THE RESTRICTION OF $Gl_n(\mathbb{C})$ MODULES TO THE SUBGROUP OF S_n

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The character of a $Gl_n(\mathbb{C})$ module is given by the formula

$$ch_{Gl_n(\mathbb{C})}(M) = \sum_{b \in M} \left[b \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & x_n \end{pmatrix} \right]_{coef\ b}$$

where the sum runs over a basis of the module M. This expression is a symmetric function in the indeterminates x_i.

The character determines the decomposition of the module into irreducibles. If the module has a character given by a Schur function in the variables $x_1, x_2, \dots, x_n$ then the module is irreducible. The character is a function of the eigenvalues of the matrix to $\mathbb{C}$.

Since the symmetric group (represented as permutation matrices) is a natural subgroup of $Gl_n(\mathbb{C})$ we ask the following question:

Question: Given an irreducible $Gl_n(\mathbb{C})$ module, how does it decompose into irreducible S_n modules?

This turns out to be easy to compute for specific examples. I don't know the answer to this question in general and it would be very useful to have a clear expression for this decomposition. The character of a $Gl_n(\mathbb{C})$ module M at a permutation matrix A_σ will be the evaluation of the symmetric function $ch_{Gl_n(\mathbb{C})}(M)(x_1, x_2, \dots, x_n)$ at the eigenvalues of the matrix A_σ.

Now the eigenvalues of the matrix A_σ are determined by the cycle structure of the permutation σ and they will be $\Xi_{\lambda_1}, \Xi_{\lambda_2}, \dots, \Xi_{\lambda_r}$ where $\Xi_m = 1, e^{2\pi i/m}, e^{4\pi i/m}, \dots, e^{2(m-1)\pi i/m}$ (the m roots of unity).

Example: So for instance, the irreducible $Gl_3(\mathbb{C})$ module with character $s_{(5)}(x_1, x_2, x_3)$ considered as an S_3 module has an S_3 character that when evaluated at the identity has character equal to $s_{(5)}(1,1,1) = 21$. The character evaluated at the permutation $(12)(3)$ will be $s_{(5)}(1,-1,1) = 3$. The character evaluated at the permutation (123) will be $s_{(5)}(1, e^{2\pi i/3}, e^{4\pi i/3}) = 0$.

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As an example, we consider M as the $Gl_n(\mathbb{C})$ module consisting of the polynomials in n variables of degree k. That is, $M = \{y_1^{\alpha_1} \cdots y_n^{\alpha_n} : \alpha_1 + \cdots + \alpha_n = k\}$. Note that M has a character given by

$$ch_{Gl_n(\mathbb{C})}(M) = \sum_{y^\alpha \in M} \left[y^\alpha \begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & x_n \end{pmatrix} \right]_{coef\ y^\alpha} = \sum_{y^\alpha \in M} x^\alpha = s_{(k)}(x_1, \dots, x_n).$$

We define the Frobenius characteristic of an S_n character as

$$\mathcal{F}_{S_n}(\chi) = \sum_{\lambda \vdash n} \chi(\sigma(\lambda)) p_\lambda / z_\lambda$$

where $\sigma(\lambda)$ is a permutation of cycle type λ. For a $Gl_n(\mathbb{C})$ character $f(x_1, x_2, \dots, x_n)$ we have that

$$\mathcal{F}_{S_n}(f(x_1, x_2, \dots, x_n)) = \sum_{\lambda \vdash n} f[\Xi_{\lambda_1} + \cdots + \Xi_{\lambda_{\ell(\lambda)}}] p_\lambda / z_\lambda.$$

Example: The example of the module with character $s_{(5)}(x_1, x_2, x_3)$, we have already calculated the Frobenius image as $\mathcal{F}_{S_3}(s_{(5)}(x_1, x_2, x_3)) = 21p_{(111)}/6 + 3p_{(21)}/2$. But we have also determined that the module is isomorphic to the polynomials of degree 5 in three variables. Since we know that the Frobenius image of the polynomial ring in 3 variables has a graded Frobenius image of $h_3\left[\frac{X}{1-q}\right]$ hence we know that the coefficient of q^5 in the expression will be equal to the Frobenius image $\mathcal{F}_{S_3}(s_{(5)}(x_1, x_2, x_3))$. Since we know that

$$h_3\left[\frac{X}{1-q}\right] = s_{(3)}[X] s_{(3)}\left[\frac{1}{1-q}\right] + s_{(21)}[X] s_{(21)}\left[\frac{1}{1-q}\right] + s_{(111)}[X] s_{(111)}\left[\frac{1}{1-q}\right]$$

then we know that the coefficient of $q^5 s_\lambda[X]$ will be the number of column strict tableaux of shape λ with entries in $0, 1, 2, 3, \dots$ whose entries sum to 5.

