LECTURE 5: ASYMPTOTIC FREENESS OF RANDOM MATRICES

We consider further the relationship between random matrices and free probability. A "random matrix" is really a measurable function

$$A : (\Omega_1, \mathfrak{B}_1, \mu_1) \to (\Omega_2, \mathfrak{B}_2, \mu_2) \tag{1}$$

between two probability spaces, where the underlying set of the target space is a set of matrices usually over some field (in our case it is always $\mathbb{C}$). Then each entry $e_{ij} \circ A$ is a $\mathbb{C}$-valued random variable. Thus we can think of a random matrix as a matrix whose entries are random variables.

Let A_N be a $GUE(N)$ random matrix. Let us recall what this means. Let

$$A_N : (\Omega, \mathfrak{B}, \mu) \to \mathcal{H}_N \tag{2}$$

be a Hermitian matrix-valued measurable function defined on a classical probability space ($\mathcal{H}_N$ is homeomorphic to the Euclidean space $\mathbb{C}^{N(N-1)/2}$ so the measure on $\mathcal{H}_N$ is just the pushforward of a Borel measure on Euclidean space under this homeomorphism). Then each entry of A_N is a complex-valued random variable h_{ij}, and $h_{ji} = \overline{h_{ij}}$ for $i \neq j$ while $h_{ii} = \overline{h_{ii}}$ thus implying that h_{ii} is in fact a real-valued random variable. A_N is said to be GUE-distributed if each h_{ij} with $i < j$ is of the form

$$h_{ij} = x_{ij} + \sqrt{-1}y_{ij}, \tag{3}$$

where $x_{ij}, y_{ij}, 1 \leq i < j \leq N$ are independent standard Gaussian random variables, each with mean 0 and variance $\frac{1}{2N}$. This determines the below-diagonal entries also. Moreover, the GUE requirement means that the diagaonal entries h_{ii} are real-valued independent Gaussian random variables which are also independent from the x_{ij}'s and the y_{ij}'s and have mean 0 and variance $\frac{1}{N}$.

Let tr be the normalized trace linear functional on the full $N \times N$ matrix algebra over $\mathbb{C}$. Then $\mathrm{tr}(A_N)$ is a random variable. In Lecture 1 we saw the proof of Wigner's semicircle law:

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}(A_N^n)] = \frac{1}{n+1}\binom{2n}{n}.$$

Date: Lecture given on Oct. 18, 2007.

In the language we have developed, this means that

$$A_N \to s \text{ as } N \to \infty,$$

where the convergence is in distribution and s is a semicircular element living in some non-commutative probability space.

We also saw Voiculescu's remarkable generalization of Wigner's semicircle law: if $A_N^{(1)}, \dots, A_N^{(p)}$ are p independent GUE random matrices (meaning that if we collect the real and imaginary parts of the above diagonal entries together with the diagonal entries we get a family of independent real Gaussians with mean 0 and variances as explained above), then

$$A_N^{(1)}, \dots, A_N^{(p)} \to s_1, \dots, s_p \text{ as } N \to \infty,$$

where $s_1, \dots, s_p$ is a family of freely independent semicircular elements. This amounts to proving that

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}(A_N^{(i(1))} \dots A_N^{(i(p))}] = \phi(s_{i(1)} \dots s_{i(p)}).$$

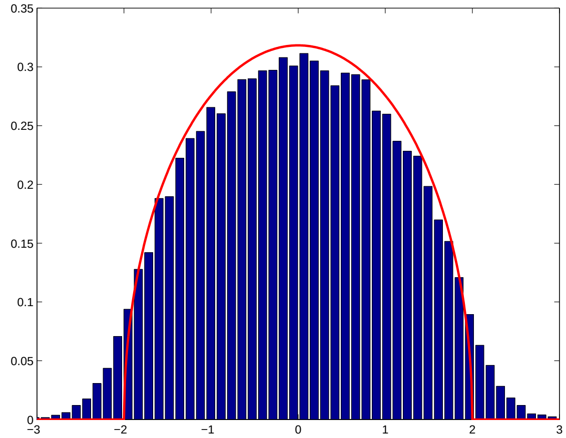
Recall that since $s_1, \dots, s_p$ are free their mixed cumulants will vanish, and only the second cumulants of the form $\kappa_2(s_i, s_i)$ will be non-zero.

The above two statements are "in distribution," i.e. with respect to the averaged trace $\mathbb{E}[\mathrm{tr}(\cdot)]$ which gives the empirical eigenvalue distribution. However they also hold in the sense of almost sure convergence, i.e. for all points ω in the underlying probability space except for a set of measure 0.

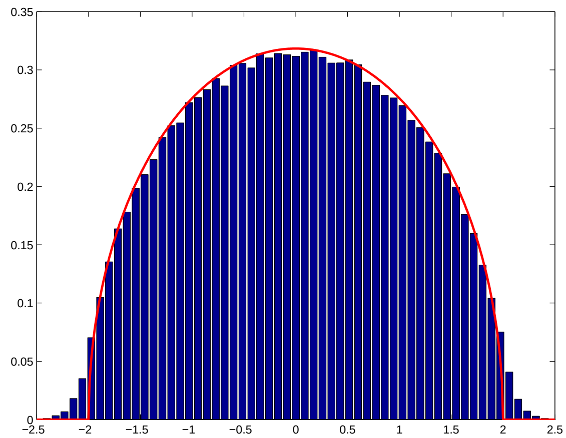
The following pictures provide numerical simulations of random matrices and show the difference between "convergence of averaged eigenvalue distribution" and "almost sure convergence".

