Nephrology

Diffusive-Convective Mass Transfer Rates for Solutes Present on Both Sides of a Dialyzer Membrane

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The transport (J) of waste products across dialyzer membranes is known to be proportional to the blood inlet concentration (C_{bi}) according to $J = KC_{bi}$, where K is the clearance. For solutes present on both sides of the membrane, like sodium chloride, it has been shown[1] that under certain conditions the transport rate will depend linearly also upon the dialysis fluid inlet concentration C_{di} according to $J = K_bC_{bi} - K_dC_{di}$. K_b and K_d are generalized clearances, which depend upon flow rates and membrane permeability but are independent of the concentrations. We have extended the results of Ross *et al.* in three ways. First, they only considered ultrafiltration (UF) that is equally distributed along the dialyzer. This is an unrealistic assumption, especially in hemodiafiltration and hemofiltration treatments with large UF rates (Q_{uf}) leading to large pressure drops along the dialyzer. Our approach allows for an arbitrary UF distribution. Second, it was possible to incorporate the more realistic model of Villaroel *et al.* for the local combination of diffusion and convection. Finally, we allow an arbitrary distribution of blood among the different fibers. All of these results are valid in both cocurrent and countercurrent configurations. With a sieving coefficient of 1, a good approximation for small solutes, we were also able to show that $K_d = K_b - Q_{uf}$, irrespective of the UF distribution along the dialyzer. This is an important result that, for example, provides a theoretical foundation for allowing a nonzero Q_{uf} in conductivity based clearance measurements. *ASAIO Journal* 2005; 51:246–251.

Early results on the diffusive-convective mass transfer rates in dialyzers for solutes present on both sides of the membrane were given by Ross *et al.*[1] In this article we have extended their results in several important directions, including less restrictions on flow distributions, membrane permeability and sieving properties, and with the more accurate treatment of the combination of diffusion and convection according to Villaroel *et al.*[2]

For solutes that are not present in the fresh dialysis fluid, it is well established that the mass transfer rate (J) across a dialyzer membrane is proportional to the solute concentration C_{bi} in the blood entering the dialyzer so that $J = KC_{bi}$. By imposing a number of restrictive conditions, a theoretical expression for the proportionality constant, the clearance K, was derived many years ago.[3] These conditions include an equal distribution of the blood flow among the fibers in the dialyzer, an instant and perfect mixing of the blood at each point in each of the fibers, an instant and perfect mixing of the dialysis fluid at each point along the dialyzer, and a constant mass transfer coefficient along the dialyzer. The early results also assumed the absence of ultrafiltration (UF) and a sieving coefficient (S) of 1.

For the case of countercurrent blood and dialysate flows, the clearance K depends upon the blood flow rate Q_b, the dialysis fluid flow rate Q_d, and the mass transfer area coefficient k_oA of the dialyzer according to the well known formula[3]

$$K = \frac{Q_b \cdot Q_d \cdot (1 - f)}{Q_b - Q_d \cdot f} \text{ with } f = e^{k_oA\left(\frac{1}{Q_b} - \frac{1}{Q_d}\right)} \qquad (1)$$

This formula has often been used also with small but nonzero UF rates, as dictated by the required weight loss of the patient, thus neglecting the added clearance from convection. The correct inclusion of the convective contribution is fairly complicated. To simply add the ultrafiltration rate to the clearance computed from pure diffusion from **Equation 1** is incorrect because ultrafiltration affects both the flow rates and the concentrations, so that the combination of diffusion and convection results in less clearance than the sum of the two. Because of the increasing popularity of hemodiafiltration (HDF), higher UF rates must be taken into account to provide more accurate clearances.

When including the effect of UF upon clearance, several new assumptions must be made. It is, for example, common to assume that the UF is evenly distributed along the dialyzer. Without this assumption, the resulting formulas become much more complicated, but it would of course be more realistic to assume a decreasing UF rate along the dialyzer, and in some cases even a negative UF rate, that is, backfiltration, at the blood outlet end. A sieving coefficient of less than 1 will decrease the effect of ultrafiltration for larger solutes, but taking this into account will complicate the formulas considerably. It is also common in the derivation of formulas to just add the diffusive and convective transports across the membrane at each point along the dialyzer.[4–8] As shown by Villaroel *et al.*,[2] this is an overly simplified summation because diffusion and convection also interact inside the membrane. The effect of this interaction upon clearance was included in the work by Waniewski *et al.*[9] also for transport backwards from dialysate to blood.[10]

