# PROBLEMS INVOLVING PRESCRIBED GRADIENT IMAGE

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Let $\Omega \subset \mathbf{R}^n$ be a bounded convex domain, $f, g$ smooth positive functions. If $u \in C^2(\Omega)$ is a uniformly convex solution of

$$\det D^2 u = \frac{f(x)}{g(Du)} \quad \text{in} \quad \Omega, \tag{1}$$

then $Du : \Omega \to \mathbf{R}^n$ is a diffeomorphism onto $\Omega^* = Du(\Omega)$.

Conversely, given bounded convex domains $\Omega, \Omega^*$ in $\mathbf{R}^n$ and smooth positive functions $f, g$ satisfying the compatibility condition

$$\int_\Omega f(x)dx = \int_{\Omega^*} g(y)dy, \tag{2}$$

is there a convex solution of

$$\begin{aligned} \det D^2 u &= \frac{f(x)}{g(Du)} \quad \text{in} \quad \Omega, \\ Du(\Omega) &= \Omega^* ? \end{aligned} \tag{3}$$

**Monge mass transport problem.** Minimize

$$\mathcal{C}(\mathbf{s}) = \int_\Omega |x - \mathbf{s}(x)|^2 f(x)dx \tag{4}$$

among all maps $\mathbf{s} : \Omega \to \Omega^*$ which push forward the measure $d\mu = f(x)dx$ onto $d\nu = g(y)dy$.

- There is a unique weak solution $\mathbf{s}$ of the problem and $\mathbf{s} = Du$ for some convex function $u$ solving (3) in a weak sense [Brenier 1990].

- If $\Omega^*$ is convex, and $f, g \in C^{0,\alpha}$ are positive, then $u \in C^{2,\alpha}(\Omega)$ for any $\alpha \in (0,1)$ [Caffarelli 1992].

- If $\Omega, \Omega^*$ are uniformly convex, $\partial\Omega, \partial\Omega^* \in C^{2,\alpha}$ and $f \in C^{0,\alpha}(\overline{\Omega})$, $g \in C^{0,\alpha}(\overline{\Omega}^*)$ are positive, then $u \in C^{2,\alpha}(\overline{\Omega})$ [Caffarelli 1996; Urbas 1997 (under slightly stronger regularity assumptions)].

**Questions** (i) What can be proved for other cost functions? Existence and uniqueness results for strictly convex costs have been proved by Gangbo and McCann, Caffarelli. Very little is known about regularity of optimal maps for nonquadratic costs.

(ii) What can be proved if $\det D^2u$ is replaced by

$$f(\lambda_1, \ldots, \lambda_n)$$

where $\lambda_1, \ldots, \lambda_n$ are either:

• the eigenvalues of $D^2u$;

• the principal curvatures of the graph of $u$, i.e., the eigenvalues of

$$\frac{D_{ij}u}{\sqrt{1+|Du|^2}} \quad \text{relative to} \quad \delta_{ij} + D_iuD_ju.$$

This leads to the boundary value problems

$$\text{(Hessian)} \qquad \begin{aligned} F(D^2u) &= g(x, u, Du) \quad \text{in} \quad \Omega, \\ Du(\Omega) &= \Omega^*. \end{aligned}$$

$$\text{(Curvature)} \qquad \begin{aligned} F(Du, D^2u) &= g(x, u) \quad \text{in} \quad \Omega, \\ Du(\Omega) &= \Omega^*. \end{aligned}$$

What are natural conditions to impose on $f$? If $u$ is a locally uniformly convex solution of

$$\begin{aligned} F(D^2u) &= g(x, u, Du) \quad \text{in} \quad \Omega, \\ Du(\Omega) &= \Omega^*, \end{aligned} \tag{5}$$

then the Legendre transform

$$u^*(y) = x \cdot y - u(x), \qquad y = Du(x),$$

is a locally uniformly convex solution of

$$\begin{aligned} \frac{1}{F([D^2u^*]^{-1})} &= \frac{1}{g(Du^*, y \cdot Du^* - u^*, y)} \quad \text{in} \quad \Omega^*, \\ Du^*(\Omega^*) &= \Omega. \end{aligned} \tag{6}$$

So $u^*$ satisfies a similar kind of problem with

$$f^*(\lambda_1, \ldots, \lambda_n) = \frac{1}{f(\lambda_1^{-1}, \ldots, \lambda_n^{-1})}.$$

We should impose conditions on $f$ such that the convexity of the solution is a natural assumption for both (5) and (6).

