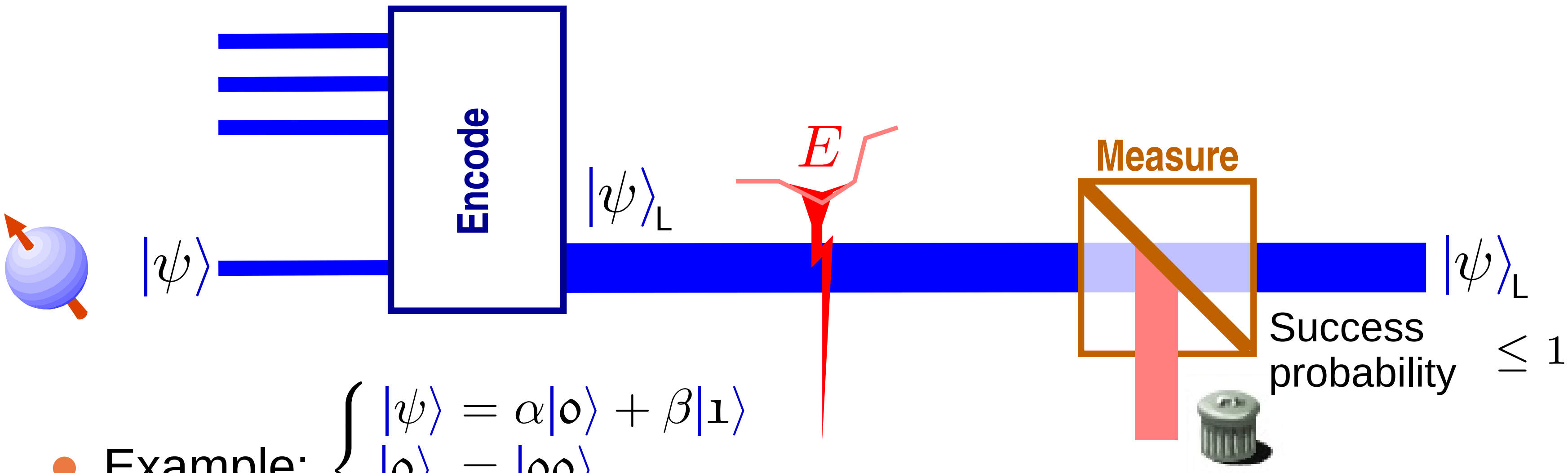

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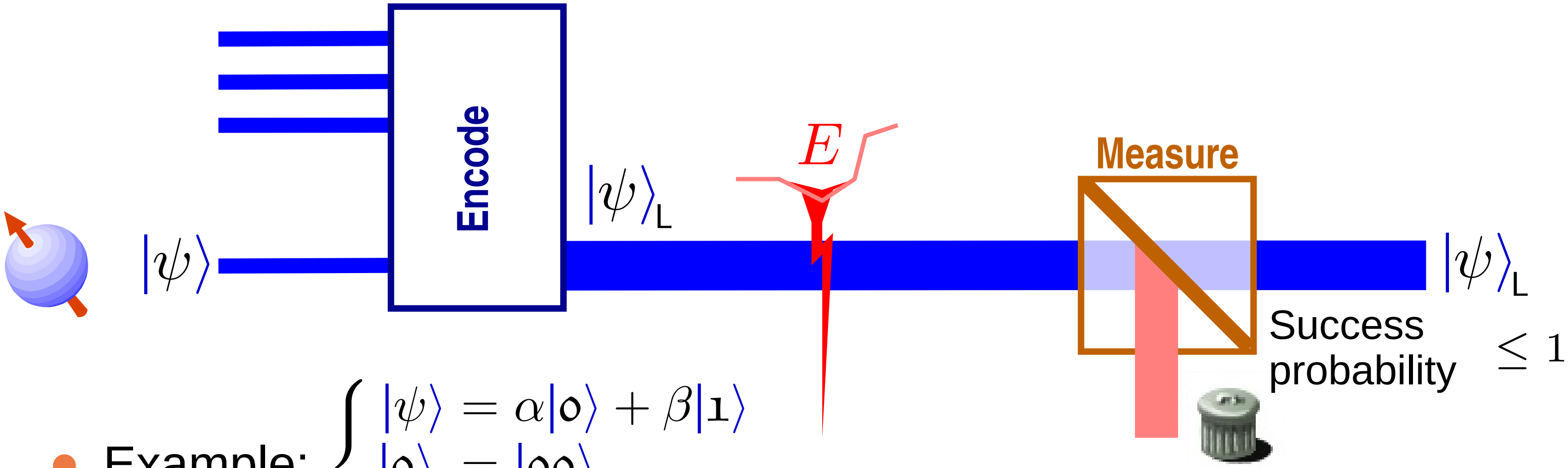