For $\lambda = (3)$ we know that the tableaux are given by $\boxed{0|0|5}$, $\boxed{0|1|4}$, $\boxed{0|2|3}$, $\boxed{1|1|3}$, $\boxed{1|2|2}$.

For $\lambda = (2,1)$ the tableaux are given by $\begin{array}{|c|c}\hline 5 & \\ \hline 0 & 0 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 4 & \\ \hline 0 & 1 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 1 & \\ \hline 0 & 4 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 3 & \\ \hline 0 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 2 & \\ \hline 0 & 3 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 2 & \\ \hline 1 & 2 \\ \hline\end{array}$, $\begin{array}{|c|c}\hline 3 & \\ \hline 1 & 1 \\ \hline\end{array}$.

For $\lambda = (1,1,1)$ the tableaux are given by $\begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline 0 \\ \hline\end{array}$ and $\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 0 \\ \hline\end{array}$.

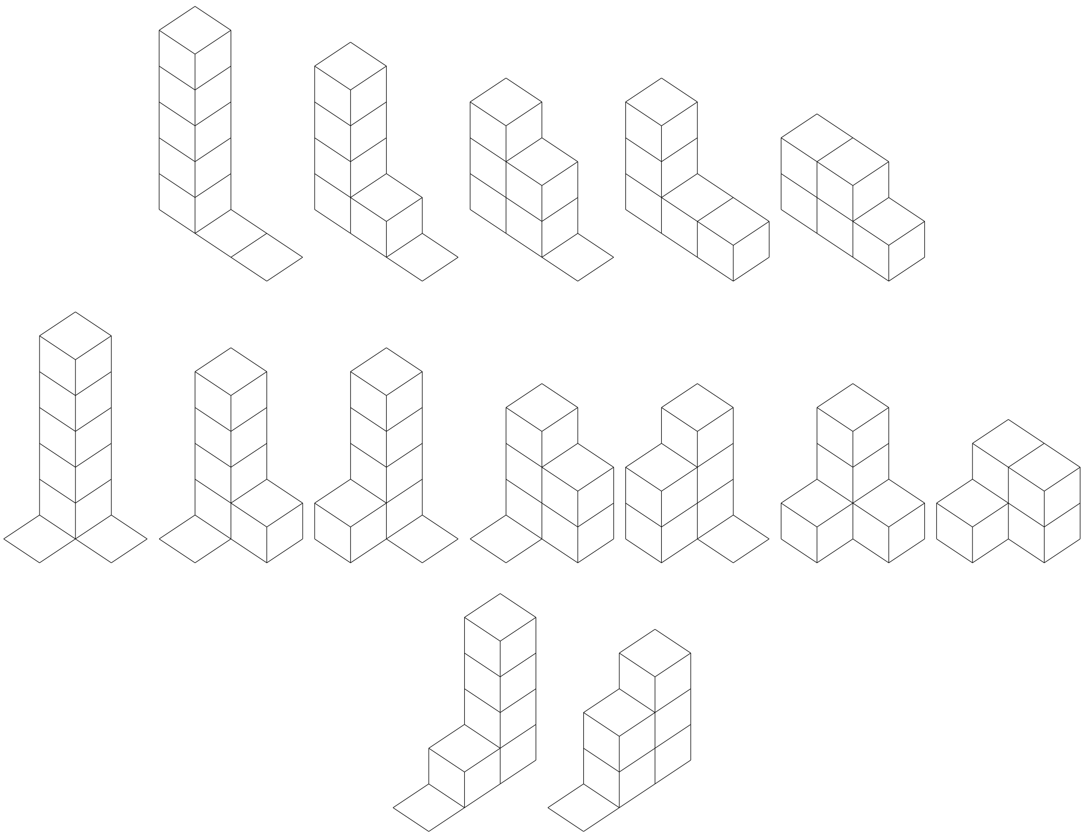
We conclude that $\mathcal{F}_{S_3}(s_{(5)}(x_1, x_2, x_3)) = 5s_{(3)} + 7s_{(21)} + 2s_{(111)}$.