**Assumptions:**

(i) $f \in C^\infty(\Gamma_+) \cap C(\overline{\Gamma}_+)$;

(ii) $f$ is symmetric;

(iii) $f > 0$ in $\Gamma_+$ and $f = 0$ on $\partial\Gamma_+$;

(iv) $f_i = \frac{\partial f}{\partial \lambda_i} > 0$ on $\Gamma_+$ for $i = 1, \ldots, n$;

(v) $f(t, \ldots, t) \to \infty$ as $t \to \infty$;

(vi) $f(\lambda', \lambda_n) \to \infty$ as $\lambda_n \to \infty$

for any $\lambda' \in \Gamma_+^{n-1} \subset \mathbf{R}^{n-1}$.

These are "natural" assumptions in the following sense.

**Proposition.** *If $f$ satisfies (i)—(vi), then so does $f^*$.*

**Additional assumptions:**

(vii) $f$ is concave;

(viii) $\sum_{i=1}^n f_i\lambda_i^2 \to \infty$ as $|\lambda| \to \infty$ on $\{\lambda : \mu_1 \le f(\lambda) \le \mu_2\}$ for any $\mu_2 \ge \mu_1 > 0$.

**Definition.** $f \in \mathcal{F} \iff f$ satisfies (i)—(viii).

**Examples.**

1. $f(\lambda) = \left(\prod_{i=1}^n \lambda_i\right)^{1/n}$ (Monge-Ampère).

2. (A nonexample) For any integer $0 < m \le n$ let

$$S_m(\lambda_1, \ldots, \lambda_n) = \sum_{1 \le i_1 < \cdots < i_m \le n} \lambda_{i_1} \cdots \lambda_{i_m}.$$

If $m < n$, $f = (S_m)^{1/m}$ satisfies all the conditions except $f = 0$ on $\partial\Gamma_+$. So $f \notin \mathcal{F}$. Convexity of the solution is not natural if $m < n$.

3. (Another nonexample) For $l = 1, \ldots, n-1$

$$f_{n,l}(\lambda) = \left(\frac{S_n(\lambda)}{S_l(\lambda)}\right)^{\frac{1}{n-l}}$$

satisfies all the conditions except (vi) (and (viii) if $l = n-1$). So $f_{n,l} \notin \mathcal{F}$.

If $f = f_{n,l}$, then $f^* = (S_{n-l})^{\frac{1}{n-l}}$.

4.

$$f_\epsilon = \epsilon (S_n)^{1/n} + f_{n,l}, \qquad \epsilon > 0,$$

and

$$\tilde{f}_\alpha = \left( \frac{S_n^{1-\alpha l/n}}{S_l^{1-\alpha}} \right)^{\frac{1}{n-l}},$$

$$l = 1, \ldots, n-1, \qquad \alpha \in (0,1],$$

belong to $\mathcal{F}$. Notice that $f_\epsilon \to f_{n,l}$ and $\tilde{f}_\alpha \to f_{n,l}$ as $\epsilon \to 0$ and $\alpha \to 0$.

5. If $\alpha_1, \ldots, \alpha_n \in [0,1]$, $\sum \alpha_k \leq 1$, $\alpha_n > 0$, then

$$\hat{f} = \prod_{k=1}^{n} S_k^{\alpha_k/k}$$

belongs to $\mathcal{F}$.

**Hessian equations.**

**Theorem 1.** *Suppose $f \in \mathcal{F}$ and $\Omega$, $\Omega^* \subset \mathbf{R}^n$ are uniformly convex with $C^\infty$ boundaries. Suppose $g \in C^\infty(\overline{\Omega} \times \mathbf{R} \times \overline{\Omega}^*)$ is positive and*

$$g(x,z,p) \to \infty \quad \text{as} \quad z \to \infty,$$
$$g(x,z,p) \to 0 \quad \text{as} \quad z \to -\infty,$$

*uniformly for all $(x,p) \in \Omega \times \Omega^*$, and $g$ is convex with respect to $Du$. Then the problem*

$$F(D^2u) = g(x,u,Du) \quad \text{in} \quad \Omega,$$
$$Du(\Omega) = \Omega^*.$$

*has a convex solution $u \in C^\infty(\overline{\Omega})$. If $g_z > 0$, the solution is unique in the class of convex functions.*