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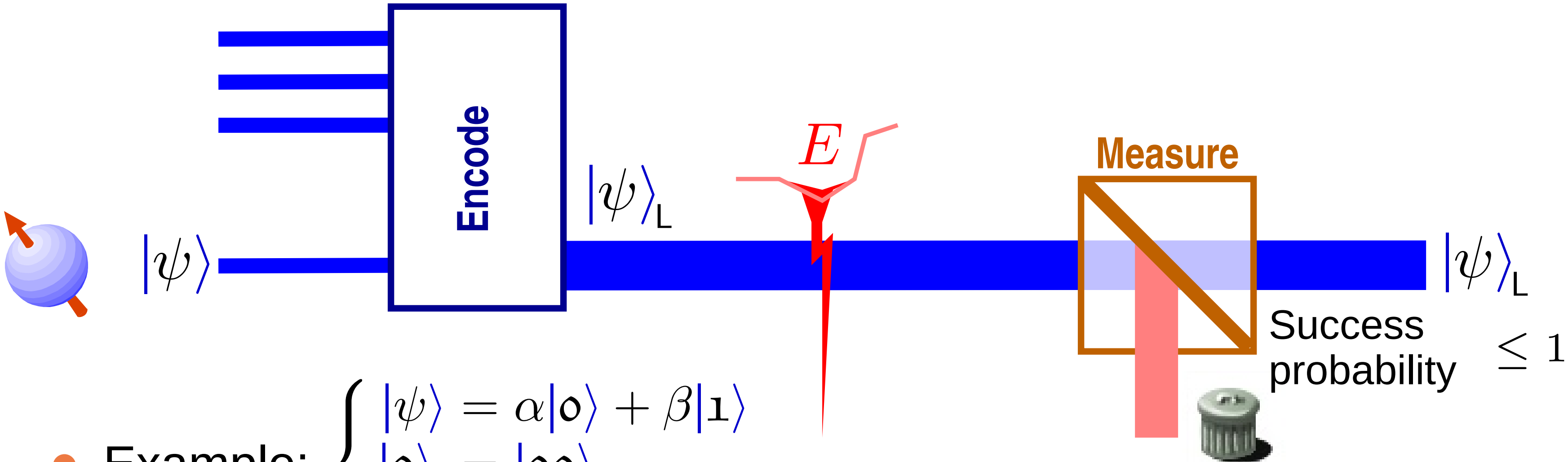

