# Gelfand-Zeitlin Actions on Classical Groups

Mark Colarusso
*University of Notre Dame*

## Introduction

**Setting:**

- Kostant and Wallach [KW1] construct an integrable system on $\mathfrak{gl}(n, \mathbb{C})$ using Gelfand-Zeitlin theory.
- Corresponding Hamiltonian vector fields are complete and integrate to an action of $\mathbb{C}^{n(n-1)/2}$ on $\mathfrak{gl}(n, \mathbb{C})$. Refer to this action as Gelfand-Zeitlin action.
- Orbits of Gelfand-Zeitlin action of dimension $\frac{n(n-1)}{2}$ form leaves of polarization of open, dense subvariety of a regular adjoint orbit.

**Sections:**

1. Describe all orbits of dimension $\frac{n(n-1)}{2}$ of the Gelfand-Zeitlin action.
2. Algebraically integrate Gelfand-Zeitlin system on covering spaces of decomposition classes.

## I: Orbit Structure of Gelfand-Zeitlin Action

### Lie-Poisson Structure

**Definition:**

A smooth variety $(X, \{\cdot,\cdot\})$ is a Poisson variety if $\{\cdot,\cdot\}$ makes the sheaf of functions $\mathcal{O}_X$ on $X$ into a sheaf of Poisson algebras.

If $\mathfrak{g}$ is a reductive finite dimensional Lie algebra, then $\mathfrak{g} \cong \mathfrak{g}^*$ is a Poisson variety with the Lie-Poisson structure.

A function $f \in \mathcal{O}_\mathfrak{g}$ defines a Hamiltonian vector field $\xi_f(g) = \{f, g\}$.

Let $G$ be the adjoint group of $\mathfrak{g}$.

**Fact:** The symplectic leaves of the Lie-Poisson structure are the adjoint orbits $G \cdot x$.

i.e. $G \cdot x$ is symplectic and its tangent space is spanned by Hamiltonian vector fields.

### Gelfand-Zeitlin Algebra

Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ with the Lie-Poisson structure. We use a Poisson analogue of the Gelfand-Zeitlin algebra to construct an integrable system on a regular adjoint orbit in $\mathfrak{g}$.

Let $\mathfrak{g}_i = \mathfrak{gl}(i, \mathbb{C})$, $G_i = GL(i, \mathbb{C})$.

$\mathfrak{g}_i$ is a subalgebra of $\mathfrak{g}$ by embedding

$$Y \hookrightarrow \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly, $G_i \hookrightarrow G$.

Poisson analogue of Gelfand-Zeitlin subalgebra:

$$J(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}_1]^{G_1} \otimes_\mathbb{C} \cdots \otimes_\mathbb{C} \mathbb{C}[\mathfrak{g}]^G.$$

### Gelfand-Zeitlin Integrable System

**Notation:** For $x \in \mathfrak{g}$, let $x_i \in \mathfrak{g}_i$ be the $i \times i$ upper left-hand corner of $x$.

$\mathbb{C}[\mathfrak{g}_i]^{G_i} = \mathbb{C}[f_{i,1}, \ldots, f_{i,i}]$, where $f_{i,j}(x) = tr(x_i^j)$.

**Define:** $J_{GZ} \subset J(\mathfrak{g})$,

$$J_{GZ} = \{f_{i,j} : 1 \leq i \leq n-1, 1 \leq j \leq i\}.$$

**Observe:**

$$|J_{GZ}| = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} = \frac{\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g})}{2},$$

is half the dimension of regular $G \cdot x$.

**Facts:**[KW1]

- $J(\mathfrak{g}) \subset \mathbb{C}[\mathfrak{g}]$ is Poisson commutative.
- The restriction of $J_{GZ}$ to a regular adjoint orbit is an integrable system.

**Fact:** An analogous Gelfand-Zeitlin integrable system exists for complex orthogonal Lie algebras $\mathfrak{so}(n, \mathbb{C})$ (see [Col2]).

### Gelfand-Zeitlin Actions

Let $\xi_{f_{i,j}}$ be the Hamiltonian vector field of $f_{i,j} \in J_{GZ}$.

Let

$$\mathfrak{a} = span\{\xi_{f_{i,j}} : 1 \leq i \leq n-1, 1 \leq j \leq i\}.$$

Kostant and Wallach prove:

**Key Theorem:** [KW1]

The Lie algebra $\mathfrak{a}$ is a commutative Lie algebra of dimension $\frac{n(n-1)}{2}$ and integrates to a global action of $\mathbb{C}^{\frac{n(n-1)}{2}}$ on $\mathfrak{g}$.

This action of $\mathbb{C}^{\frac{n(n-1)}{2}}$ on $\mathfrak{g}$ is sometimes referred to as the Gelfand-Zeitlin action.

**Notation:**

Following [KW1], we define $A := \mathbb{C}^{\frac{n(n-1)}{2}}$.

In [Col2] we prove analogous results for $\mathfrak{so}(n, \mathbb{C})$.

### Strongly Regular Elements

**Definition:** $x \in \mathfrak{g}$ is called *strongly regular* if

$$\dim(A \cdot x) = \frac{n(n-1)}{2}.$$

If $x \in \mathfrak{g}_{sreg}$, then $A \cdot x \subset G \cdot x$ is Lagrangian.

**Goal:** Describe all strongly regular $A$-orbits.

**Strategy:** Study the Kostant-Wallach map $\Phi : \mathfrak{g} \to \mathbb{C}^{\frac{n(n+1)}{2}}$,

$$\Phi(x) = (p_{1,1}(x), \ldots, p_{i,j}(x), \ldots, p_{n,n}(x)),$$

where $p_{i,j}$ is the coefficient of $t^{j-1}$ in the characteristic polynomial of $x_i$.