Convergence of averaged eigenvalue distribution of $N \times N$ Gaussian random matrices to Wigner's semicircle

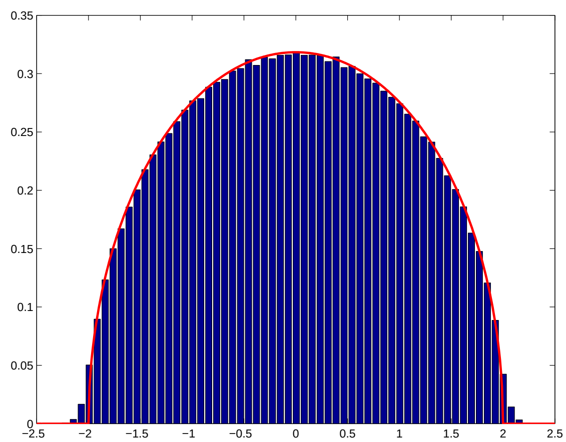
(number of realizations in each case: 10,000)



N=5



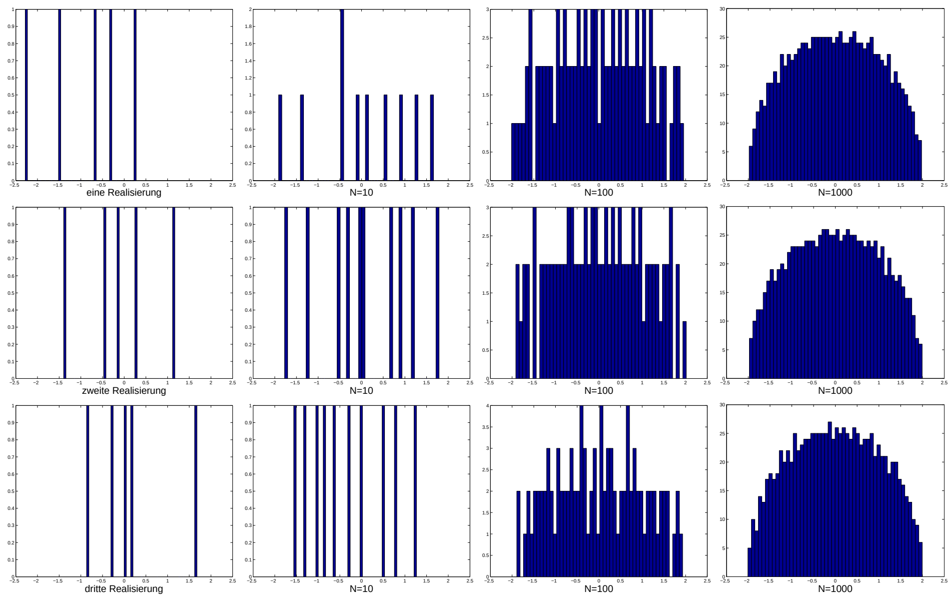
N=20



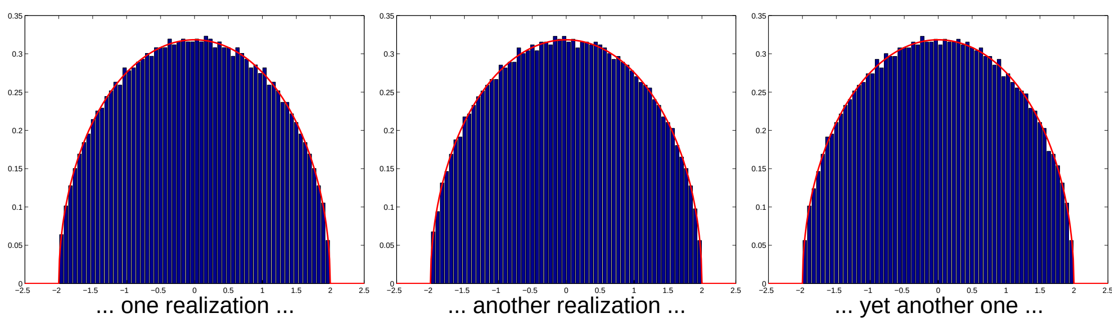
N=50

Almost sure convergence to Wigner's semicircle.

three different realizations:



$N = 4000$



We have asymptotic freeness of two independent Gaussian random matrices A_N, B_N:

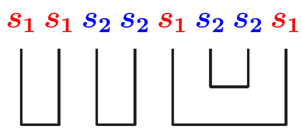
$$A_N, B_N \xrightarrow{\text{distr}} s_1, s_2,$$

where s_1, s_2 are free semicircular elements.