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For solutes that are present on both sides of the membrane, like sodium chloride, the transport rate will under certain conditions[1] depend linearly also on the concentration C_{di} in the fresh dialysis fluid according to $J = K_b C_{bi} - K_d C_{di}$. The proportionality constants K_b and K_d do not depend upon the concentrations, but are, in the general case, complicated functions of the blood and dialysis fluid flow rates, Q_{uf}, and the membrane properties. By setting $C_{di} = 0$, we see that K_b equals the classical clearance, K. In certain cases $K_d = K_b - Q_{uf}$, but this is not true in general when the sieving coefficient is less than 1.

In this article, the results of Ross *et al.*[1] have been extended by eliminating several of the assumptions they made. The linearity of the mass transport rate as a function of the concentrations as well as the relation between the proportionality constants K_b and K_d have thus been established under less restrictive conditions. The initial derivation assumes a countercurrent configuration. The same results are then obtained for the cocurrent case by minor changes in the derivation.

Materials and Methods

The transport rate J of solutes from blood to dialysis fluid can, in the general case, be calculated as

$$J = Q_b C_{bi} - (Q_b - Q_{uf})C_{bo} \qquad (2)$$

where C_{bo} is the solute concentration at the blood outlet. This mass balance equation can be used to determine clearance both experimentally and theoretically. It is important that concentrations and flows all refer to the same type of fluid, that is, whole blood, plasma, or blood water, depending upon where the solute is distributed. Laboratory values are usually given as plasma concentrations and thus usually need to be converted. This is particularly important when Q_{uf} is large, like in HDF. In this article, the term "blood flow" is used to denote the relevant flow on the blood side of the dialyzer.

For a theoretical calculation, the inlet concentration and the flow rates are assumed to be known. The only unknown entity in **Equation 2** is then the solute concentration C_{bo}. Under certain conditions, this can be theoretically derived from a system of differential equations describing the evolution of the blood and dialysate concentrations and flow rates along the dialyzer. The basic principles are discussed in the following sections, and the full derivation is given in the Appendix.

The transport may be driven by both diffusion and convection. The size of a pure diffusive transport is proportional to the concentration difference between blood and dialysis fluid, and a pure convective transport would be proportional to the blood concentration. It is common to assume that the total transport across the membrane at each point along the dialyzer is simply the sum of the pure diffusive and convective terms. However, the two transport mechanisms will interact[2] so that the total transport is less than the sum of the two applied separately. The reason for this is that the ultrafiltrate flow will enter into the membrane fluid with the same solute concentration as the blood. This will decrease the concentration gradient within the membrane and thus reduce the diffusive transport. The effect of this local combination of diffusion and convection is included in the derivation in the Appendix and is an extension of the results by Ross *et al.*[1]

It is assumed in the derivation that there is an immediate mixing within both the blood flow and the dialysate flow at each point along the dialyzer. This is unrealistic, and instead concentration profiles will develop where the concentration decreases gradually from the center of the fibers to the bulk of the dialysis fluid. The diffusive transport is at each point driven by the concentration gradient, and the assumption of immediate mixing then means that the total effect of the gradient is summarized as a difference between the mean concentrations.

The effect of ultrafiltration upon the transport is twofold. It contributes directly to the mass transport across the membrane by convection, and it changes the flow rates in the blood and the dialysate, which affects the size of the concentration changes caused by the mass transport. These changes in the flow rates complicate the theoretical derivation slightly, but as shown in the Appendix, it is possible to get general results even for an arbitrary distribution of the ultrafiltration within the dialyzer. This is also an extension of the results by Ross *et al.*[1]

For theoretical clearance calculations, it is usually assumed that the blood flow is equally distributed among the different fibers of the dialyzer. This is, however, an unrealistic assumption. Because of the production process, there will inevitably be variations in inner diameter among the fibers. Because the flow resistance is inversely proportional to the fourth power of the diameter, even small diameter variations will produce fairly large flow variations. In addition, the fiber diameter may be reduced by varying degrees of clotting. It is, therefore, of interest to get results also for the case of different flow rates through different fibers. The flow rate in the fiber affects the concentration of solute along the fiber, and it is thus necessary to treat each fiber separately, which leads to a very large system of equations, as shown in the Appendix. The dialysate compartment, on the other hand, where there are no separating walls, is treated as one compartment with perfect mixing. This arbitrary distribution of the blood flow among fibers is the third extension of the results by Ross *et al.*[1]

