Ideals in parabolic subalgebras of simple Lie algebras

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Notation, Definitions and Preliminary Results

- *For a finite dimensional simple Lie algebra $\mathfrak{g}$ of rank n with fixed Cartan subalgebra $\mathfrak{h}$, we use the standard partial ordering $\leq$ on the set of roots R: $\alpha \leq \beta \iff \beta - \alpha \in Q^+$.*
- *Define $d_i : Q \longrightarrow \mathbb{Z}$ by $\eta = \sum_{i=1}^{n} d_i(\eta)\,\alpha_i$, and let $J \subset [n]$.*
- *Set $R(J) = \{\alpha \in R : d_i(\alpha) = 0 \text{ if } i \notin J\}$, $R^+(J) = R(J) \cap R^+$.*
- *A subset Φ of R^+ is a J-**ideal** if $\Phi \cap R^+(J) = \emptyset$ and*

$$\alpha \in \Phi, \beta \in R^+ \cup R(J), \beta + \alpha \in R \Rightarrow \beta + \alpha \in \Phi.$$

- *The set of J-ideals in R^+ is in one-to-one correspondence with the set of ad-nilpotent ideals of the parabolic subalgebra*

$$\mathfrak{p}_J = \mathfrak{h} \oplus_{\alpha \in R^+} \mathfrak{g}_\alpha \oplus_{\alpha \in R^+(J)} \mathfrak{g}_{-\alpha},$$

where $\Phi \mapsto \oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$.

- *$A \subset R^+$ is a J-**antichain** if $A \cap R^+(J) = \emptyset$ and, for all $\alpha, \beta \in A$ and $j \in J$, we have $\alpha \nleq \beta, \beta \nleq \alpha$ and $\alpha - \alpha_j \notin R$.*
- *There is a one-to-one correspondence between J-antichains of R^+ and J-ideals of R^+:*

$$A \mapsto \Phi(A) = \bigcup_{\alpha \in A} \left\{\beta \in R^+ : \beta \geq \alpha\right\}.$$

- *So by enumerating all J-antichains for a fixed J, we enumerate all ad-nilpotent ideals for a fixed $\mathfrak{p}_J$.*
- *An ideal Φ is of **nilpotence** k if for any $\beta_1, \cdots, \beta_{k+1} \in \Phi$ (not necessarily distinct), $\sum_{s=1}^{k+1} \beta_s \notin R$. Φ is **abelian** if it is of nilpotence 1.*
- ***Theorem 1** Let A be an antichain in R^+, θ the highest root of $\mathfrak{g}$. Then $\Phi(A)$ is a k-nilpotent ideal if and only if for any $\beta_1, \cdots, \beta_{k+1} \in A$ (not necessarily distinct), $\sum_{s=1}^{k+1} \beta_s \nleq \theta$.*
- ***Proposition 1** Let $J \subset [n]$. A J-antichain A is abelian if and only if the following hold:*
  1. *for all $\alpha \in A$, there exists $i \in J$ such that $2d_i(\alpha) > d_i(\theta)$.*
  2. *for all $\alpha, \beta \in A$ with $\alpha \neq \beta$, there exists $i \in J$ such that $d_i(\alpha) + d_i(\beta) > d_i(\theta)$; in particular, $d_i(\alpha) \neq 0, d_i(\beta) \neq 0$.*

Counting for A_n

Some notation:

- *$\mathbf{A}_{s,J}$ is the set of abelian J-antichains with s elements ($\mathbf{A}_{0,J} = \{\emptyset\}$).*
- *For $i, j \in [n]$, $i \leq j$, $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$ ($\alpha_{i,i} = \alpha_i$).*
- *For calculational purposes, $\binom{n}{k} = 0$ when $k < 0$ or $k > n$.*

For A_n, $R^+ = \left\{\alpha_{i,j} : 1 \leq i \leq j \leq n\right\}$ and $\theta = \alpha_{1,n}$. Let $A \subset R^+$, and let $J \subset [n]$. To have $A \in \mathbf{A}_{s,J}$, we need to see how the elements of A behave with respect to the definition of a J-antichain, as well as conditions (1) and (2) of the Proposition.

- *J-antichain: Consider two roots $\alpha_{i,j}, \alpha_{k,l} \in A, \alpha_{i,j} \neq \alpha_{k,l}$. If $i = k$, then $\alpha_{i,\min(j,l)} \leq \alpha_{i,\max(j,l)}$, a contradiction. So assume without loss of generality that $i < k$. Then $j < l$, since otherwise $\alpha_{k,l} \leq \alpha_{i,j}$. Now let $\alpha_{k,l} \in A, j \in J$. We need $\alpha_{k,l} - \alpha_{j,j} \notin R$. If j is not between k and l, then $\alpha_{k,l} - \alpha_{j,j}$ is a $\mathbb{Z}$-linear combination of simple roots with both positive and negative coefficients, which is not a root. If $k < j < l$, then $\alpha_{k,l} - \alpha_{j,j} = \alpha_{k,j-1} + \alpha_{j+1,l}$, which is not a root. So the only time $\alpha_{k,l} - \alpha_{j,j}$ is a root is if $j = k$ or $j = l$. Thus, if $\alpha_{k,l} \in A$, then $k, l \notin J$.*
- *Condition 1: Since $2d_i(\alpha_{i,j}) > d_i(\theta)$ for any $\alpha_{i,j} \in R^+$, there are no roots which are excluded a priori from A by this condition.*
- *Condition 2: From above, we already know that if $\alpha_{i,j}, \alpha_{k,l} \in A, \alpha_{i,j} \neq \alpha_{k,l}$, then $i < j$ and $k < l$. If $j < k$, then $d_i(\alpha_{i,j}) + d_i(\alpha_{k,l}) \leq d_i(\theta)$ for all i, contradicting condition 2. So $i < k \leq j < l$.*