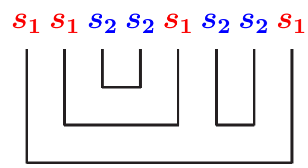
This means, for example, that

$$\lim_{N\to\infty} \mathrm{tr}(A_N A_N \boldsymbol{B}_N \boldsymbol{B}_N A_N \boldsymbol{B}_N \boldsymbol{B}_N A_N) = \varphi(\boldsymbol{s_1 s_1 s_2 s_2 s_1 s_2 s_2 s_1})$$

We have $\varphi(\boldsymbol{s_1 s_1 s_2 s_2 s_1 s_2 s_2 s_1}) = 2$, since there are two non-crossing pairings which respect the color:



and

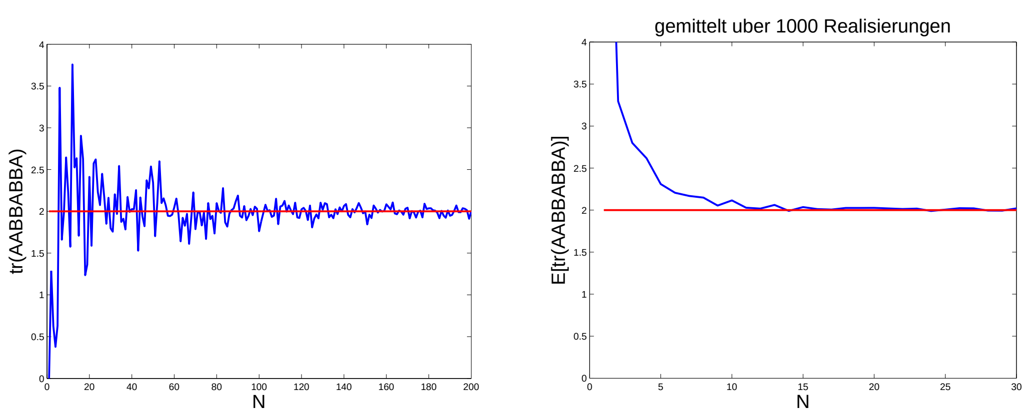


Here are numerical simulations for two independent Gaussian random matrices for

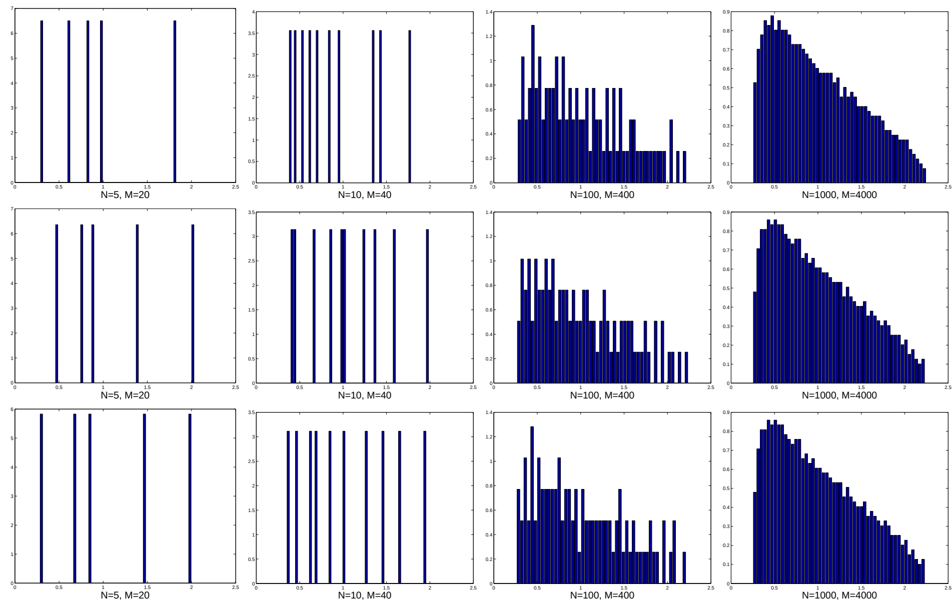
$$\mathrm{tr}(A_N A_N B_N B_N A_N B_N B_N A_N)$$

one realization

averaged over 1000 realizations

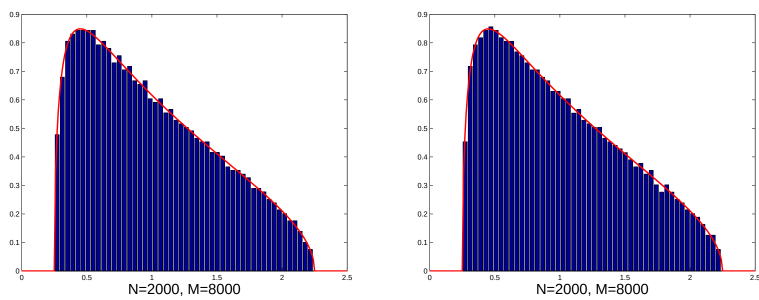


Consider **Wishart random matrix** $A = XX^*$, where X is $N \times M$ random matrix with independent Gaussian entries.