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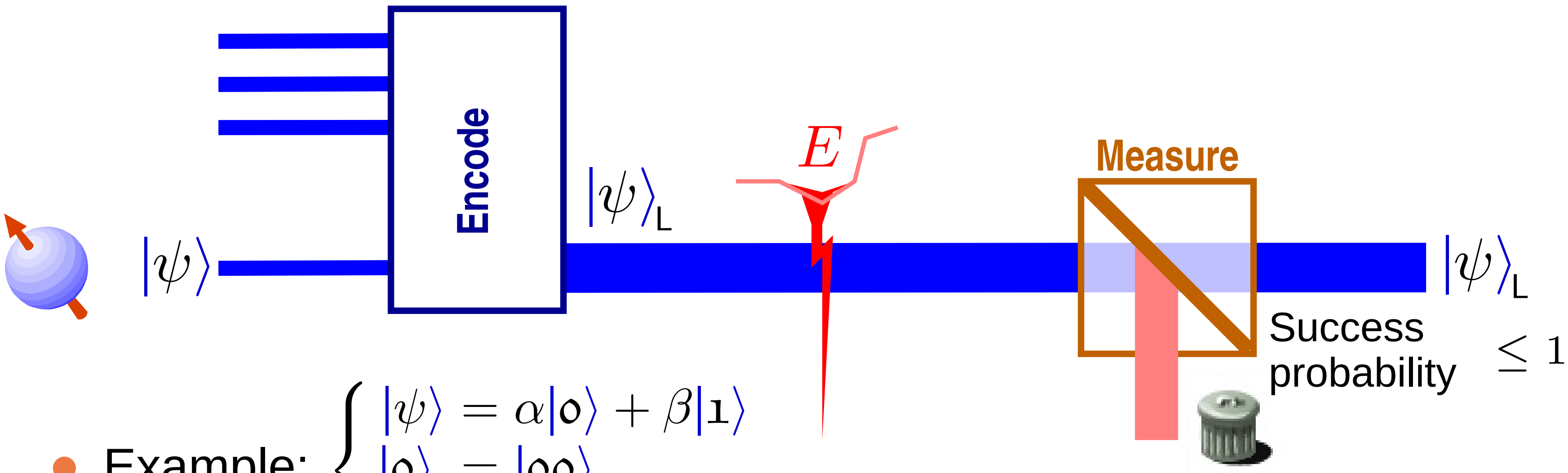

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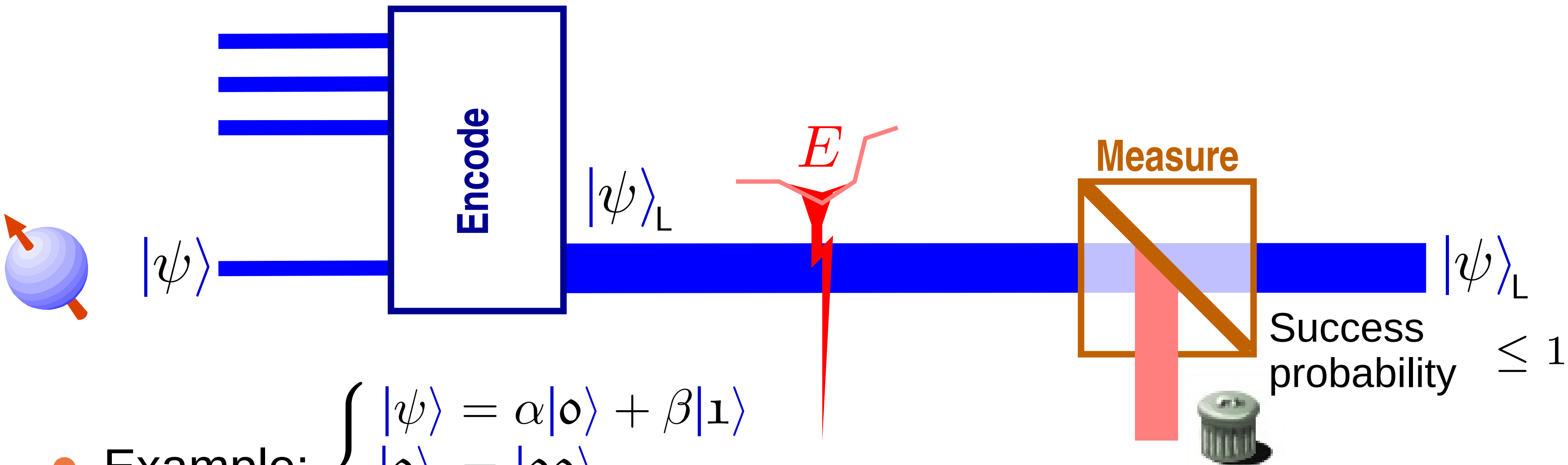