Notice that in general that we can express the special case

$$\mathcal{F}_{S_n}(s_{(k)}(x_1, \dots, x_n)) = \sum_{\lambda \vdash n} c_\lambda^{(k)} s_\lambda$$

where $c_\lambda^{(k)}$ is the number of column strict tableaux of shape λ whose entries sum to k.

Macdonald also gives another interpretation of this coefficient (p. 81 example 14). $c_\lambda^{(k)}$ is also the number of column strict plane partitions of shape λ. A plane partition is a stack of blocks with adjacent stacks which are weakly decreasing in height. A plane partition is column strict if the stacks are strictly decreasing in the columns.



Note that this is similar to the last interpretation except that the order of the alphabet is reversed since these objects are equivalent to column strict tableaux of shape λ with entries that are weakly decreasing rows and strictly decreasing in columns.

We remark that given two $Gl_n(\mathbb{C})$ modules, their (inner) tensor product will have character as the product of the product of the modules. That is, for modules M, N,

$$ch_{Gl_n(\mathbb{C})}(M \otimes N) = ch_{Gl_n(\mathbb{C})}(M) ch_{Gl_n(\mathbb{C})}(N).$$

This follows directly from the definition of the character.

We also have that the Frobenius image satisfies $\mathcal{F}_{S_n}(fg) = \mathcal{F}_{S_n}(f) * \mathcal{F}_{S_n}(g)$ where $*$ is the inner tensor product (Kronecker product). The Kronecker product on symmetric functions is defined as $p_\lambda/z_\lambda * p_\mu/z_\mu = \delta_{\lambda\mu} p_\lambda/z_\lambda$. It then follows that

$$\begin{aligned}
\mathcal{F}_{S_n}(f) * \mathcal{F}_{S_n}(g) &= \left(\sum_{\lambda \vdash n} f[\Xi_{\lambda_1} + \cdots + \Xi_{\lambda_{\ell(\lambda)}}] p_\lambda/z_\lambda \right) * \left(\sum_{\lambda \vdash n} g[\Xi_{\lambda_1} + \cdots + \Xi_{\lambda_{\ell(\lambda)}}] p_\lambda/z_\lambda \right) \\
&= \sum_{\lambda \vdash n} f[\Xi_{\lambda_1} + \cdots + \Xi_{\lambda_{\ell(\lambda)}}] g[\Xi_{\lambda_1} + \cdots + \Xi_{\lambda_{\ell(\lambda)}}] p_\lambda/z_\lambda \\
&= \mathcal{F}_{S_n}(fg)
\end{aligned}$$

Macdonald (p. 50 example 17) talks about the evaluation of a Schur function at the sum of roots of unity. $s_\lambda[\Xi_m] = \pm 1$ if λ has an empty m-core (i.e. it can be tiled with ribbons of length m) and $s_\lambda[\Xi_m] = 0$ otherwise. The sign of $s_\lambda[\Xi_m]$ will be the sign of the unique permutation σ such that $\lambda + \delta_m = \sigma(\delta_m) \pmod m$ where $\delta_m = (m-1, m-2, \ldots, 1, 0)$.

It is quite simple to write a short program which accepts a symmetric function and a value of n (determining which copy of $Gl_n(\mathbb{C})$ on is working in) and which returns a symmetric function which returns the Frobenius image of this function as an S_n character.

```
> with(SF)
> psum:=proc(lst,k) local i;
add(lst[i],i=1..k);
end:

> toSnFrob:=proc(expr,n) local i,lambda,j;
add( mul( cat(p,lambda[i]), i=1..nops(lambda))/zee(lambda)*
simplify(subs(seq(seq(x[psum(lambda,i-1)+j+1]=exp(2*Pi*I*j/lambda[i]),
j=0..lambda[i]-1), i=1..nops(lambda)),
evalsf(expr, add(x[i],i=1..convert(lambda,`+`))))),
lambda=Par(n));
end:
```

The program above substitutes the roots of unity in for the variables of the symmetric function after evaluating the symmetric function at n variables.