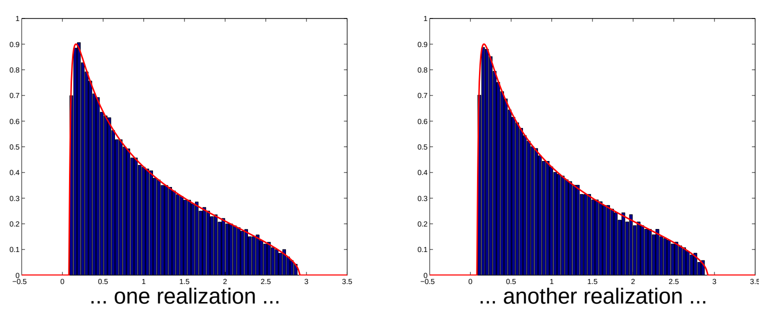


Its eigenvalue distribution converges almost surely towards

Marchenko-Pastur distribution.



$$N = 2000, M = 8000$$



$$N = 3000, M = 6000$$

1. Products of Free Random Variables

Let $\{a_1, \dots, a_r\}$ and $\{b_1, \dots, b_r\}$ be free random variables, and consider

$$\phi(a_1 b_1 a_2 b_2 \dots a_r b_r) = \sum_{\pi \in NC(2r)} \kappa_\pi[a_1, b_1, a_2, b_2, \dots, a_r, b_r].$$

Since the a's are free from the b's, we only need to sum over those partitions π which do not connect the a's with the b's. Each such partition may be written as $\pi = \pi_a \cup \pi_b$, where π_a denotes the blocks consisting of a's and π_b the blocks consisting of b's. Hence by the definition of free cumulants

$$\begin{aligned}\phi(a_1 b_1 a_2 b_2 \dots a_r b_r) &= \sum_{\pi_a \cup \pi_b \in NC(2r)} \kappa_{\pi_a}[a_1, \dots, a_r] \kappa_{\pi_b}[b_1, \dots, b_r] \\ &= \sum_{\pi_a \in NC(r)} \kappa_{\pi_a}[a_1, \dots, a_r] \Bigg(\sum_{\substack{\pi_b \in NC(r) \\ \pi_a \cup \pi_b \in NC(2r)}} \kappa_{\pi_b}[b_1, \dots, b_r] \Bigg).\end{aligned}$$

Now, the summation condition on the internal sum is equivalent to the condition $\pi_b \leq K(\pi_a)$, where K denotes the Kreweras complement (which is an order-reversing bijection) on the lattice $NC(r)$. Thus in particular we can write

$$\phi(a_1 b_1 a_2 b_2 \dots a_r b_r) = \sum_{\pi \in NC(2r)} \kappa_\pi[a_1, \dots, a_r] \cdot \phi_{K(\pi)}[b_1, \dots, b_r].$$

and in a similar way

$$\phi(a_1 b_1 a_2 b_2 \dots a_r b_r) = \sum_{\pi \in NC(2r)} \phi_{K^{-1}(\pi)}[a_1, \dots, a_r] \cdot \kappa_\pi[b_1, \dots, b_r].$$

Note that K^2 is not the identity, but a cyclic rotation of π.

These formulas are particularly useful when one of the sets of variables has simple cumulants, as is the case for semicircular random variables (where only the second order cumulants are non-vanishing, i.e. the sum is effectively only over non-crossing pairings).

2. Asymptotic Freeness Between Gaussian and Constant Random Matrices

Consider a sequence $(A_N)_{N \geq 1}$ of random matrices with A_N a $GUE(N)$ random matrix. Also let $(D_N)_{N \geq 1}$ be a sequence of constant (i.e., non-random) matrices with $D_N \in \mathbb{C}^{N \times N}$. Assume that

$$\lim_{N \to \infty} \operatorname{tr}(D_N^m) \tag{4}$$

exists for all $m \geq 1$. Then we have

$$A_N \to s \text{ and } D_N \to d \text{ as } N \to \infty, \tag{5}$$

where s, d live in some non-commutative probability space. What is the relation between s and d?

In order to answer this question we need to find the limiting mixed moments

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}(D_N^{q(1)} A_N D_N^{q(2)} \dots D_N^{q(m)} A_N)] \tag{6}$$

for all $m \geq 1$ where $q(k)$ can be 0. In the calculation let us suppress the dependence on N to reduce the number of indices, and write

$$D^q = (d_{ij}^q)_{i,j=1}^N \text{ and } A = (a_{ij})_{i,j=1}^N. \tag{7}$$

The Wick formula allows us to calculate mixed moments in the entries of A :

$$\mathbb{E}[a_{i(1)j(1)} a_{i(2)j(2)} \dots a_{i(m)j(m)}] = \sum_{\pi \in \mathcal{P}_2(m)} \prod_{(r,s)\in\pi} \mathbb{E}[a_{i(r)j(r)} a_{i(s)j(s)}] \tag{8}$$