**Theorem 2.** *Let $\Omega$, $\Omega^*$ be as above and $g \in C^\infty(\overline{\Omega} \times \mathbf{R})$ positive and satisfying*

$$g_z \geq 0 \quad \text{on} \quad \Omega \times \mathbf{R}$$

*and*

$$g(x,z) \to \infty \quad \text{as} \quad z \to \infty,$$
$$g(x,z) \to 0 \quad \text{as} \quad z \to -\infty,$$

*uniformly for $x \in \Omega$. Then for $l = 1, \ldots, n-1$, the problem*

$$F_{n,l}[u] = g(x, u) \quad in \quad \Omega,$$
$$Du(\Omega) = \Omega^*,$$

*has a convex solution $u \in C^\infty(\overline{\Omega})$. If $g_z > 0$, the solution is unique in the class of convex functions.*

$$F_{n,l} \longleftrightarrow f_{n,l} = \left(\frac{S_n}{S_l}\right)^{\frac{1}{n-l}}$$

**Theorem 3.** *Let $\Omega$, $\Omega^*$ be as above and assume that $g \in C^\infty(\mathbf{R} \times \overline{\Omega}^*)$ is a positive function satisfying*

$$g_z \geq 0 \quad on\ \mathbf{R} \times \Omega^*,$$

$$g(z, p) \to \infty \quad as \quad z \to \infty,$$
$$g(z, p) \to 0 \quad as \quad z \to -\infty,$$

*uniformly for $p \in \Omega^*$. Then for $k = 1, \ldots, n-1$, the problem*

$$F_k[u] = \frac{1}{g(x \cdot Du - u, Du)} \quad in \quad \Omega,$$
$$Du(\Omega) = \Omega^*,$$

*has a convex solution $u \in C^\infty(\overline{\Omega})$. If $g_z > 0$, the solution is unique in the class of convex functions.*

$$F_k \longleftrightarrow (S_k)^{1/k}$$

**Curvature equations.** We need an extra technical assumption:

$$\text{(ix)} \qquad \sum f_i(\lambda)\lambda_i \geq c(\mu_1, \mu_2) > 0$$

$$\text{in} \quad \{\lambda : \mu_1 \leq f(\lambda) \leq \mu_2\}$$

for any $\mu_2 \geq \mu_1 > 0$ (e.g., $f$ is homogeneous).

**Theorem 4.** *Suppose that $f \in \mathcal{F}$ satisfies (ix), $\Omega$, $\Omega^*$ are uniformly convex domains in $\mathbf{R}^n$ with $C^\infty$ boundaries, and $g \in C^\infty(\overline{\Omega} \times \mathbf{R})$ is positive and satisfies*

$$g(x, z) \to \infty \quad \text{as} \quad z \to \infty,$$
$$g(x, z) \to 0 \quad \text{as} \quad z \to -\infty,$$

*uniformly for all $x \in \Omega$. Then the problem*

$$F(Du, D^2u) = g(x, u) \quad \text{in} \quad \Omega,$$
$$Du(\Omega) = \Omega^*,$$

*has a convex solution $u \in C^\infty(\overline{\Omega})$. If $g_z > 0$, the solution is unique in the class of convex functions.*

**Sketch of proofs.**

1. Reformulate $Du(\Omega) = \Omega^*$ as

$$h(Du) = 0 \quad \text{on} \quad \partial\Omega \tag{7}$$

where $h \in C^\infty(\mathbf{R}^n)$ is a uniformly concave defining function for $\Omega^*$:

$$\Omega^* = \{p \in \mathbf{R}^n : h(p) > 0\}, \quad |Dh| = 1 \quad \text{on} \quad \partial\Omega.$$

2. Use continuity method to reduce to a priori estimates in $C^{2,\alpha}(\overline{\Omega})$.

3. Degenerate obliqueness: If $u \in C^2(\overline{\Omega})$ is uniformly convex, then

$$\chi := h_p(Du) \cdot \nu = \sqrt{u^{ij}\nu_i\nu_j \; u_{kl}\nu_k^*\nu_l^*} \geq 0,$$

where $\nu, \nu^*$ are the inner unit normal vector fields to $\partial\Omega, \partial\Omega^*$. The boundary condition is not a priori strictly oblique.