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Error Detection II

- A *quantum code* is a subspace $\mathcal{C}$.
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- Equivalently:

$$E = \begin{pmatrix} \mathcal{C}\left\{ \overbrace{\begin{matrix} \lambda_E & 0 & \ldots & 0 \\ 0 & \lambda_E & & \vdots \\ \vdots & & \ddots & \\ 0 & & \ldots & \lambda_E \end{matrix}}^{\mathcal{C}} \right. & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$$

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- Equivalently: For all $|\phi\rangle_{\mathsf{L}}, |\psi\rangle_{\mathsf{L}}$

$$|\phi\rangle_{\mathsf{L}} \perp |\psi\rangle_{\mathsf{L}} \Rightarrow E|\phi\rangle_{\mathsf{L}} \perp |\psi\rangle_{\mathsf{L}}$$

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- $\mathcal{C}$ is a $[[n,k,d]]_{\mathcal{E}_1}$ code means:
  - *Length* n: Total number of qubits is n.
  - k encoded qubits, $\dim \mathcal{C} = 2^k$.
  - Minimum distance at least d for $\mathcal{E}_1$.

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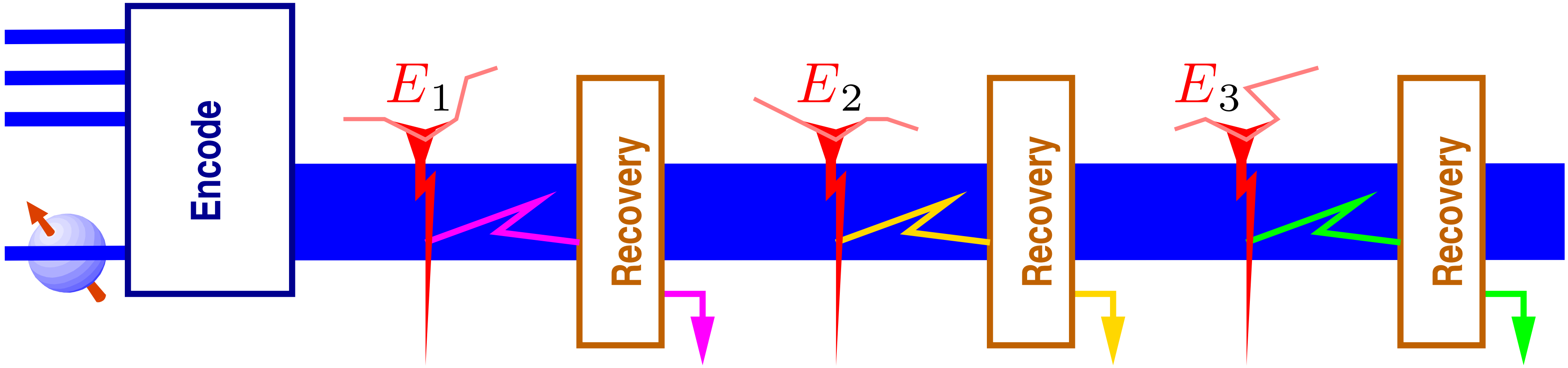
  - Encode $\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$.
- . . . the three bit repetition code.