where

$$\mathbb{E}[a_{ij} a_{kl}] = \delta_{il}\delta_{jk}\frac{1}{N}. \tag{9}$$

Thus we have

$$\begin{aligned}
&\mathbb{E}[\mathrm{tr}(D_N^{q(1)} A_N D_N^{q(2)} \dots D_N^{q(m)} A_N)] = \\
&= \frac{1}{N} \sum_{i,j\in[N]^{[m]}} \mathbb{E}[d^{q(1)}_{j(1)i(1)} a_{i(1)j(2)} d^{q(2)}_{j(2)i(2)} a_{i(2)j(3)} \dots d^{q(m)}_{j(m)i(m)} a_{i(m)j(1)}] \\
&= \frac{1}{N} \sum_{i,j\in[N]^{[m]}} \mathbb{E}[a_{i(1)j(2)} a_{i(2)j(3)} \dots a_{i(m)j(1)}] d^{q(1)}_{j(1)i(1)} \dots d^{q(m)}_{j(m)i(m)} \\
&= \frac{1}{N^{1+m/2}} \sum_{\pi\in P_2(m)} \sum_{i,j\in[N]^{[m]}} \prod_{r=1}^{m} \delta_{i(r)j(\gamma\pi(r))} d^{q(1)}_{j(1)i(1)} \dots d^{q(m)}_{j(m)i(m)} \\
&= \frac{1}{N^{1+m/2}} \sum_{\pi\in P_2(m)} \sum_{i,j\in[N]^{[m]}} d^{q(1)}_{j(1)j(\gamma\pi(1))} \dots d^{q(m)}_{j(m)j(\gamma\pi(m))} \\
&= \sum_{\pi\in P_2(m)} N^{\sharp(\gamma\pi)-1-m/2}\, \mathrm{tr}_{\gamma\pi}[D^{q(1)}, \dots, D^{q(m)}].
\end{aligned}$$

In the above calculation, we regard a pairing $\pi \in P_2(m)$ as a product of disjoint transpositions in $S(m)$ (i.e. an involution). Also $\gamma \in S(m)$ denotes the "long cycle" $\gamma = (1, 2, \dots, m)$ and $\sharp(\sigma)$ is the number of cycles in the factorization of $\sigma \in S(m)$ as a product of disjoint cycles. tr is the normalized trace, as always, and we have extended it multiplicatively as a functional on non-crossing partitions. For example if $\sigma = (1,3,6)(4)(2,5) \in S(6)$ then

$$\mathrm{Tr}_\sigma[D^{(1)}, D^{(2)}, D^{(3)}, D^{(4)}, D^{(5)}, D^{(6)}] = N^3 \,\mathrm{tr}(D^{(1)}D^{(3)}D^{(6)})\,\mathrm{tr}(D^{(4)})\,\mathrm{tr}(D^{(2)}D^{(5)}).$$

Now since

$$\lim_{N\to\infty} N^{\sharp(\gamma\pi)-1-m/2} = \begin{cases} 1 \text{ if } \pi \in NC_2(m) \\ 0 \text{ otherwise} \end{cases}$$

we have that

$$\lim_{N\to\infty} \mathbb{E}[\mathrm{tr}(D_N^{q(1)} A_N D_N^{q(2)} \dots D_N^{q(m)} A_N)] = \sum_{\pi \in NC_2(m)} \phi_{\gamma\pi}[d^{q(1)}, \dots, d^{q(m)}].$$

Now compare this to the formula for d, s free with s semicircular:

$$\phi[d^{q(1)} s d^{q(2)} s \dots d^{q(m)} s] = \sum_{\pi \in NC_2(m)} \phi_{K^{-1}(\pi)}[d^{q(1)}, \dots, d^{q(m)}].$$

The two are the same provided $K^{-1}(\pi) = \gamma\pi$ where K is the Kreweras complement. But this is true for all $\pi \in NC_2(m)$. Consider for example $\pi = \{1,10\} \cup \{2,3\} \cup \{4,7\} \cup \{5,6\} \cup \{8,9\} \in NC_2(10)$. Regard this as the involution $\pi = (1,10)(2,3)(4,7)(5,6)(8,9) \in S(10)$. $\gamma\pi = (1)(2,4,8,10)(3)(5,7)(6)(9)$ which is exactly $K^{-1}(\pi)$.

Hence we have the following theorem.

Theorem: Let $A_N^{(1)}, \dots, A_N^{(p)}$ be p independent $GUE(N)$ random matrices and let $D_N^{(1)}, \dots, D_N^{(q)}$ be q constant non-random matrices such that

$$D_N^{(1)}, \dots, D_N^{(q)} \to d_1, \dots, d_q \text{ as } N \to \infty.$$

Then

$$A_N^{(1)}, \dots, A_N^{(p)}, D_N^{(1)}, \dots, D_N^{(q)} \to s_1, \dots, s_p, d_1, \dots, d_p \text{ as } N \to \infty$$

where each s_i is semicircular and $s_1, \dots, s_p, \{d_1, \dots, d_p\}$ are free. That is, $A_N^{(1)}, \dots, A_N^{(p)}, \{D_N^{(1)}, \dots, D_N^{(q)}\}$ are asymptotically free.

3. Asymptotic Freeness Between Haar Unitary Random Matrices and Constant Matrices

Let $U(N) \subset \mathbb{C}^{N\times N}$ denote the group of unitary matrices, i.e. $N \times N$ complex matrices which satisfy $U^*U = UU^* = I_N$. Since $U(N)$ is a compact group, one can take dU to be Haar measure on $U(N)$ normalized so that $\int_{U(N)} dU = 1$, which gives a probability measure on N. A "Haar unitary" is a matrix U_N chosen at random from $U(N)$ with respect to Haar measure. There is a useful theoretical and practical way to construct Haar unitaries: take an $N \times N$ random matrix whose entries are independent standard complex Gaussians and apply the Gram-Schmidt orthogonalization procedure; the resulting matrix is then a Haar unitary.