**Notation:** Let $\sigma_i(x_i)$ be the collection of eigenvalues of $x_i \in \mathfrak{g}_i$ counted with multiplicity.

**Observe:** $\Phi(x) = \Phi(y)$ if and only if $\sigma_i(x_i) = \sigma_i(y_i)$ for all $i$.

### Results on $A$-Orbit Structure

**Key Theorem:** [Col1]

1. Let $c \in \mathbb{C}^{\frac{n(n+1)}{2}}$ be such that for $x \in \Phi^{-1}(c)$, $|\sigma_i(x_i) \cap \sigma_{i+1}(x_{i+1})| = j_i$ for $1 \leq i \leq n-1$. Then there are $2^{\sum_{i=1}^{n-1} j_i}$ strongly regular $A$-orbits in $\Phi^{-1}(c)$.
2. Let $x \in \Phi^{-1}(c)$ be strongly regular and let $Z_i$ denote the centralizer of the Jordan form of $x_i$ in $G_i$.
   Then $Z_1 \times \cdots \times Z_{n-1}$ acts freely and algebraically on the variety of strongly regular elements $\Phi^{-1}(c)$ and its orbits coincide with the $A$-orbits in (1).

**Remark:** A similar result was reached by Bielawski and Pidstrygach in [BP].

In [Col2] we prove an analogous result for elements $x \in \mathfrak{so}(n, \mathbb{C})$ where $x_i$ is regular semisimple and $j_i = 0$ for all $i$.

## II: Algebraic Integrability of Gelfand-Zeitlin Fields

### Decomposition Classes and the $A$-action

Let $\mathfrak{l}_i \subset \mathfrak{g}_i$ be a Levi subalgebra, let $\mathfrak{z}_i$ be the centre of $\mathfrak{l}_i$, and let $u_i \in \mathfrak{l}_i$ be principal nilpotent.

Denote by $\mathfrak{z}_{i,gen} = \{z \in \mathfrak{z}_i : \mathfrak{z}_{\mathfrak{g}_i}(z) = \mathfrak{l}_i\}$.

**Definition:** The variety

$$D_i = G_i \cdot (\mathfrak{z}_{i,gen} + u_i) \subset \mathfrak{g}_i$$

is called a *regular decomposition class*.

Let $D_i \subset \mathfrak{g}_i$ be a regular decomposition class, $1 \leq i \leq n-1$.

**Define:**

$$X_\mathcal{D} := \{x : x \text{ is strongly regular}, x_i \in D_i \text{ for all } i\},$$

**Fact:** $X_\mathcal{D}$ is $A$-invariant.

**Goal:** To realize the action of $A$ as the action of an algebraic group on a covering space of $X_\mathcal{D}$.

### Covering Space

Let $\mathfrak{z}_\mathcal{D} := \mathfrak{z}_{1,gen} \oplus \cdots \oplus \mathfrak{z}_{n,gen}$.

**Define:** $\hat{\mathfrak{g}}_\mathcal{D} \subset X_\mathcal{D} \times \mathfrak{z}_\mathcal{D}$,

$$\hat{\mathfrak{g}}_\mathcal{D} = \{(x, (z_1, \ldots, z_n)) : x_i \in G_i \cdot (z_i + u_i)\}.$$

Have projections

$$\mu : \hat{\mathfrak{g}}_\mathcal{D} \to X_\mathcal{D},\ \kappa : \hat{\mathfrak{g}}_\mathcal{D} \to \mathfrak{z}_\mathcal{D}.$$

**Proposition:** [CE]

1. $\hat{\mathfrak{g}}_\mathcal{D}$ is smooth and $\mu : \hat{\mathfrak{g}}_\mathcal{D} \to X_\mathcal{D}$ is an étale covering.
2. Moreover, $\hat{\mathfrak{g}}_\mathcal{D}$ is a subvariety of a Poisson variety $\dot{\mathfrak{g}}_\mathcal{D}$.

### Algebraic Integrability

Let $Z_{D_i} = Z_{L_i}(u_i)$ be the centralizer of $u_i$ in $L_i$.

Define $Z_\mathcal{D} = Z_{D_1} \times \cdots \times Z_{D_{n-1}}$.

**Key Theorem:** [CE]

1. There exists a Lie algebra $\hat{\mathfrak{a}}$ of Hamiltonian vector fields on $\dot{\mathfrak{g}}_\mathcal{D}$ of dimension $\frac{n(n-1)}{2}$, which integrates to a free, algebraic action of $Z_\mathcal{D}$ on $\hat{\mathfrak{g}}_\mathcal{D}$. The $Z_\mathcal{D}$-action lifts the $A$-action on $X_\mathcal{D}$ to $\hat{\mathfrak{g}}_\mathcal{D}$.
2. The action of $Z_\mathcal{D}$ preserves the fibres $\kappa^{-1}(z_1, \ldots, z_n)$. If $j_i = |\sigma_i(z_i) \cap \sigma_{i+1}(z_{i+1})|$, then there are $2^{\sum_{i=1}^{n-1} j_i}$ $Z_\mathcal{D}$-orbits in $\kappa^{-1}(z_1, \ldots, z_n)$.

In the case where each $D_i$ consists of regular semisimple elements, this results generalizes results in [KW2].

### References

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Contact information: Mark Colarusso, 742 S 32nd St. South Bend, IN, 46615 Email: *mcolarus@nd.edu*; Web: *http://www.nd.edu/∼mcolarus*

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