The theoretical results were verified in an experimental study[11] where a 30 L tank of dialysate, representing a patient on dialysis, was dialyzed for approximately 180 minutes against a dialysis fluid of different conductivity. The "dialysis session" was conducted in countercurrent configuration with blood and dialysate flow rates set at 300 and 500 ml/min, respectively, and an ultrafiltration rate of 1.0 L/h. Six sessions were performed: (set A) three with the initial "patient" (tank) conductivity, C_{pt}, set at 17.0 mS/cm and the dialysis fluid conductivity, C_{di}, set at 13.0 mS/cm, and (set B) three with $C_{pt} = 13.0$ and $C_{di} = 17.0$ mS/cm, thus obtaining diffusive and convective mass transfer in the same (set A) and in opposing directions (set B). Flow through cells permitted continuous conductivity measurement in the blood and dialysis fluid entering and exiting the dialyzer. Plots were made of $J\text{-}Q_{uf}C_{di}$ *versus* $C_{bi}\text{-}C_{di}$.

Results

The Countercurrent Case

Based on the above assumptions outlined in the previous section, a model for all of the concentrations along the dialyzer is derived in the Appendix. It is shown that the concentrations are determined by a system of first order linear differential equations with coefficients that may vary with the position in

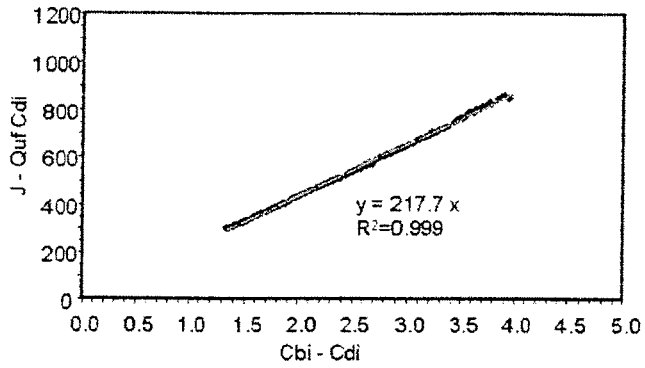

Figure 1. Plot of $J\text{-}Q_{uf}C_{di}$ *versus* $C_{bi}\text{-}C_{di}$ for an experimental study session where both diffusion and convection occurred in the same direction (C_{pt} = 17.0 and C_{di} = 13.0 mS/cm). The trendline represents **equation 5** with k_b = 217.7 mL/min.

the dialyzer. The solution to this system of equations produces an expression for the blood concentration at the dialyzer outlet, from which the transport rate J of solute across the membrane can be written as

$$J = K_b\, C_{bi} - K_d\, C_{di} \tag{3}$$

This linear dependence of the transport rate upon the inlet concentrations has been shown under widely varying general conditions:

- Arbitrary distribution of the blood flow among different fibers
- Arbitrary distribution of the UF among different fibers and along the dialyzer length
- Inclusion of the interaction between diffusion and convection across the membrane[2]
- Arbitrary variation of the membrane permeability among different fibers and along the dialyzer length
- Arbitrary variation in the sieving properties among different fibers and along the dialyzer length

The proportionality constants (which are independent of the concentrations) are in the general case quite complicated functions of the flow rates and the membrane permeability. For solutes that are small enough to pass the membrane freely (so that Staverman's reflection coefficient $\sigma = 0$), it is shown that

$$K_b - K_d = Q_{uf} \tag{4}$$

We can then write

$$J = K_b(C_{bi} - C_{di}) + Q_{uf}\, C_{di} \tag{5}$$

This result was verified in the experimental study[11] by plotting $J\text{-}Q_{uf}C_{di}$ *versus* $C_{bi}\text{-}C_{di}$. **Figure 1** represents one of the sessions from set A where both diffusion and convection were in the same direction, from the "blood" to the dialysis fluid. All six plots of J-QufCdi *versus* Cbi-Cdi were shown to be perfectly linear ($R^2 > 0.999$), thus validating **Equation 5**.