Error Correction Process



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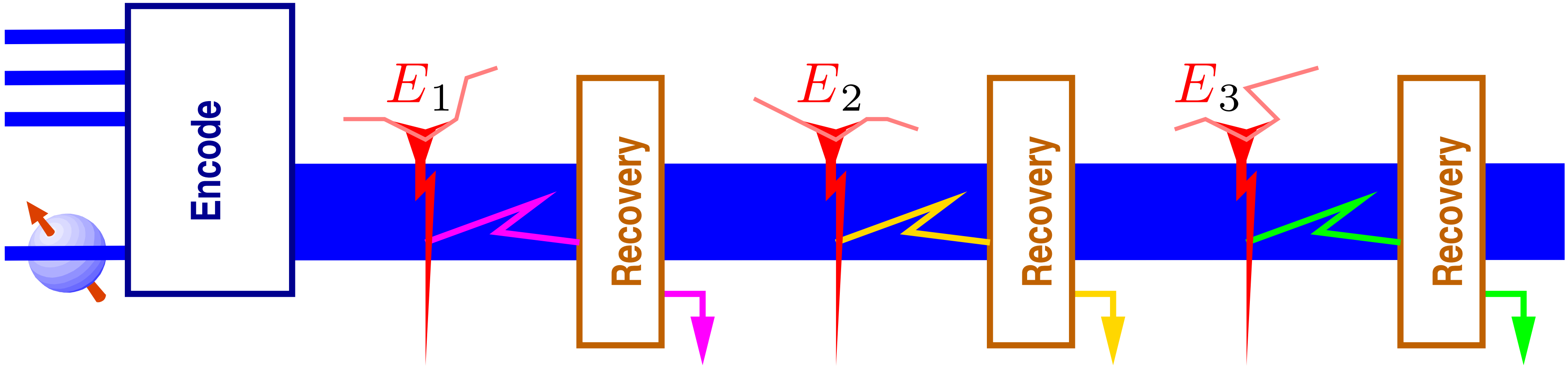

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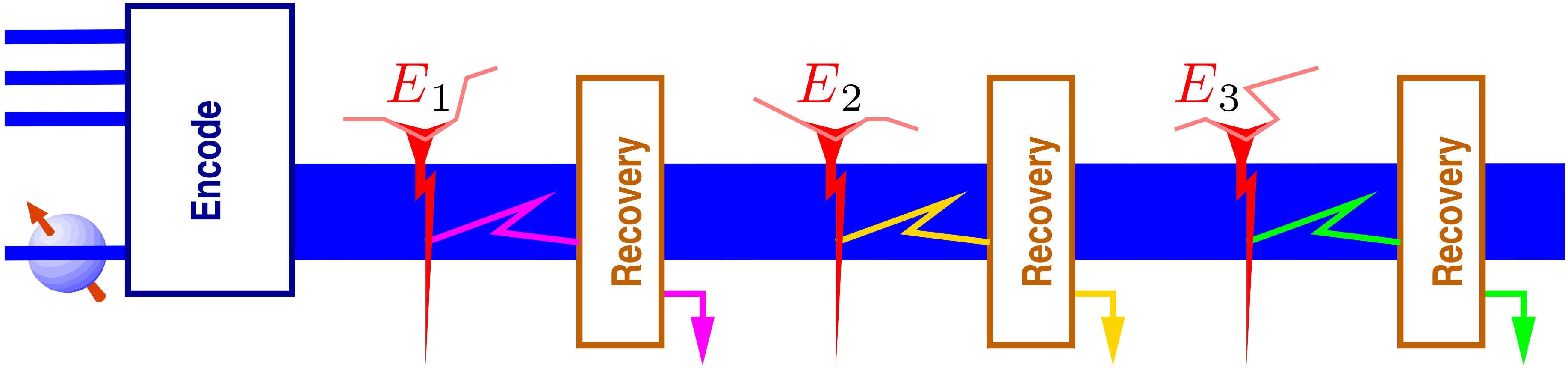