We compute an example using this program:

```
> for i from 1 to 6 do
> tos(toSnFrob(s[1],i));
> od;
```

$$s_{(1)}$$
$$s_{(2)} + s_{(11)}$$
$$s_{(3)} + s_{(21)}$$
$$s_{(4)} + s_{(31)}$$

$$s_{(5)} + s_{(41)}$$
$$s_{(6)} + s_{(51)}$$

Observing the data below, one simple conjecture to make (that should not be that hard to prove) is

Conjecture 1.

$$\mathcal{F}_{S_n}(s_{(1^k)}(x_1, x_2, \ldots, x_n)) = s_{(n-k,1^k)} + s_{(n-k+1,1^{k-1})}$$

Here is data for $\mathcal{F}_{S_n}(s_\lambda(x_1, x_2, \ldots, x_n))$ for $n \le 6$ and $|\lambda| \le 5$. Note that if $n \le \ell(\lambda)$ then $s_\lambda(x_1, x_2, \ldots, x_n) = 0$.

$$F_{S_1}(s_{(1)}) = s_{(1)}$$
$$F_{S_1}(s_{(2)}) = s_{(1)}$$
$$F_{S_1}(s_{(3)}) = s_{(1)}$$
$$F_{S_1}(s_{(4)}) = s_{(1)}$$
$$F_{S_1}(s_{(5)}) = s_{(1)}$$
$$F_{S_2}(s_{(1)}) = s_{(2)} \ + \ s_{(11)}$$
$$F_{S_2}(s_{(2)}) = 2s_{(2)} \ + \ s_{(11)}$$
$$F_{S_2}(s_{(11)}) = s_{(11)}$$
$$F_{S_2}(s_{(3)}) = 2s_{(2)} \ + \ 2s_{(11)}$$
$$F_{S_2}(s_{(21)}) = s_{(2)} \ + \ s_{(11)}$$
$$F_{S_2}(s_{(4)}) = 3s_{(2)} \ + \ 2s_{(11)}$$
$$F_{S_2}(s_{(31)}) = s_{(2)} \ + \ 2s_{(11)}$$
$$F_{S_2}(s_{(22)}) = s_{(2)}$$
$$F_{S_2}(s_{(5)}) = 3s_{(2)} \ + \ 3s_{(11)}$$
$$F_{S_2}(s_{(41)}) = 2s_{(2)} \ + \ 2s_{(11)}$$
$$F_{S_2}(s_{(32)}) = s_{(2)} \ + \ s_{(11)}$$
$$F_{S_3}(s_{(1)}) = s_{(21)} \ + \ s_{(3)}$$
$$F_{S_3}(s_{(2)}) = 2s_{(3)} \ + \ 2s_{(21)}$$
$$F_{S_3}(s_{(11)}) = s_{(111)} \ + \ s_{(21)}$$
$$F_{S_3}(s_{(3)}) = s_{(111)} \ + \ 3s_{(3)} \ + \ 3s_{(21)}$$
$$F_{S_3}(s_{(21)}) = s_{(111)} \ + \ s_{(3)} \ + \ 3s_{(21)}$$
$$F_{S_3}(s_{(111)}) = s_{(111)}$$
$$F_{S_3}(s_{(4)}) = s_{(111)} \ + \ 4s_{(3)} \ + \ 5s_{(21)}$$
$$F_{S_3}(s_{(31)}) = 3s_{(111)} \ + \ 2s_{(3)} \ + \ 5s_{(21)}$$
$$F_{S_3}(s_{(22)}) = 2s_{(3)} \ + \ 2s_{(21)}$$
$$F_{S_3}(s_{(211)}) = s_{(111)} \ + \ s_{(21)}$$

