Incorporating Estimation Error into Optimal Consumption and Portfolio Allocation in Continuous Time

The problem is concerned with incorporating estimation error into optimal portfolio consumption and investment in continuous time. As with any consumption and investment problem, the two primary steps in the solution are:

1) Characterize the consumption/investment opportunity set available to the investor. This amounts to specifying:
   i. the set of stochastic price processes $\{S_i(\cdot)\}_{i=1}^{N}$ of the N assets which are available for investment;
   ii. the amount of initial wealth W_0 available to be invested;
   iii. the set of admissible pairs $\{(c(\cdot), \boldsymbol{\pi}(\cdot))\}$ of consumption/portfolio processes, where the consumption process $c(\cdot)$ is nonnegative[1] and the portfolio process $\boldsymbol{\pi}(\cdot)$ is $\mathbb{R}^N$–valued. Generally, this will require specifying constraints, such as the budget constraint (wealth invested must equal total wealth minus consumption), wealth must be nonnegative, constraints on short-selling and/or borrowing, etc.
2) Characterize the preferences of the investor over the set $\{(c(\cdot), \boldsymbol{\pi}(\cdot))\}$, and identify the pair(s) $(c^*(\cdot), \boldsymbol{\pi}^*(\cdot))$ that is (are) most preferred (the optimum may not be unique).

This problem was originally solved by Robert C. Merton in [Merton, 1969] and [Merton, 1971] (cf. also Chapter 6 of the book [Merton, 1992]), and has come to be known as "Merton's Problem." Merton made the assumption that the asset price processes $\{S_i(\cdot)\}_{i=1}^{N}$ are given by Geometric Brownian Motion, and proved that the investment opportunity set can be generated by two portfolios or "mutual funds" of assets[2], which themselves obey Geometric Brownian Motion. This result is sometimes known as a two-fund separation theorem, and does not depend on the market being in equilibrium[3] (Merton utilized the separation theorem in his development of the Intertemporal Capital Asset Pricing Model[4] (a general equilibrium model) in [Merton, 1973], but the mutual fund theorem is only dependent on the assumed properties of the asset price processes).

Also, Merton assumed that the investor's preferences were given by a time-separable expected utility function, of the form

---

[1] The requirement that the consumption process be nonnegative is tantamount to assuming that the investor has no source of income other than his/her initial wealth. The more general case of positive income (in particular, stochastic income) is much more difficult, but should eventually be addressed.

[2] If one of the assets is the bank account process, which is instantaneously risk-free, then it can be taken to be one of the mutual funds.

[3] The solution of the optimal consumption and portfolio allocation problem does require the absence of (unlimited) arbitrage opportunities for its solution, however.

[4] Specifically, the versions of the ICAPM in which interest rates are constant, or in which there is no risk-free asset.

$$
\begin{aligned}
\mathsf{U}(c,\boldsymbol{\pi}) &= \int_0^T U\left(c(t),t\right)dt \\
&= e^{-\rho t}\int_0^T U_{\gamma,\beta,\eta}\left(c(t)\right)dt
\end{aligned}
$$

where

$$U_{\gamma,\beta,\eta}(x) = \frac{1-\gamma}{\gamma}\left(\frac{\beta x}{1-\gamma}+\eta\right)^{\gamma}$$

is a utility function of HARA (Hyperbolic Absolute Risk Aversion, a.k.a. Linear Risk Tolerance) type. This form of preference structure assumes that the investor sees no benefit in end-of-period wealth, and chooses to consume all of his/her wealth by time *T*. By formulating the problem as a stochastic optimal control problem, Merton was able to solve the resulting Hamilton-Jacobi-Bellman equation, and found that the optimal strategy $\left(c^*,\boldsymbol{\pi}^*\right)$ is given by

$$c^*(t) = \frac{(\rho-\gamma\nu)\left(W(t)+\dfrac{\delta\eta}{\beta r}\left\{1-\exp\left[r(t-T)\right]\right\}\right)}{\delta\left\{1-\exp\left[\dfrac{(\rho-\gamma\nu)(t-T)}{\delta}\right]\right\}} - \frac{\delta\eta}{\beta}$$

$$\boldsymbol{\pi}^*(t) = \frac{\alpha-r}{\delta\sigma^2}W(t) + \frac{\eta(\alpha-r)}{\beta r\sigma^2}\left\{1-\exp\left[r(t-T)\right]\right\}$$