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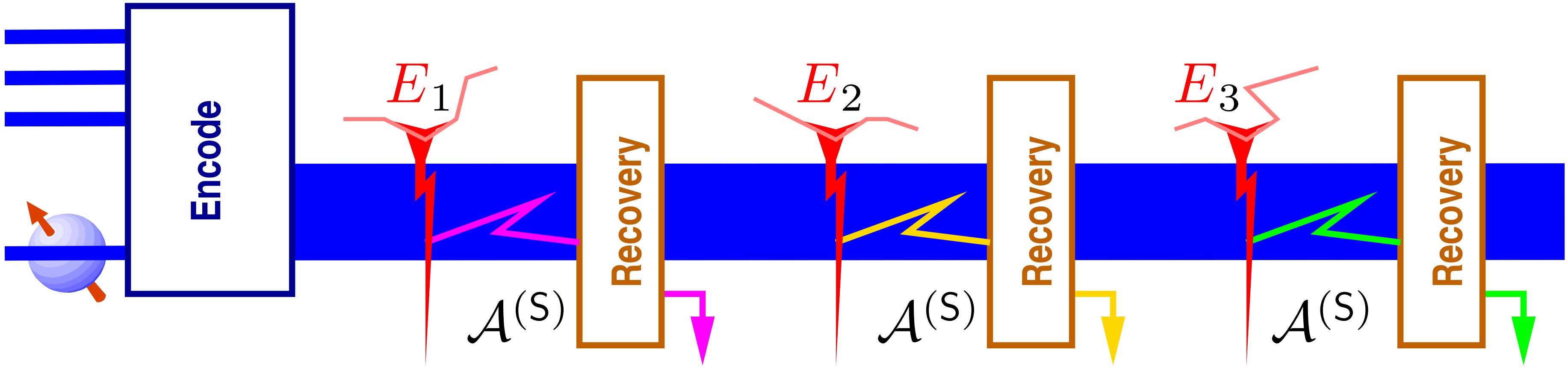

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  - $\mathcal{A}^{(\mathsf{S})}$: Algebra between errors and recovery.

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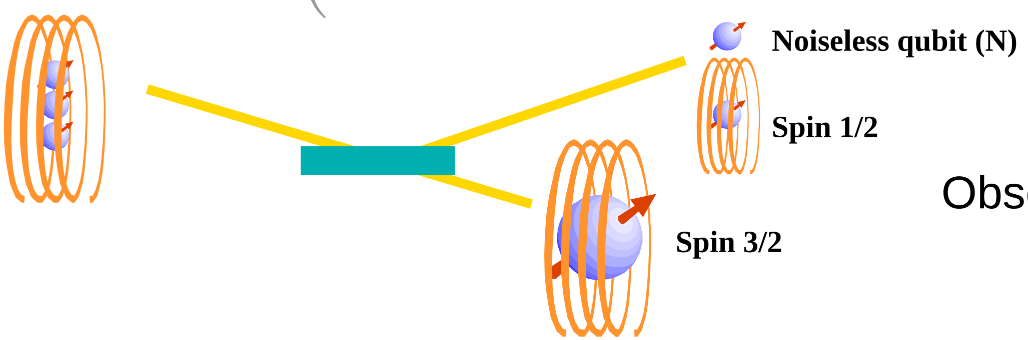

Observables: $\left\{ \begin{array}{l} {\sigma_U}^{(\mathsf{N})} = \sigma^{(\mathsf{A})} \cdot \sigma^{(\mathsf{B})} \\ {\sigma_V}^{(\mathsf{N})} = \sigma^{(\mathsf{A})} \cdot \sigma^{(\mathsf{C})} \end{array} \right.$

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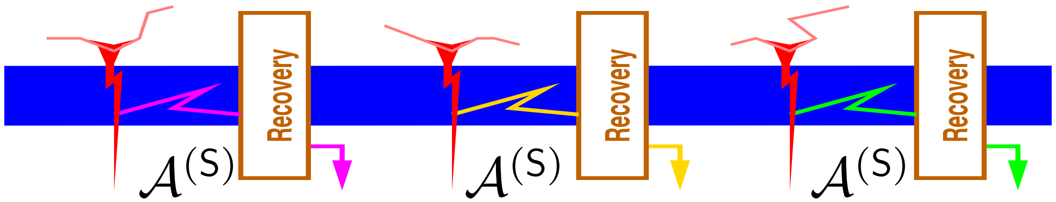