4. $C^0$ estimate: Use assumptions on $g$ and degenerate obliqueness.

5. $C^1$ estimate: Immediate from the boundary condition.

6. Once the second derivatives are bounded, global $C^{2,\alpha}$ estimates follow from strictly oblique, uniformly elliptic theory [Lieberman & Trudinger, Evans, Krylov]. So only the second derivatives need to be estimated.

7. Strict obliqueness: Suppose $\chi|_{\partial\Omega}$ has its minimum at $x_0 \in \partial\Omega$. Rotate coordinates so that $\nu(x_0) = e_n$. Then

$$h_{p_k} D_{k\alpha} u = 0 \quad \text{at} \quad x_0 \qquad \text{for} \quad \alpha = 1, \ldots, n-1, \tag{8}$$

$$h_{p_k} D_{kn} u \geq 0 \quad \text{at} \quad x_0. \tag{9}$$

Let $v = \chi + Ah(Du)$ where $A$ is a positive constant to be chosen. Then $v\,|_{\partial\Omega}$ has its minimum at $x_0$, so $D_\alpha v(x_0) = 0$ for $\alpha = 1, \ldots, n-1$, which can be written as

(10)
$$h_{p_n p_l} D_{l\alpha} u + h_{p_k} D_\alpha \nu_k + A h_{p_k} D_{k\alpha} u = 0 \quad \text{at} \quad x_0$$
$$\text{for} \quad \alpha = 1, \ldots, n-1.$$

We claim that

$$D_n v(x_0) \geq -C(A). \tag{11}$$

This can be rewritten as

$$\begin{aligned} &h_{p_n p_l} D_{ln} u + h_{p_k} D_n \nu_k \\ &\qquad + A h_{p_k} D_{kn} u \geq -C \quad \text{at} \quad x_0. \end{aligned} \tag{12}$$

Assuming this, multiply (10) by $h_{p_\alpha}$ and sum over $\alpha$ from 1 to $n-1$, and add this to $h_{p_n}$ times (12), to get

$$\begin{aligned} &A D_{kl} u h_{p_k} h_{p_l} \\ &\geq -C h_{p_n} - (D_k \nu_l) h_{p_k} h_{p_l} - h_{p_k} h_{p_n p_l} D_{kl} u \\ &= -C h_{p_n} - (D_k \nu_l) h_{p_k} h_{p_l} - h_{p_k} h_{p_n p_n} D_{kn} u \\ &\geq -C h_{p_n} - (D_k \nu_l) h_{p_k} h_{p_l}. \end{aligned}$$

In the last two lines we have used (8) and (9), together with $-h_{p_n p_n} \geq 0$. If $\chi(x_0) = h_{p_n}$ is small, we get

$$D_{kl} u \, \nu_k^* \nu_l^* = D_{kl} u \, h_{p_k} h_{p_l} \geq c. \tag{13}$$

To prove (11) compute differential inequality for $v$ and use a barrier construction.

To prove the dual estimate

$$u^{ij}\,\nu_i\nu_j \geq c, \qquad [u^{ij}] = [D^2u]^{-1}$$

use same argument applied to equation for $u^*$.

8. Second derivative bounds.

(i) Let $\beta = h_p(Du)$. Differentiate boundary condition in any tangential direction $\tau$ to get

$$D_{\tau\beta}u = 0 \quad \text{on} \quad \partial\Omega \tag{14}$$

for any tangential vectorfield $\tau$ on $\partial\Omega$.

(ii) Compute a differential inequality for $H = h(Du)$:

$$\begin{aligned}\mathcal{L}H &= F_{ij}D_{ij}H - g_{p_i}D_iH \\ &= g_{x_k}h_{p_k} + g_zh_{p_k}D_ku + F_{ij}D_{ik}uD_{jl}uh_{p_kp_l} \\ &\geq -(C(\epsilon) + \epsilon M)\mathcal{T}\end{aligned}$$

for any $\epsilon > 0$, where $\mathcal{T} = \sum F_{ii}$ and $M = \sup_\Omega |D^2u|$. Since $H = 0$ on $\partial\Omega$, a barrier argument implies

$$D_\nu H \leq C(\epsilon) + \epsilon M \quad \text{on} \quad \partial\Omega$$

for any $\epsilon > 0$. Combining this with (14) we get

$$0 \leq D_{\beta\beta} u \leq C(\epsilon) + \epsilon M \quad \text{on} \quad \partial\Omega \tag{15}$$

for any $\epsilon > 0$.