For special cases, it is possible to derive explicit formulas for clearance. In addition to having $\sigma = 0$, we may assume that all fibers are the same so that they can be lumped together as one fiber and that the membrane permeability k_o and the ultrafiltration rate are constant along the dialyzer. The system of differential equations in the Appendix then reduces to just one equation, the matrix Φ reduces to a scalar integrating factor, and K_b is[9,12]

$$K_b = \frac{Q_b \cdot Q_d - f \cdot (Q_b - Q_{uf}) \cdot (Q_d + Q_{uf})}{Q_d - f \cdot (Q_b - Q_{uf})} \tag{6}$$

where

$$f = \left[\frac{Q_b - Q_{uf}}{Q_b} \cdot \frac{Q_d + Q_{uf}}{Q_d}\right]^{\frac{1}{\gamma}} \text{ with } \gamma = e^{\frac{Q_o}{k_o A}} - 1 \tag{7}$$

A slightly different formula results if the effects[2] of the local combination of diffusion and convection are neglected and replaced by a direct sum of the two.[5] The only difference is that γ in **Equation 7** is replaced by $\gamma = Q_{uf}/k_oA$. It should be noted that both of these formulas are nonlinear in Q_{uf}. It is also possible to derive a clearance formula for the realistic case with a UF rate that changes linearly along the dialyzer, but it has a more complicated appearance and is therefore not presented here.

The Cocurrent Case

By changing the sign of the dialysate flow, it is shown at the end of the Appendix that all of the basic results, **Equations 3, 4, and 5**, are valid also in the cocurrent case. For special cases, it is again also possible to derive explicit formulas for clearance. With the same assumptions as for **Equations 6 and 7**, we get

$$K_b = \frac{Q_b \cdot (Q_d + Q_{uf}) - f \cdot Q_d \cdot (Q_b - Q_{uf})}{Q_b + Q_d} \tag{8}$$

where

$$f = \left[\frac{Q_b - Q_{uf}}{Q_b} \cdot \frac{Q_d}{Q_d + Q_{uf}}\right]^{\frac{1}{\gamma}} \text{ with } \gamma = e^{\frac{Q_o}{k_o A}} - 1 \tag{9}$$

Discussion

The result (**Equation 3**) that the mass transport rate is linear in both inlet concentrations should be expected from the linear nature of the driving forces for diffusion and convection. The same result could, therefore, be expected to hold under even more general conditions, such as without the assumption of perfect mixing within the fibers and in the dialysate. The resulting partial differential equations for the concentrations within the fibers could probably be handled. The main difficulty lies in the distribution of the fibers within the dialysate compartment, and some regularity conditions on this distribution would probably be needed to obtain any theoretical results.

Equation 5 is an important result that is needed, for example, in the derivation of the formulas for conductivity based clearance measurements. Because K_b is independent of the concentrations, we can set $C_{di} = 0$ to see that K_b equals the normal clearance. Setting $Q_{uf} = 0$, we see that K_b also equals the dialysance (by the definition of the term) in this case, and clearance and dialysance are then numerically equal. For nonzero Q_{uf}, K_b changes with Q_{uf}, and dialysance is not well defined because it would not be independent of the concentrations. Thus, contrary to common belief, a conductivity based clearance measurement that gives a value for K_b should not be considered a dialysance measurement but rather a clearance measurement.

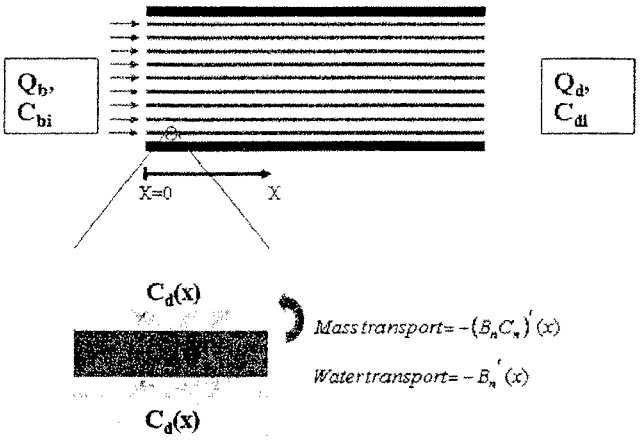

Figure 2. A longitudinal view through a dialyzer with local transports of mass and water from blood in one fiber to dialysate.