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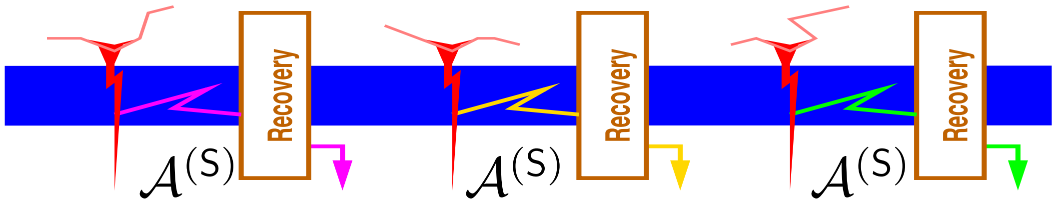

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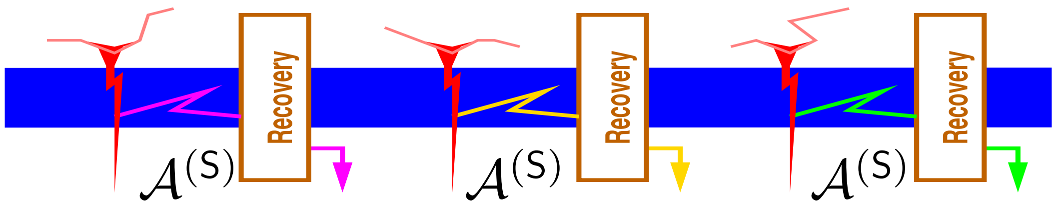

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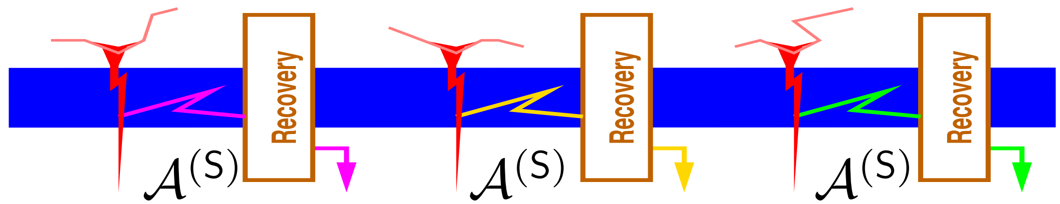

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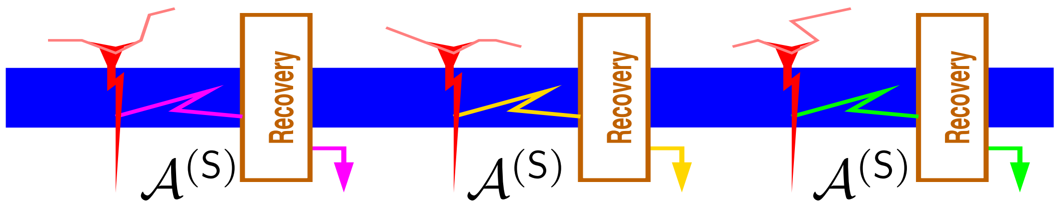

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Example.

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- Coding theory lingo: A $[[n, k, 2e+1]]$ code is e-error correcting.

Stabilizer Codes I

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- Properties:

$$\mathcal{C} \quad \text{detects} \quad IXI$$

$$IXI \cdot ZZI \quad = \quad -ZZI \cdot IXI$$

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$$\begin{aligned} \rho|\psi\rangle &= \lambda(\rho)|\psi\rangle \\ \rho\,\sigma|\psi\rangle &= -\sigma\,\rho|\psi\rangle \\ &= -\lambda(\rho)\sigma|\psi\rangle \end{aligned}$$

$\ldots\sigma|\psi\rangle$ is orthogonal to the code.