(iii) Assume that the maximal tangential second derivative of $u$ over $\partial\Omega$ occurs at $0 \in \partial\Omega$ in the tangential direction $e_1$. At any boundary point we may write any direction $e_1$ in terms of a tangential component $\tau(e_1)$ and a component in the direction of $\beta$, namely

$$e_1 = \tau(e_1) + \frac{\nu \cdot e_1}{\beta \cdot \nu} \beta$$

where

$$\tau(e_1) = e_1 - (\nu \cdot e_1)\nu - \frac{\nu \cdot e_1}{\beta \cdot \nu} \beta^\top$$

and

$$\beta^\top = \beta - (\beta \cdot \nu)\nu.$$

For $\tau = \tau(e_1)$ we have

$$\begin{aligned} D_{11} u &= D_{\tau\tau} u + \frac{2\nu_1}{\beta \cdot \nu} D_{\tau\beta} u + \frac{\nu_1^2}{(\beta \cdot \nu)^2} D_{\beta\beta} u \\ &\leq |\tau|^2 D_{11} u(0) + (C(\epsilon) + \epsilon M)\nu_1^2 \\ &\leq \left\{ 1 + C\nu_1^2 - \frac{2\nu_1 \beta_1^\top}{\beta \cdot \nu} \right\} D_{11} u(0) \\ &\quad + (C(\epsilon) + \epsilon M)\nu_1^2 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Therefore, for any constant $A > 0$

$$w = \frac{D_{11}u}{D_{11}u(0)} + \frac{2\nu_1\beta_1^\top}{\beta\cdot\nu} - Ah(Du)$$

satisfies

$$\begin{aligned} w &\le 1 + \left\{C + \frac{C(\epsilon) + \epsilon M}{D_{11}u(0)}\right\}\nu_1^2 \quad \text{on} \quad \partial\Omega \\ &\le 1 + C(\epsilon)|x'|^2 \quad \text{on} \quad \partial\Omega \quad \text{near} \quad 0. \end{aligned}$$

Here we use that for small $\epsilon > 0$

$$M = \sup_\Omega |D^2u| \le C(\epsilon) + CD_{11}u(0).$$

Next we compute

$$F_{ij}D_{ij}w - g_{p_i}D_iw \ge -C(C(\epsilon) + \epsilon M)\mathcal{T} \quad \text{in} \quad \Omega$$

for any $\epsilon > 0$. A barrier argument implies $D_\beta w(0) \le C(\epsilon) + \epsilon M$, which simplifies to

$$D_{11\beta}u(0) \le (C(\epsilon) + \epsilon M)D_{11}u(0)$$

for all sufficiently small $\epsilon > 0$.

Finally, tangentially differentiate the boundary condition twice in the $e_1$ direction at 0, to get

$$D_{11\beta}u + h_{p_kp_l}D_{1k}uD_{1l}u + \kappa_1 D_{\nu\beta}u = 0 \quad \text{at} \quad 0,$$

where $\kappa_1 > 0$ is the normal curvature of $\partial\Omega$ at 0 in the direction $e_1$. Then

$$-h_{p_k p_l} D_{1k} u D_{1l} u \leq (C(\epsilon) + \epsilon M) D_{11} u \quad \text{at} \quad 0.$$

Since $h$ is uniformly concave,

$$D_{11} u(0) \leq C(\epsilon) + \epsilon M.$$

The full second derivative bound now follows by finally fixing $\epsilon > 0$ sufficiently small and using that $M$ is controlled by $D_{11}u(0)$: for small $\epsilon > 0$

$$M = \sup_{\Omega} |D^2 u| \leq C(\epsilon) + C D_{11} u(0).$$

**Remarks** (i) Modifications in some parts of the argument are needed to get Theorems 2 and 3.

(ii) For the curvature case the argument is more complicated. There is less symmetry in the problem than in the Hessian case. For the second derivative estimates

$$D_{ij} u \longleftrightarrow \sqrt{1 + |Du|^2}\ h_{ij}$$

where $h_{ij}$ are the components of the second fundamental form in a local orthonormal frame on graph $u$.