From this information, we can now construct a general abelian J-antichain of size s:

$$A = \left\{\alpha_{i_k, j_k} : 1 \leq k \leq s;\ i_k, j_k \in [n]\backslash J;\ i_1 < i_2 < \cdots < i_s \leq j_1 < j_2 < \cdots < j_s\right\}.$$

Given this description, $\mathbf{A}_{s,J}$ breaks into 2 cases: $i_s < j_1$, and $i_s = j_1$. These correspond to subsets of $[n]\backslash J$ of size $2s$ in the first case (2 endpoints for each of the s elements of A), and $2s - 1$ in the second (one less than in case 1 since $i_s = j_1$). Thus

$$\#\mathbf{A}_{s,J} = \binom{n - \#J}{2s - 1} + \binom{n - \#J}{2s},$$

and the number of abelian J-antichains is

$$\sum_s \#\mathbf{A}_{s,J} = \sum_s \left(\binom{n-\#J}{2s-1} + \binom{n-\#J}{2s}\right) = \sum_p \binom{n-\#J}{p} = 2^{n-\#J}.$$

Results for the other simple algebras

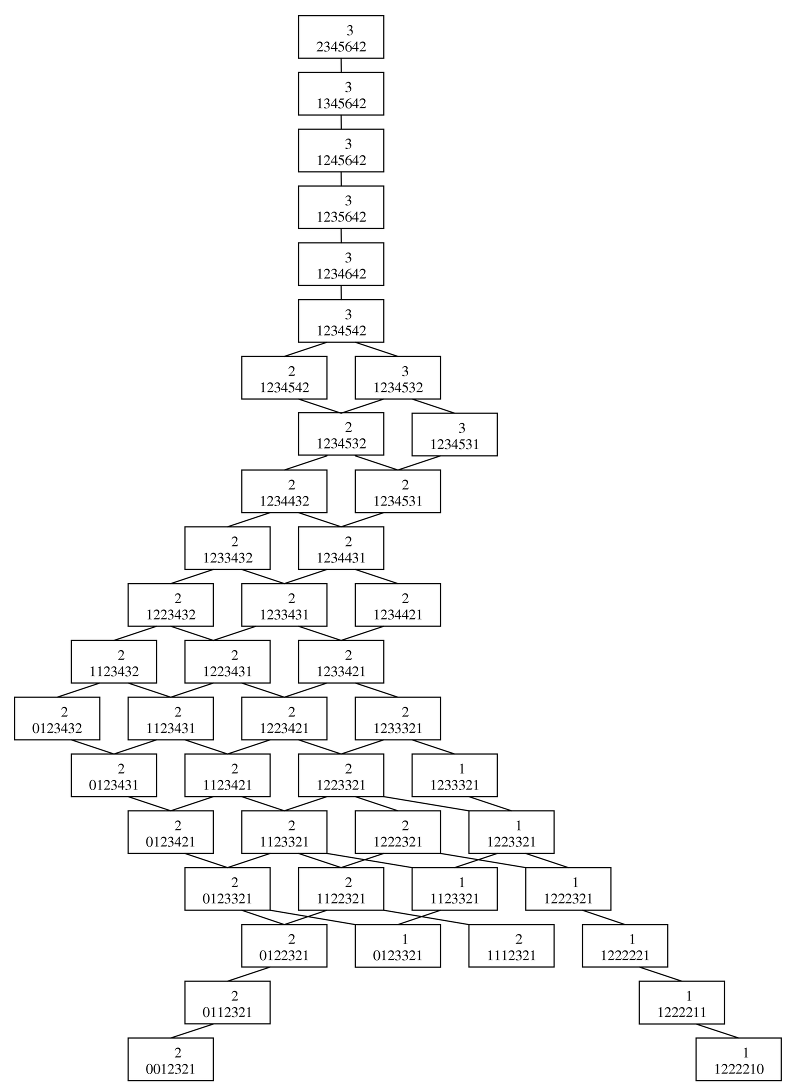
- For the other classical Lie algebras, similar arguments to those used for A_n allow us to describe all abelian J-antichains.
- Of the classical Lie algebras, only C_n admits a nice, simple formula for the total number ($2^{n-\#J}$).
- In both classical and exceptional cases, we use this combinatorial approach for the case when $J = \emptyset$ (i.e. counting abelian ideals for the Borel subalgebra $\mathfrak{p}_\emptyset = \mathfrak{b} = \mathfrak{h} \oplus_{\alpha \in R^+} \mathfrak{g}_\alpha$) to recover Peterson's result of having 2^n abelian ideals for $\mathfrak{b}$. This is obvious for A_n and C_n from the closed formulas, since $\#J = 0$.
- For abelian antichains in the case of the exceptional Lie algebras, one can simply draw the poset of roots, eliminating those roots that don't meet condition 1 of the Proposition, and count the antichains. The E_8 poset is to the right.

Current Research

- RJ: generalizing the results outlined in this poster to $\mathbb{Z}_2$-graded Lie algebras; and an inductive approach to Panyushev's antichain dualization algorithm that generalizes to D_n.
- Tim: expanding on results regarding specific antichains A which correspond to infinite-dimensional associative Koszul algebras whose global dimension is $\#\Phi(A)$.

The E_8 poset

The poset of roots of E_8 admissible by condition 1 of the Proposition:



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