The result of **Equation 5** should be expected not to hold for larger solutes with $\sigma > 0$ because it requires that all components of Γ in Equation A14 equal 0. This cannot happen if the ultrafiltration is positive all along the dialyzer and would be a rare event also in other cases.

The effects of protein in the blood deserve special attention. Concentration polarization of protein that has been described for high UF rates leads to changes in the membrane permeability for solutes and fluid, which may vary among fibers and with position in the dialyzer. The conditions outlined previously show that all such effects of protein layers on the membrane are included in our derivation. The Donnan equilibrium[13–15] that, in the presence of protein, affects ionic substances (such as sodium chloride) is more complex to analyze. The result will depend upon the model used to describe the Donnan effects. The results described previously will still hold if we assume that a fixed amount of the solute is attached to the protein and that the blood concentration refers only to the removable fraction of the solute.

Conclusion

The mass transport rate across a dialyzer membrane has been shown to be linear in the inlet concentrations of both blood and dialysis fluid under quite general conditions. Under the same general conditions, but with complete convective permeability (zero reflection coefficient), the difference between the proportionality constants has been shown to equal the UF rate. These results form the basis for conductivity based clearance measurements.

Appendix

Derivation of the Mass Transport Rate: The Countercurrent Case

The basic principle is to calculate the theoretical mass transfer rate by solving equations for the blood concentrations along the dialyzer. The concentrations and flows are viewed as functions of the distance x ($x \leq L$) from the blood inlet along the dialyzer. To minimize notation, the conventional Q_b and Q_d are only used for the inlet flows, whereas the internal blood and dialysate flows are denoted just B and D. **Figure 2** is a longitudinal view through a dialyzer with local transports of mass and water from blood in one fiber to dialysate. The following notations are used:

$B_n(x)$ = blood flow rate at distance x in fiber n

Q_b = total blood flow rate at blood inlet

$D(x)$ = dialysate flow rate at distance x

Q_d = dialysate flow rate at dialysate inlet

Q_{do} = dialysate flow rate at dialysate outlet

Q_{uf} = total UF rate

$C_n(x)$ = solute concentration at distance x in fiber n

C_{bi} = inlet blood concentration

$C_d(x)$ = mean solute concentration in dialysate at distance x

C_{di} = inlet dialysate concentration

C_{do} = outlet dialysate concentration

$\sigma_n(x)$ = Staverman's reflection coefficient at distance x in fiber n

$Pe_n(x)$ = Peclet number at distance x in fiber n

$k_0a_n(x)$ = mass transfer area coefficient per unit length at distance x in fiber n

$k_n(x)$ = mass transfer area coefficient with Villaroel correction

For ease of notation, the argument (x) will be dropped whenever clarity permits.

The decrease in solute mass flow rate in fiber n can be written in two ways. On the one hand, it equals the derivative of the mass flow rate (left hand side of Equation A1). On the other hand, it can be written as the interaction of diffusive and convective transport across the membrane.[2] Rearranging the expression for this interaction,[2] we get the differential equation (prime denotes derivative with respect to x)

$$(B_nC_n)' = -k_n(C_n - C_d) + B_n'(1 - \sigma_n)C_n \qquad \text{(A1)}$$

where[2]

$$k_n = k_0a_n\frac{Pe_n}{e^{Pe_n} - 1} = \frac{-B_n'(1 - \sigma_n)}{e^{-B_n'(1-\sigma_n)/k_0a_n} - 1} \qquad \text{(A2)}$$

The last equality follows from the definition of the Peclet number

$$Pe_n = \frac{-B_n'(1 - \sigma_n)}{k_0a_n} \qquad \text{(A3)}$$

For continuity reasons, the change in solute mass flow rate in the dialysate must be

$$(DC_d)' = \sum_n (B_nC_n)' \qquad \text{(A4)}$$

so that with an α independent of x (but dependent upon flows and concentrations), we have

$$DC_d = \sum_n (B_nC_n) + \alpha \qquad \text{(A5)}$$

The physical meaning of α is the net mass flow of solute along the dialyzer in the direction of the dialysate flow. This

xmust be constant along the dialyzer, because no solute is generated, and may be related to the net solute flow leaving the dialyzer at the blood entrance.

$$\alpha = Q_{do}C_{do} - Q_bC_{bi} = Q_dC_{di} - Q_bC_{bi} + J \qquad (A6)$$

where J is the rate of solute mass transfer across the membrane.