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- Rules:

$$\begin{array}{ccccccc}
X\cdot Y & \sim & Z & & 01\oplus 10 & = & 11 \\
Y\cdot Z & \sim & X & \Leftrightarrow & 10\oplus 11 & = & 01 \\
Z\cdot X & \sim & Y & & 11\oplus 01 & = & 10
\end{array}$$

References

[1] D. Aharonov and M. Ben-Or. Fault-tolerant quantum computation with constant error. In *Proceedings of the 29th Annual ACM Symposium on the Theory of Computation (STOC)*, pages 176–188, New York, New York, 1996. ACM Press.

[2] D. Aharonov and M. Ben-Or. Fault-tolerant quantum computation with constant error. quant-ph/9906129, 1999.

[3] A. Barenco, A. Berthiaume, D. Deutsch, A. Ekert, R. Jozsa, and C. Macchiavello. Stabilization of quantum computations by symmetrization. *SIAM J. Comput.*, 26:1541–1557, 1997.

[4] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters. Mixed state entanglement and quantum error-correcting codes. *Phys. Rev. A*, 54:3824–3851, 1996.

[5] A.R. Calderbank, E.M. Rains, P.W. Shor, and N.J.A. Sloane. Quantum error correction and orthogonal geometry. *Phys. Rev. A*, 78:405–408, 1997.

[6] A.R. Calderbank and P.W. Shor. Good quantum error-correcting codes exist. *Phys. Rev. A*, 54:1098–1105, 1996.

[7] D. Gottesman. A class of quantum error-correcting codes saturating the quantum hamming bound. *Phys. Rev. A*, 54:1862–1868, 1996.

[8] D. Gottesman. A theory of fault-tolerant quantum computation. *Phys. Rev. A*, 57:127–137, 1998.

[9] A. Yu. Kitaev. Quantum computations: algorithms and error correction. *Russian Math. Surveys*, 52:1191–1249, 1997.

[10] E. Knill and R. Laflamme. A theory of quantum error correcting codes. *Phys. Rev. A*, 55:900–911, 1997.

[11] E. Knill, R. Laflamme, and L. Viola. Theory of quantum error correction for general noise. *Phys. Rev. Lett.*, 84:2525–2528, 2000.

[12] E. Knill, R. Laflamme, and W. Zurek. Resilient quantum computation: Error models and thresholds. *Proc. R. Soc. Lond. A*, 454:365–384, 1998.

[13] D. A. Lidar, I. L. Chuang, and K. B. Whaley. Decoherence-free subspaces for quantum computation. *Phys. Rev. Lett.*, 81:2594–2597, 1998.

[14] B. Misra and E. C. G. Sudarshan. Zenos paradox in quantum theory. *J. Math. Phys.*, 18:756–763, 1977.

[15] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cmabridge University Press, 2001.

[16] J. Preskill. Reliable quantum computers. *Proc. R. Soc. Lond. A*, 454:385–410, 1998.

[17] P. W. Shor. Scheme for reducing decoherence in quantum computer memory. *Phys. Rev. A*, 52:2493–2496, 1995.

[18] P. W. Shor. Fault-tolerant quantum computation. In *Proceedings of the 37th Symposium on the Foundations of Computer Science (FOCS)*, pages 56–65, Los Alamitos, California, 1996. IEEE press.

[19] A. Steane. Multiple particle interference and quantum error correction. *Proc. R. Soc. Lond. A*, 452:2551–2577, 1996.

[20] L. Viola, E. Knill, and S. Lloyd. Dynamical decoupling of open quantum systems. *Phys. Rev. Lett.*, 82:2417–2421, 1999.

[21] L. Viola and S. Lloyd. Dynamical suppression of decoherence in two-state quantum systems. *Phys. Rev. A*, 58:2733–2744, 1998.

[22] L. Viola, S. Lloyd, and E. Knill. Universal control of decoupled quantum systems. *Phys. Rev. Lett.*, 83:4888–4891, 1999.

[23] P. Zanardi and M. Rasetti. Noiseless quantum codes. *Phys. Rev. Lett.*, 79:3306–3309, 1997.