What is the $*$-distribution of a Haar unitary with respect to the state $\phi = \mathbb{E} \circ \mathrm{tr}$? Since $U_N^* U_N = I_N = U_N U_N^*$, the $*$-distribution is determined by the sequences $\phi(U_N^m)$ for $m \in Z$. Note that for any complex number $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, λU_n is again a Haar unitary. Thus, $\phi(\lambda^m U_N^m) = \phi(U_N^m)$ for all $m \in \mathbb{Z}$. This implies that we must have $\phi(U_N^m) = 0$ for $m \neq 0$.

Definition: Let $(A, \phi))$ be a non-commutative probability space with A a unital $*$-algebra. $u \in A$ is called a Haar unitary if

- u is unitary, i.e. $u^* u = 1_A = uu^*$;
- $\phi(u^k) = \delta_{0,k}$ for $k \in \mathbb{Z}$.

Thus a Haar unitary random matrix $U_N \in U(N)$ is a Haar unitary for each $N \geq 1$ (with $\phi = \mathbb{E} \circ \mathrm{tr}$).

We want to see that asymptotic freeness occurs between Haar unitary random matrices and constant non-random matrices, as was the case with GUE random matrices. The crucial element in the Gaussian setting was the Wick formula, which of course does not apply when dealing with Haar unitary random matrices whose entries are neither independent nor Gaussian. However, we do have a replacement for the Wick formula in this context, which is known as the "Weingarten convolution formula."

Let $i, j, k, l \in [N]^{[n]}$ be functions. The Weingarten convolution formula asserts the existence of a sequence of functions $(\mathrm{Wg}_N)_{N=1}^{\infty}$ with each Wg_N a function from permutations to complex numbers which satisfies

$$\mathbb{E}[u_{i(1)j(1)} \dots u_{i(n)j(n)} \overline{u_{k(1)l(1)}} \dots \overline{u_{k(n)l(n)}}] = \sum_{\sigma,\tau \in S(n)} \prod_{r=1}^{n} \delta_{i(r)k(\sigma(r))} \delta_{j(r)l(\tau(r))} \, \mathrm{Wg}_N(\tau\sigma^{-1})$$

as well as the asymptotic growth condition

$$\mathrm{Wg}_N(\pi) \sim \frac{1}{N^{2n-\sharp(\pi)}} \text{ as } N \to \infty$$

for any $\pi \in S(n)$. Here the u's are the entries of a Haar unitary U_N. The precise definition of the Weingarten function is quite complicated and uneccessary for our purposes; we only need the convolution formula and the asymptotic estimate. This allows us to prove the following.

Theorem: Let $U_N^{(1)}, \dots, U_N^{(p)}$ be p independent Haar unitary random matrices, and let $D_N^{(1)}, \dots, D_N^{(q)}$ be q constant non-random matrices

such that
$$D_N^{(1)}, \dots, D_N^{(q)} \to d_1, \dots, d_q \text{ as } N \to \infty.$$
Then
$$U_N^{(1)}, \dots, U_N^{(p)}, D_N^{(1)}, \dots, D_N^{(q)} \to u_1, \dots, u_p, d_1, \dots, d_q \text{ as } N \to \infty$$
where each u_i is a Haar unitary and $u_1, \dots, u_p, \{d_1, \dots, d_q\}$ are free.

The proof proceeds in a similar fashion as in the Gaussian setting.

Note that in general if u is a Haar unitary living in a non-commutative probability space which is $*$-free from elements $\{a, b\}$, then a and ubu^* are free. In order to prove this, consider
$$\phi(p_1(a)q_1(ubu^*) \dots p_r(a)q_r(ubu^*))$$
where p_i, q_i are polynomials such that
$$\phi(p_i(a)) = 0 = \phi(q_i(ubu^*)).$$
Note that by the unitary condition we have $q_i(ubu^*) = uq_i(b)u^*$. Thus
$$0 = \phi((q_i(ubu^*)) = \phi(uq_i(b)u^*) = \phi(uu^*)\phi(q(b)) = \phi(q(b)).$$
Hence
$$\phi(p_1(a)q_1(ubu^*) \dots p_r(a)q_r(ubu^*)) = \\ \phi(p_1(a)uq_1(b)u^*p_2(a) \cdots p_r(a)uq_r(b)u^*) = 0$$
since u is $*$-free from $\{a, b\}$ and $\phi(u) = \phi(u^*) = 0$.

Thus our Theorem from above yields also the following as a corollary.

Theorem: Let A_N and B_N be two sequences of constant $N \times N$ matrices with $A_N \to a$ and $B_N \to b$. Let U_N be a sequence of Haar unitary random matrices. Then
$$A_N, U_N B_N U_N^* \to a, b$$
where a and b are free.

Conjugation by a Haar unitary random matrix corresponds to a "random rotation" (i.e. $U(N)$ is the autmorphism group of the Euclidean space $\mathbb{C}^N$.) Thus the above theorem says that randomly rotated constant matrices become asymptotically free in the limit of large matrix dimension.

The following pictures show how the machinery of free probability can be used to calculate asymptotic eigenvalue distributions of some random matrices.

Recall that we have the following theorem.