where

$$\delta = 1-\gamma$$

$$\nu = r + \frac{(\alpha-r)^2}{2\delta\sigma^2}$$

$$W(t) = \boldsymbol{\pi}(t)'\,\mathbf{1}$$

($W(t)$ is the investor's total wealth at time *t*). Merton noted that the solution could be easily extended to include the case that the investor derives benefit from end-of-period wealth if preferences are defined by

$$\mathsf{U}(c,\boldsymbol{\pi}) = e^{-\rho t}\int_0^T U_{\gamma,\beta,\eta}\left(c(t)\right)dt + H(T)V_{\gamma,a,b}\left(W(T)\right)$$

where

$$V_{\gamma,a,b}(x) = \left(aW(x)+b\right)^{\gamma}$$

since the structure of the value function is unchanged; for more general preferences, the solution involves systematic time horizon effects.

Subsequent work on this problem has sought to generalize Merton's work in numerous ways, including (but not limited to) the following[5]:

---

[5] Details of these extensions and references to the original papers may be found in [Korn, 1997], [Karatzas and Shreve, 1998], [Duffie, 2001], and [Dana and Jeanblanc, 2003].

1) More general asset price processes than GBM, for instance, Ito processes with deterministic (and even stochastic) time-dependent drift and diffusion parameters, even more general diffusion and Markov processes[6], or general semi-martingales. In the latter case, the additional assumption that the market is complete (or more generally, effectively complete) is required, and the method of solution uses the so-called Cox-Huang-Pliska method, which involves the use of the Martingale Representation Theorem.
2) Inclusion of stochastic income for the investor, with various degrees of generality regarding the structure of the income process[7].
3) Inclusion of transaction costs.
4) Investor preferences given by expected utility functions that are non-time-separable (for instance, recursive or stochastic differential utility), or even non-expected utility preference orderings.

However, the one way in which Merton's original work has not been generalized is the *explicit* incorporation of estimation risk into the characterization of the asset price processes, *and* the translation (or propagation) of that risk into the optimal choice of consumption/investment strategy[8]. In other words, to date, all work on the consumption/portfolio allocation problem in continuous time has assumed that the parameters of the asset price processes are known *with perfect certainty*[9]. In reality, however, these parameters must be estimated, and there will *always* be some measure of estimation risk[10]. Some might counter, and say that the quadratic cross-variation processes

$$\left\langle S_i, S_j \right\rangle_t = \int_0^t \sigma_{ij}(s)\, ds$$

are observable, and hence the diffusion coefficient $\sigma_{ij}(s)$ can in principal be estimated with perfect certainty. However, this is a fallacy for two reasons:

1) Assuming that the asset price processes are *fully observed stochastic processes*, to eliminate error in the estimation of $\sigma_{ij}(s)$ would require storing the uncountable number of values $\left\{ S_i(u), S_j(u) \middle| \, |s-u| < \varepsilon \right\}$ for some $\varepsilon > 0$, and this would require an infinite amount of memory for storage, which would bankrupt the investor. Practically, therefore, only a finite number of values of the asset

[6] Merton himself considered more general asset price processes than GBM in [Merton, 1971].
[7] Merton included the case of stochastic income in [Merton, 1971], but since he did not impose the constraint that wealth be nonnegative (even in the absence of income), it is easy to make this generalization. With the wealth nonnegativity restriction in place, incorporating income is much more difficult.
[8] The number of references that deal with improved methods of parameter estimation for continuous-time diffusion processes is far to numerous to list here. Techniques include (among others) Maximum Likelihood Estimation (MLE), quasi/pseudo MLE, Generalized Method of Moments, Efficient Method of Moments, Markov Chain Monte Carlo (MCMC), etc. A representative list of references may be found in Chapter 12 of [Gourieroux and Jasiak, 2001].
[9] Cf. [Korn, 1997], [Karatzas and Shreve, 1998], [Duffie, 2001] or [Dana and Jeanblanc, 2003].
[10] Much work has recently been done on the incorporation of estimation error into single-period optimal portfolio selection. See, for instance, [Scherer, 2004] or [Meucci, 2005] for details.

price processes can be used in the estimation, i.e. *the asset price processes are partially observed*.

