A PROJECT FOR THE SUMMER SCHOOL IN IWASAWA THEORY

Consider an imaginary quadratic field F and a prime p. We let F_∞ denote the cyclotomic $\mathbf{Z}_p$-extension of F and let $\Gamma = \mathrm{Gal}(F_\infty/F)$. Let L_∞ denote the maximal, unramified, abelian pro-p extension of F_∞. In classical Iwasawa theory, one considers $X = \mathrm{Gal}(L_\infty/F_\infty)$ as a module over the formal power series ring $\Lambda = \mathbf{Z}_p[[T]]$. The Λ-module X turns out to be a torsion module. One can associate with the module X a nonzero ideal, the so-called characteristic ideal for X. It is a principal ideal. We let $f_X(T)$ be a generator of that ideal. The purpose of this project is to study the relationship between $f_X(T)$ and the ideal class group of the field F.

To be specific, consider the following question: Let $h_F^{(p)}$ denote the largest power of p dividing the class number of F. Can one determine $h_F^{(p)}$ from $f_X(T)$? This turns out to be a nontrivial question and answering it in an explicit way is the objective of this project.

First consider the case where p is an odd prime which is inert or ramified in $F/\mathbf{Q}$. Then one can show that $f_X(0) \neq 0$ and that $h_F^{(p)}$ is the largest power of p dividing the p-adic integer $f_X(0)$. There are a number of different ingredients in the proof of this result. The first part of the project is to find a complete proof.

If p is odd and splits in the imaginary quadratic field F, then it turns out that $f_X(0) = 0$. Furthermore, one can prove that $f_X(T)$ only has a simple zero at $T = 0$. This is interesting in itself and is part of the project. There still is a way of recovering $h_F^{(p)}$ from the power series $f_X(T)$. This involves finding a formula for the power of p dividing the coefficient of T in $f_X(T)$. It is much more subtle than the first case in that the formula involves other arithmetic information about the imaginary quadraic field F. For example, if $F = \mathbf{Q}(i)$ and if $\pi = a + bi$ is a prime factor of p in the ring of integers of F, then $log_\pi(a - bi)$ enters into the formula somehow. The function log_π is just the usual p-adic logarithm function on the units in the π-adic completion of F. The final part of the project is to derive the formula.