Theorem [Voiculescu 1986, Speicher 1994]:
Put
$$G(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\varphi(A^m)}{z^{m+1}} \qquad \text{Cauchy transform}$$
and
$$\mathcal{R}(z) = \sum_{m=1}^{\infty} \kappa_m z^{m-1} \qquad \boldsymbol{\mathcal{R}}\textbf{-transform}$$
Then we have
$$\frac{1}{G(z)} + \mathcal{R}(G(z)) = z.$$

Let A and B be free. Then one has
$$\mathcal{R}^{A+B}(z) = \mathcal{R}^{A}(z) + \mathcal{R}^{B}(z),$$
or equivalently
$$\kappa_m^{A+B} = \kappa_m^{A} + \kappa_m^{B} \qquad \forall\, m.$$

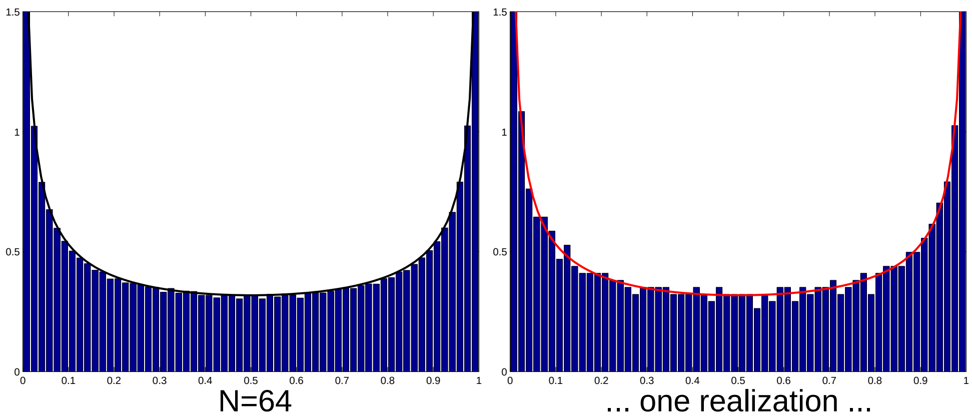
This, together with the Stieltjes inversion formula, gives an effective algorithm for calculating free convolutions, i.e., the **asymptotic eigenvalue distribution** of sums of random matrices in **generic position**:

$$\begin{tikzcd}
A \arrow[r, rightsquigarrow] & G^A \arrow[r, rightsquigarrow] & R^A \arrow[d] & & & \\
 & & R^A + R^B = R^{A+B} \arrow[r, rightsquigarrow] & G^{A+B} \arrow[r, rightsquigarrow] & A+B \\
B \arrow[r, rightsquigarrow] & G^B \arrow[r, rightsquigarrow] & R^B \arrow[u] & & &
\end{tikzcd}$$

Consider $A + UBU^*$, where U is Haar random matrix and A and B are diagonal matrices, each with $N/2$ eigenvalues 0 and $N/2$ eigenvalues $1/2$

Thus the asymptotic eigenvalue distribution of the sum should be the same as the distribution of the sum of two free Bernoulli distributions, i.e., an arc-sine distribution

Here are corresponding simulations for random matrices; the first is averaged over the ensemble, the second is for one realization.