Correspondingly, the changes in fluid flow rate and blood flow rate are related

$$D' = \sum_n B_n' \qquad (A7)$$

so that the difference between D and the sum of the B_n is constant along the dialyzer. This constant, β, is the net volumetric flow in the dialysate flow direction and can be related to the net flow leaving the dialyzer at the blood entrance,

$$\beta = Q_{do} - Q_b = Q_d - Q_b + Q_{uf} \qquad (A8)$$

To simplify notations we also introduce the constant δ as

$$\delta = \frac{\alpha}{\beta} = \frac{\alpha}{D - \sum_n B_n} \qquad (A9)$$

The physical meaning of δ is the mean concentration across the dialyzer (blood and dialysate compartments together), and it will be constant along the dialyzer because both numerator and denominator are constant. This requires the denominator to be nonzero. The case where the denominator of **Equation A9** equals 0, or equivalently $Q_d = Q_b - Q_{uf}$, can be handled by a continuity argument in the final result. We can now insert C_d from **Equation A5** into **Equation A1** and subtract the constant δ from each occurrence of C_n to get

$$(C_n - \delta)' = -\left(\frac{k_n}{B_n} + \sigma_n\frac{B_n'}{B_n}\right)(C_n - \delta) + \frac{k_n}{DB_n}\sum_m B_m(C_m - \delta) - \delta\sigma_n\frac{B_n'}{B_n} \qquad (A10)$$

The summation index m in the first term of the second line covers all the fibers in the dialyzer. This term represents the influence upon each fiber concentration from all of the other fibers (*via* the dialysate). Introducing the vector **C** having all $(C_n - \delta)$ as its elements, **Equation A10** can be written

$$\mathbf{C}' = \mathbf{F}(x)\cdot\mathbf{C} + \delta\cdot\mathbf{G}(x) \qquad (A11)$$

where the elements of the matrix **F** and the vector **G** are

$$\mathbf{F}_{nn} = -\left(\frac{k_n}{B_n} + \sigma_n\frac{B_n'}{B_n}\right) + \frac{k_n}{D}$$

$$\mathbf{F}_{nm} = \frac{k_n}{DB_n}B_m \text{ for } n \neq m \qquad (A12)$$

$$\mathbf{G}_n = -\sigma_n\frac{B_n'}{B_n}$$

The solution to **Equation A11** can be written as[16]

$$\mathbf{C}(x) = \mathbf{\Phi}(x)\cdot\mathbf{C}(0) + \delta\cdot\mathbf{\Gamma}(x) \qquad (A13)$$

where the vector Γ is defined by

$$\mathbf{\Gamma}(x) = \mathbf{\Phi}(x)^x\int_0 \mathbf{\Phi}^{-1}(s)\mathbf{G}(s)ds \qquad (A14)$$

and the matrix $\Phi(x)$ has the property

$$\mathbf{\Phi}'(x) = \mathbf{F}(x)\cdot\mathbf{\Phi}(x) \qquad (A15)$$

Returning now to the blood concentrations at the outlet (x = L) we expand **Equation A13** to write

$$C_n(L) - \delta = \sum_m \mathbf{\Phi}_{nm}(L)(C_m(0) - \delta) + \delta\cdot\mathbf{\Gamma}_n(L) \qquad (A16)$$

Both Φ and Γ depend only upon the flow rates and the material constants k_n and σ_n and not upon the concentrations and can, in principle (although it is difficult in the general case), be calculated if the variations along the dialyzer are known. Using **Equation A5** at x = L, together with **Equation A16** and the fact that all the inlet blood concentrations are equal, we can now calculate δ [and thereby α as $\alpha = \delta(Q_d - Q_b + Q_{uf})$]

$$Q_dC_{di} = \sum_n B_n(L)\cdot\delta + \sum_n B_n(L)\sum_m \mathbf{\Phi}_{nm}(L)(C_{bi} - \delta) + \delta\cdot\sum_n B_n(L)\mathbf{\Gamma}_n(L) + \alpha = \delta(Q_b - Q_{uf}) + (C_{bi} - \delta)\Psi + \delta\Lambda + \delta(Q_d - Q_b + Q_{uf}) \qquad (A17)$$