$$F_{S_3}(s_{(5)}) = 2s_{(111)} + 5s_{(3)} + 7s_{(21)}$$
$$F_{S_3}(s_{(41)}) = 4s_{(111)} + 4s_{(3)} + 8s_{(21)}$$
$$F_{S_3}(s_{(32)}) = 2s_{(111)} + 3s_{(3)} + 5s_{(21)}$$
$$F_{S_3}(s_{(311)}) = 2s_{(111)} + 2s_{(21)}$$
$$F_{S_3}(s_{(221)}) = s_{(3)} + s_{(21)}$$
$$F_{S_4}(s_{(1)}) = s_{(4)} + s_{(31)}$$
$$F_{S_4}(s_{(2)}) = 2s_{(4)} + 2s_{(31)} + s_{(22)}$$
$$F_{S_4}(s_{(11)}) = s_{(211)} + s_{(31)}$$
$$F_{S_4}(s_{(3)}) = s_{(211)} + 3s_{(4)} + 4s_{(31)} + s_{(22)}$$
$$F_{S_4}(s_{(21)}) = 2s_{(211)} + s_{(4)} + 3s_{(31)} + 2s_{(22)}$$
$$F_{S_4}(s_{(111)}) = s_{(1111)} + s_{(211)}$$
$$F_{S_4}(s_{(4)}) = 2s_{(211)} + 5s_{(4)} + 6s_{(31)} + 3s_{(22)}$$
$$F_{S_4}(s_{(31)}) = 2s_{(4)} + 3s_{(22)} + 5s_{(211)} + s_{(1111)} + 7s_{(31)}$$
$$F_{S_4}(s_{(22)}) = 2s_{(4)} + 3s_{(22)} + s_{(211)} + 3s_{(31)}$$
$$F_{S_4}(s_{(211)}) = s_{(22)} + 3s_{(211)} + s_{(1111)} + s_{(31)}$$
$$F_{S_4}(s_{(1111)}) = s_{(1111)}$$
$$F_{S_4}(s_{(5)}) = 6s_{(4)} + 4s_{(22)} + 4s_{(211)} + 10s_{(31)}$$
$$F_{S_4}(s_{(41)}) = 5s_{(4)} + 7s_{(22)} + 9s_{(211)} + 2s_{(1111)} + 12s_{(31)}$$
$$F_{S_4}(s_{(32)}) = 4s_{(4)} + 5s_{(22)} + 6s_{(211)} + s_{(1111)} + 9s_{(31)}$$
$$F_{S_4}(s_{(311)}) = 3s_{(22)} + 6s_{(211)} + 3s_{(1111)} + 3s_{(31)}$$
$$F_{S_4}(s_{(221)}) = s_{(4)} + 2s_{(22)} + 2s_{(211)} + 3s_{(31)}$$
$$F_{S_4}(s_{(2111)}) = s_{(211)} + s_{(1111)}$$
$$F_{S_5}(s_{(1)}) = s_{(5)} + s_{(41)}$$
$$F_{S_5}(s_{(2)}) = 2s_{(5)} + 2s_{(41)} + s_{(32)}$$
$$F_{S_5}(s_{(11)}) = s_{(311)} + s_{(41)}$$
$$F_{S_5}(s_{(3)}) = s_{(311)} + 3s_{(5)} + 2s_{(32)} + 4s_{(41)}$$
$$F_{S_5}(s_{(21)}) = 2s_{(311)} + s_{(221)} + s_{(5)} + 2s_{(32)} + 3s_{(41)}$$
$$F_{S_5}(s_{(111)}) = s_{(311)} + s_{(2111)}$$
$$F_{S_5}(s_{(4)}) = 2s_{(311)} + s_{(221)} + 5s_{(5)} + 4s_{(32)} + 7s_{(41)}$$
$$F_{S_5}(s_{(31)}) = 2s_{(5)} + 7s_{(41)} + 5s_{(32)} + 6s_{(311)} + s_{(2111)} + 2s_{(221)}$$
$$F_{S_5}(s_{(22)}) = 2s_{(5)} + 3s_{(41)} + 4s_{(32)} + s_{(311)} + 2s_{(221)}$$
$$F_{S_5}(s_{(211)}) = s_{(41)} + s_{(32)} + 3s_{(311)} + 2s_{(2111)} + 2s_{(221)}$$
$$F_{S_5}(s_{(1111)}) = s_{(2111)} + s_{(11111)}$$
$$F_{S_5}(s_{(5)}) = 11s_{(41)} + 7s_{(32)} + 5s_{(311)} + 2s_{(221)} + 7s_{(5)}$$
$$F_{S_5}(s_{(41)}) = 14s_{(41)} + 3s_{(2111)} + 6s_{(221)} + 5s_{(5)} + 11s_{(32)} + 11s_{(311)}$$