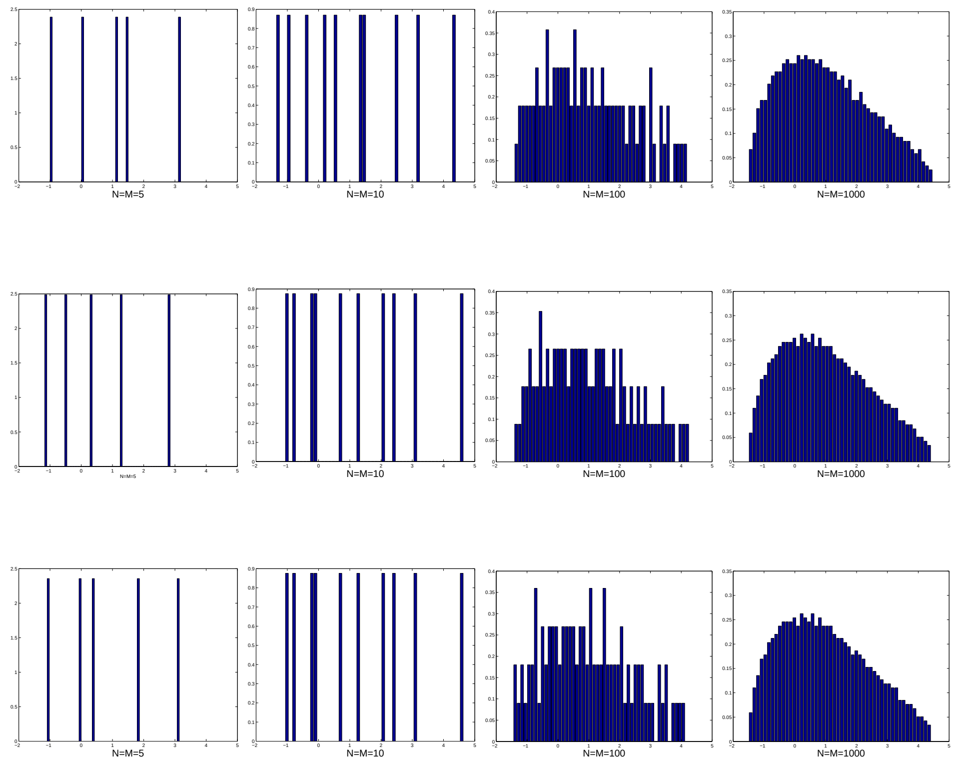


trials=5000 N=2048

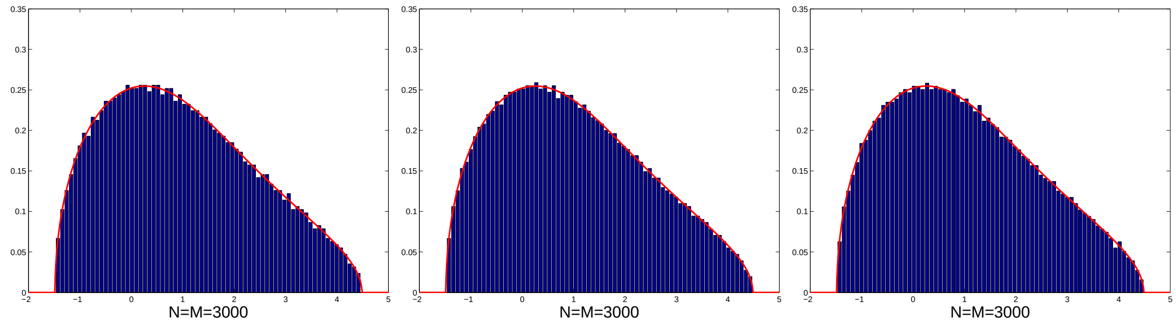
Consider now independent Wigner and Wishart matrices. They are in generic position, thus asymptotically free.

So the asymptotic eigenvalue distribution of the sum of Wigner + Wishart should be given by the distribution of the sum of a free semicircle and Marchenko-Pastur

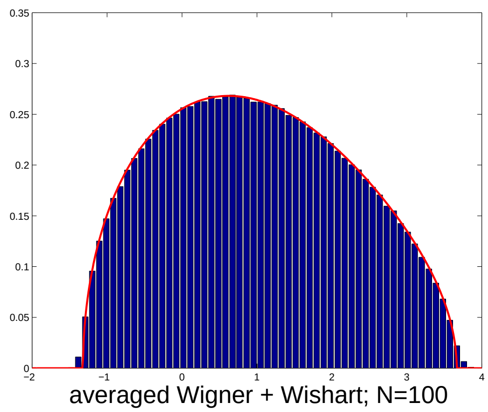
Here are three different realizations of such random matrices:

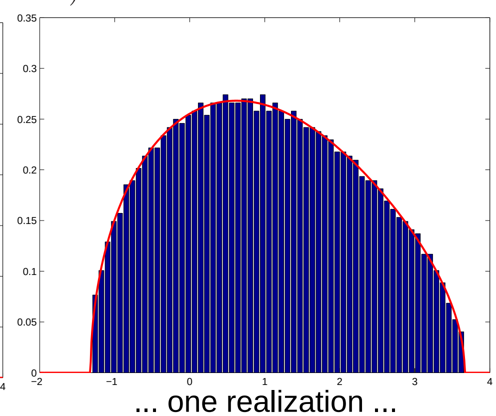


Wigner(3000) + Wishart(3000,3000)



Example: Wigner + Wishart ($M = 2N$)





trials=4000

N=3000

One has similar analytic description for product.

Theorem [Voiculescu 1987, Haagerup 1997, Nica + Speicher 1997]:
Put

$$M_A(z) := \sum_{m=0}^{\infty} \varphi(A^m) z^m$$

and define

$$S_A(z) := \frac{1+z}{z} M_A^{<-1>}(z) \qquad \textbf{\textit{S}-transform of \textit{A}}$$

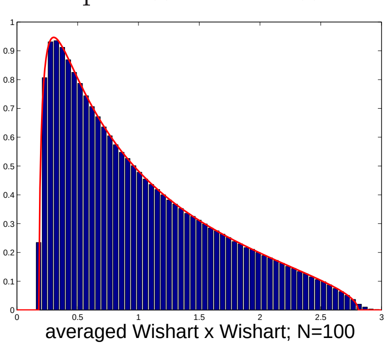
Then: If A and B are free, we have

$$S_{AB}(z) = S_A(z) \cdot S_B(z).$$

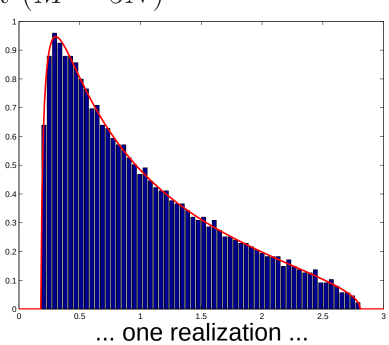
Consider two independent Wishart matrices. They are in generic position, thus asymptotically free.

So the asymptotic eigenvalue distribution of their product should be given by the distribution of the product of two free Marchenko-Pastur distributions.

Example: Wishart x Wishart $(M = 5N)$



trials=10000



N=2000