2) The microstructure of the financial markets implies that assets do not trade continuously, and hence the market price process (representing the most recent traded price) has piecewise constant paths, and most likely deviates from the "true" intrinsic price of the assets between trade times. The empirical asset price process is therefore a *partially observed stochastic process* ("observations" take place at the trading times), and hence the diffusion parameters cannot be estimated without error.

The result of the unavoidable nature of estimation risk[11] is that the optimized consumption/portfolio selection pair $\left(c^*(\cdot), \boldsymbol{\pi}^*(\cdot)\right)$ will only truly be optimal in the unlikely event that there is no estimation error; in all other cases, it will be suboptimal.

The goal, therefore, is to formulate the optimal consumption and portfolio investment problem in decision-theoretic form[12] such that, given any data sample $X = \left\{S_i\left(t_j\right) \middle| i = 1,\ldots,N;\ j = 1,\ldots,J\right\}$ of the asset price processes, we have a prescription

$$\left(c_X^*(\cdot), \boldsymbol{\pi}_X^*(\cdot)\right) = \alpha(X)$$

that associates to that sample an optimal[13] strategy; note that this prescription is dynamic, since the sample *X* will enlarge over time, likely resulting in a different optimal strategy from the previous one, going forward. The decision theoretic problem could be formulated in classical or Bayesian form. To eliminate unnecessary complications, it would be easiest to work within the assumptions of the original Merton model, except that we wish to consider the case that investors ***do*** derive benefit from end-of-period wealth, and investor wealth ***is*** constrained to be nonnegative.

In addition, much recent work has been done[14] on generalizing the expected utility framework to include the separation of uncertainty and ambiguity[15], in order to explain Ellsberg's paradox [Ellsberg, 1961]. It may be appropriate to consider such a more general (i.e. non-expected utility) preference structure for the investor.

REFERENCES

Berger, J.O. (1985), **Statistical Decision Theory and Bayesian Analysis**, 2nd edition, Springer, New York, NY.

---

[11] We shall assume that the parameter estimates have as many desired properties as possible, e.g. unbiased, consistent and efficient.

[12] Good accounts of formal statistical decision theory, both classical and Bayesian, can be found in [Berger, 1985] and [Shao, 2003].

[13] Optimality here is defined by minimization of a loss function.

[14] Cf. [Gajdos et al., 2003] and references therein, as well as [Taboga, 2005] for a direct application to portfolio selection.

[15] In the portfolio selection context, asset returns would be uncertain *even if* the distribution of returns were known with certainty; since the distribution of returns is not known with certainty, this introduces an additional ambiguity that the investor may wish to consider separately (i.e. it may be necessary to account for aversion to uncertainty and aversion to ambiguity separately). Cf. [Taboga, 2005].

Dana, R.-A. and M. Jeanblanc (2003), **Financial Markets in Continuous Time**, Springer, Berlin.

Duffie, D. (2001), **Dynamic Asset Pricing Theory**, Princeton University Press, Princeton, NJ.

Ellsberg, D. (1961), "Risk, Ambiguity, and the Savage Axioms," *Quarterly Journal of Economics*, 75: 643-669.

Gajdos, T., J.-M. Tallon and J.-C. Vergnaud (2003), "Decision Making with Imprecise Probabilistic Information," ICER Working Paper Series.

Karatzas, I. and S.E. Shreve (1998), **Methods of Mathematical Finance**, Springer, New York, NY.

Korn, R. (1997), **OPTIMAL PORTFOLIOS: Stochastic Models for Optimal Investment and Risk Management in Continuous Time**, World Scientific, River Edge, NJ.

Merton, R.C. (1969), "Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case," *Review of Economics and Statistics*, 51 (August): 247-257.

Merton, R.C. (1971), "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, 3 (December): 373-413.

Merton, R.C. (1973), "An Intertemporal Capital Asset Pricing Model," Econometrica, 41 (September): 867-887.

Merton, R.C. (1992), **Continuous-Time Finance**, revised edition, Blackwell Publishers, Cambridge, MA.

Meucci, A. (2005), **Risk and Asset Allocation**, Springer, New York, NY.

Scherer, B. (2004), **Portfolio Construction and Risk Budgeting**, 2nd edition, Risk Books, London, UK.

Shao, J. (2003), **Mathematical Statistics**, 2nd edition, Springer, New York, NY.

Taboga, M. (2005), "Portfolio Selection with Two-Stage Preferences," Working Paper.