$$F_{S_5}(s_{(32)}) = 10s_{(41)} + 2s_{(2111)} + 5s_{(221)} + 4s_{(5)} + 10s_{(32)} + 8s_{(311)}$$
$$F_{S_5}(s_{(311)}) = 3s_{(41)} + 5s_{(2111)} + s_{(11111)} + 5s_{(221)} + 4s_{(32)} + 8s_{(311)}$$
$$F_{S_5}(s_{(221)}) = 3s_{(41)} + s_{(2111)} + 4s_{(221)} + s_{(5)} + 4s_{(32)} + 3s_{(311)}$$
$$F_{S_5}(s_{(2111)}) = 3s_{(2111)} + s_{(11111)} + s_{(221)} + s_{(311)}$$
$$F_{S_5}(s_{(11111)}) = s_{(11111)}$$
$$F_{S_6}(s_{(1)}) = s_{(51)} + s_{(6)}$$
$$F_{S_6}(s_{(2)}) = 2s_{(51)} + s_{(42)} + 2s_{(6)}$$
$$F_{S_6}(s_{(11)}) = s_{(51)} + s_{(411)}$$
$$F_{S_6}(s_{(3)}) = 4s_{(51)} + 2s_{(42)} + s_{(33)} + 3s_{(6)} + s_{(411)}$$
$$F_{S_6}(s_{(21)}) = 3s_{(51)} + 2s_{(42)} + s_{(6)} + 2s_{(411)} + s_{(321)}$$
$$F_{S_6}(s_{(111)}) = s_{(3111)} + s_{(411)}$$
$$F_{S_6}(s_{(4)}) = 5s_{(6)} + 7s_{(51)} + 5s_{(42)} + s_{(33)} + s_{(321)} + 2s_{(411)}$$
$$F_{S_6}(s_{(31)}) = s_{(3111)} + 2s_{(6)} + 7s_{(51)} + 5s_{(42)} + 2s_{(33)} + 3s_{(321)} + 6s_{(411)}$$
$$F_{S_6}(s_{(22)}) = 2s_{(6)} + 3s_{(51)} + 4s_{(42)} + s_{(411)} + s_{(33)} + 2s_{(321)} + s_{(222)}$$
$$F_{S_6}(s_{(211)}) = s_{(51)} + s_{(42)} + 3s_{(411)} + 2s_{(321)} + 2s_{(3111)} + s_{(2211)}$$
$$F_{S_6}(s_{(1111)}) = s_{(21111)} + s_{(3111)}$$
$$F_{S_6}(s_{(5)}) = 12s_{(51)} + 8s_{(42)} + 5s_{(411)} + 3s_{(321)} + 7s_{(6)} + 3s_{(33)}$$
$$F_{S_6}(s_{(41)}) = 5s_{(6)} + 14s_{(51)} + 13s_{(42)} + 12s_{(411)} + 4s_{(33)} + 8s_{(321)} + 3s_{(3111)} + s_{(222)} + s_{(2211)}$$
$$F_{S_6}(s_{(32)}) = 4s_{(6)} + 10s_{(51)} + 11s_{(42)} + 8s_{(411)} + 5s_{(33)} + 8s_{(321)} + 2s_{(3111)} + s_{(222)} + s_{(2211)}$$
$$F_{S_6}(s_{(311)}) = 3s_{(51)} + 4s_{(42)} + 8s_{(411)} + s_{(33)} + 7s_{(321)} + 6s_{(3111)} + s_{(222)} + 2s_{(2211)} + s_{(21111)}$$
$$F_{S_6}(s_{(221)}) = s_{(6)} + 3s_{(51)} + 4s_{(42)} + 3s_{(411)} + 2s_{(33)} + 5s_{(321)} + s_{(3111)} + 2s_{(222)} + 2s_{(2211)}$$
$$F_{S_6}(s_{(2111)}) = s_{(411)} + s_{(321)} + 3s_{(3111)} + 2s_{(2211)} + 2s_{(21111)}$$
$$F_{S_6}(s_{(11111)}) = s_{(21111)} + s_{(111111)}$$