Exercice 3

$$\Gamma(x,y) = \frac{1}{2\pi} \log |x-y|$$

$$\Gamma \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2 / \{y=x\})$$

$$\nabla_x \Gamma = \frac{1}{2\pi |x-y|^2} (x-y)$$

$$D^2_{xx} \Gamma = \frac{I}{2\pi |x-y|^2} - \frac{2}{2\pi |x-y|^4} (x-y) \otimes (x-y)$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \quad u \otimes u = \begin{pmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{pmatrix}$$

We check that $\Delta \Gamma = \frac{1}{2\pi |x-y|^2} \left(1+1 - \frac{2}{|x-y|^2} (|x_1-y_1|^2 + |x_2-y_2|^2) \right)$

$= 0$ on $\mathbb{R}^2 / \{y\}$

Fix $y=0$

Take $f \in C_c^\infty(\{x\} < R)$ for some R large enough.

we have $\lim_{\rho \to 0} \int_{B_R / B_\rho} \Delta f(x) \Gamma(x,0)\, dx = \int_{B_R} \Delta f(x) \Gamma(x,0)\, dx$

$= \int_{\mathbb{R}^2} \Delta f(x) \Gamma(x,0)\, dx$

since $\Gamma(x,0) \in L^1(B_R)$

Then, $\int_{B_R / B_\rho} f \Delta \Gamma = 0$ since $\Delta \Gamma \equiv 0$ on $\mathbb{R} / \{0\}$

We will compute $\lim_{\rho \to 0} \int_{B_R / B_\rho} \Delta f(x) \Gamma(x,0)\, dx$

and show that it goe to $f(0)$ as $\rho \to 0$.

$$\int_{B_R / B_\rho} \Delta f \,\Gamma = \int_{\partial B_R \cup \partial B_\rho} \Gamma \nabla f \cdot \vec{n} - \int_{B_R / B_\rho} \nabla f \nabla \Gamma$$

$$\int_{B_R / B_\rho} \nabla f \nabla \Gamma = \int_{\partial B_R \cup \partial B_\rho} f \nabla \Gamma \cdot \vec{n} - \int_{B_R / B_\rho} f \Delta \Gamma$$

$\vec{n}$ the outward unit normal.

We conclude that since $f \equiv 0$ near ∂B_R

$$\int_{B_R / B_\rho} \Delta f \,\Gamma = \int_{\partial B_\rho} \Gamma \nabla f \cdot \vec{n} - \int_{\partial B_\rho} f \nabla \Gamma \cdot \vec{n}$$

$\Gamma_{|\partial B_\rho} = \ln \rho \cdot \frac{1}{2\pi}$, so that

$$\int_{\partial B_\rho} \Gamma \nabla f \cdot \vec{n} = \frac{1}{2\pi} \ln \rho \int_{\partial B_\rho} \nabla f \cdot \vec{n}$$

$$\leq \rho \ln \rho \, \frac{1}{2\pi \rho} \int_{\partial B_\rho} \nabla f \cdot \vec{n}$$

$$= \rho \ln \rho \, ⨍_{\partial B_\rho} \nabla f \cdot \vec{n} \longrightarrow 0 \quad \text{as } \rho \to$$

and $$-\int_{\partial B_\rho} f \nabla \Gamma \cdot \vec{n} = \int_{\partial B_\rho} \frac{1}{2\pi \rho} f \longrightarrow f(0) \quad \text{as } \rho \to 0.$$

which ends the proof #

The question after contains a typing mistake.

: It was show for all $u \in C^{\infty}(\mathbb{R}^2)$ (not C_c^{∞}!) <u>such</u> <u>that</u> $\Delta u \in C_c^{\infty}(\mathbb{R}^2)$, we have $u(x) = \int \Gamma(x,y)\,\Delta u(y)\,dy + h(x)$. . . .

for this compute Δu and

$$\Delta \int \Gamma(x,y)\,\Delta u(y)\,dy = \int \Delta_x \Gamma(x,y)\,\Delta u(y)\,dy$$

$$= \int \Delta_y \Gamma(x,y)\,\Delta u(y)\,dy = \Delta u(x)$$

from last question

thus $\Delta\left(u(x) - \int \Gamma(x,y)\,\Delta u(y)\,dy\right) = 0 = h.$

We know that $\Delta_x \Gamma = \delta(y=x)$

thus $\frac{1}{N} \sum \Delta_x \Gamma(x, X_i(t)) = \frac{1}{N} \sum \delta(y = X_i(t))$

thus $\Psi(x,t) = \frac{1}{N} \sum \Gamma(x, X_i(t))$ satisfies

$$\Delta \Psi = \frac{1}{N} \sum \delta(y = X_i(t)).$$

$$\nabla \Psi^{\perp}(x) = \frac{1}{2\pi N} \sum \frac{(x - X_i)^{\perp}}{|x - X_i|^2}$$

If we accept that ~~the~~ we neglect

the contribution of the term $\dfrac{(x - X_i)^{\perp}}{|x - X_i|^2}$

as x goes to X_i, we find that

$\dot{X}_i(t)$ as defined is $\nabla\psi^{\perp}(X_i)$

Thus $\omega = \frac{1}{N}\sum \delta(x - X_i(t))$ satisfies

$$\begin{cases} \partial_t \omega(t,x) + \nabla\psi^{\perp}\cdot\nabla\omega = 0 \\ \Delta\psi = \omega \end{cases}$$

which is the Euler equation in vorticity form.

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