In **Equation A17** we have introduced Ψ and Λ as

$$\Psi = \sum_n B_n(L)\sum_m \mathbf{\Phi}_{nm}(L)$$
$$\Lambda = \sum_n B_n(L)\mathbf{\Gamma}_n(L) \qquad (A18)$$

With α from **Equation A8 and A9** the mass transfer rate J is obtained from **Equation A6** as

$$J = Q_bC_{bi} - Q_dC_{di} + \delta(Q_d - Q_b + Q_{uf}) \qquad (A19)$$

Inserting δ from **Equation A17** and rearranging we get

$$J = K_bC_{bi} - K_dC_{di} \qquad (A20)$$

where

$$K_b = \frac{Q_b(Q_d + \Lambda) - \Psi(Q_d + Q_{uf})}{Q_d + \Lambda - \Psi}$$
$$K_d = \frac{Q_d(Q_b - Q_{uf} + \Lambda - \Psi)}{Q_d + \Lambda - \Psi} \qquad (A21)$$

To study the relationship between K_b and K_d we can analyze their difference

$$K_b - K_d = Q_{uf} - \frac{\Lambda(Q_d - Q_b + Q_{uf})}{Q_d + \Lambda - \Psi} \qquad (A22)$$

Unless $Q_d = Q_b - Q_{uf}$ (which would be very uncommon, but can be handled by continuity) we see that if and only if $\Lambda = 0$ we can write

$$J = K_b(C_{bi} - C_{di}) + Q_{uf}C_{di} \qquad (A23)$$

Note that for $C_{di} = 0$ the explicit dependence upon Q_{uf} disappears from Equation A23. However, Q_{uf} still affects the clearance K_b (through Ψ in **Equation A21**). One way to have $\Lambda = 0$ is that $\sigma_n = 0$ for all the fibers, that is, that the solute is small enough for a free convective passage through the membrane.

The Cocurrent Case

In this case, the dialysate flow is assumed to enter at the same end as the blood flow, that is, at x = 0. **Equations A1, A2, and A3** for the blood concentrations will be unchanged, but the sign of the dialysate flow D in the following equations will change, as will the sign of α because it is interpreted as the solute flow in the dialysate direction. The modified **Equations A5, A8, and A9** will read

$$DC_d = -\sum_n (B_n C_n) + \alpha \qquad \text{(A5')}$$

$$\beta = Q_d + Q_b \qquad \text{(A8')}$$

$$\delta = \frac{\alpha}{\beta} = \frac{\alpha}{D + \sum_n B_n} \qquad \text{(A9')}$$

The changed sign of D will also change the sign of all terms in **Equations A10 and A12** that contain D. The next difference is in the boundary conditions leading to **Equation A17**. In the cocurrent case we use **Equations A5', A8', and A9'** at x = 0 to get the much simpler expression

$$\delta(Q_d + Q_b) = Q_d C_{di} + Q_b C_{bi} \qquad \text{(A17')}$$

With the same definitions of Ψ and Λ as before, the mass transfer rate J will be the difference between inlet and outlet mass flow rates (using **Equation A16**)

$$J = Q_b C_{bi} - \sum_n B_n(L) C_n(L) = Q_b C_{bi} - \sum_n B_n(L) \cdot \delta -$$

$$\sum_n B_n(L) \sum_m \Phi_{nm}(L)(C_{bi} - \delta) - \delta \cdot \sum_n B_n(L) \Gamma_n(L) =$$

$$Q_b C_{bi} - \delta(Q_b - Q_{uf}) - (C_{bi} - \delta)\Psi - \delta\Lambda \qquad \text{(A19')}$$

Inserting δ from **Equation A17'** and rearranging we again get **Equation A20** where

$$K_b = \frac{Q_d(Q_b - \Psi) + Q_b(Q_{uf} - \Lambda)}{Q_b + Q_d} \qquad \text{(A21')}$$

$$K_d = \frac{Q_d(Q_b - Q_{uf} + \Lambda - \Psi)}{Q_b + Q_d}$$

To study the relationship between K_b and K_d we can analyze their difference

$$K_b - K_d = Q_{uf} - \Lambda \qquad \text{(A22')}$$

We see that, if and only if $\Lambda = 0$, we again get **Equation A23**.

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