Natural questions concerning Lie algebras of quotients

Juana Sánchez Ortega

Dpto. Álgebra, Geometría y Topología. Facultad de Ciencias.
Universidad de Málaga. 29071 Málaga.
jsanchez@agt.cie.uma.es

These definitions are consistent with the non-graded ones (see [6, Definitions 2.1 and 2.5]) in the sense that they coincide when considering the trivial grading.

A useful tool will provide with examples of graded algebras of quotients (See EXAMPLE 1 and [5, Theorem 2.9 and Corollary 2.10])

Inspired by Utumi's construction [7] and making use an idea of C. Martínez [3], M. Siles Molina built in [6] the maximal algebra of quotients for every semiprime Lie algebra. We follow here her construction.

1 Graded quotients

GRADED QUOTIENTS

L = graded subalgebra of a graded Lie algebra **Q**

Q is a **graded algebra of quotients** of **L**

$q_\tau \in Q$, ${}_L(q_\tau)$ = linear span in Q of q_τ and $\mathrm{ad}\, x_1 \ldots \mathrm{ad}\, x_n q_\tau$, with $n \in \mathbb{N}$, $x_1, \ldots, x_n \in L$

$\forall\, p_\sigma, q_\tau \in Q$, $p_\sigma \neq 0$, $\exists\, x_\alpha \in L : [x_\alpha, p_\sigma] \neq 0$, $[x_\alpha, {}_L(q_\tau)] \subseteq L$

Q is a **graded weak algebra of quotients** of **L**

$\forall\, 0 \neq p_\sigma \in Q_\sigma, \exists\, x_\alpha \in L : 0 \neq [x_\alpha, p_\sigma] \in L$

BEING A GRADED ALGEBRA OF QUOTIENTS ⇄ BEING A GRADED WEAK ALGEBRA OF QUOTIENTS (See [5, Remark 2.3])

As it happened in the non-graded case some **PROPERTIES** of a graded Lie algebra are **INHERITED** by each of its algebras of quotients (See [6, Lemma 2.11])

RELATIONSHIP BETWEEN GRADED (WEAK) ALGEBRAS OF QUOTIENTS AND (WEAK) ALGEBRAS OF QUOTIENTS

Proposition. *Let $L \subseteq Q$ be graded Lie algebras. Consider the following conditions:*

(i) *Q is an algebra of quotients of L.*

(ii) *Q is a graded algebra of quotients of L.*

Then (i) implies (ii). Moreover, if L is graded semiprime then (ii) implies (i).

Lemma. *Let $L \subseteq Q$ be graded Lie algebras. If Q is a weak algebra of quotients of L then Q is also a graded weak algebra of quotients of L.*

EXAMPLE 1 R = *- prime associative pair with involution

Q(R) = associative Martindale pair of symmetric quotients

Then **TKK(H(Q(R),*)** is a **3-graded algebra of quotients of TKK(H(R,*)),** where H(. , *) is the set consisting of all symmetric elements.

MAXIMAL GRADED QUOTIENTS

Suppose **L** is G-graded and take **I** a graded ideal of L

A derivation $\delta : I \to L$ has *degree* $\sigma \in G$ if it satisfies $\delta(I_\tau) \subseteq L_{\tau\sigma}$ $(\forall \tau \in G)$. In this case, δ is called a *graded derivation of degree* σ.

$\mathrm{Der}_{gr}(I, L)_\sigma$ = the set of all graded derivations of degree σ

It is a Φ-module with the natural operations and hence

$\mathrm{Der}_{gr}(I, L) := \oplus_{\sigma \in G} \mathrm{Der}_{gr}(I, L)_\sigma$ is also a Φ-module.

$\mathcal{I}_{gr-e}(L)$ = the set of all graded essential ideals of L

For L graded semiprime, the direct limit

$$Q_{gr-m}(L) := \varinjlim_{I \in \mathcal{I}_{gr-e}(L)} \mathrm{Der}_{gr}(I, L)$$

is a graded algebra of quotients of L containing L as a graded subalgebra via the following graded Lie monomorphism:

$$\varphi : \begin{array}{ccc} L & \to & Q_{gr-m}(L) \\ x & \mapsto & (\mathrm{ad}\, x)_L \end{array}$$

Moreover, it is maximal among the graded algebras of quotients of L and is called the ***MAXIMAL GRADED ALGEBRA OF QUOTIENTS OF* L.** This notion extends that of maximal algebra of quotients given in [6].

NOTE !!

THE MAXIMAL ALGEBRA OF QUOTIENTS OF A 3-GRADED LIE ALGEBRA IS 3-GRADED TOO AND COINCIDES WITH ITS MAXIMAL GRADED ALGEBRA OF QUOTIENTS (See [5, Theorem 3.2])

2 Jordan pairs of quotients and 3-graded Lie quotients

FIRST TARGET: Analyze the relationship between notion of Jordan pairs of quotients in the sense of [2 ,2.5] and 3-graded Lie quotients, via the TKK-construction

Theorem. *Let V be a semiprime subpair of a Jordan pair W. Then the following conditions are equivalent:*

(i) *W is a pair of $\mathfrak{M}$-quotients of V.*

(ii) *$\mathrm{TKK}(W)$ is an algebra of quotients of $\mathrm{TKK}(V)$.*

(i) implies (ii) was proved in [2, Theorem 2.10]

NEXT OBJECTIVE: Analyze the relationship between:
- Maximal Jordan pairs of $\mathcal{M}$-quotients (See [2])
- Maximal 3-graded Lie quotients

$Q_m(V)$ = maximal pair $\mathcal{M}$-quotients of V

Theorem. *Assume that $\frac{1}{6} \in \Phi$.*

(i) *Let V be a strongly nondegenerate Jordan pair. Then*

$$Q_m(V) = \Big(\big(Q_m(\mathrm{TKK}(V))\big)_1, \big(Q_m(\mathrm{TKK}(V))\big)_{-1} \Big)$$

is the maximal Jordan pair of $\mathfrak{M}$-quotients of V.

(ii) *If $L = L_{-1} \oplus L_0 \oplus L_1$ is a strongly non-degenerate Jordan 3-graded Lie algebra satisfying that $Q_m(L)$ is Jordan 3-graded, then*

$$Q_m(L) \cong Q_m(\mathrm{TKK}(V)) \cong \mathrm{TKK}(Q_m(V)),$$

where $V = (L_1, L_{-1})$ is the associated Jordan pair of L.

Is the CONDITION on L NECESSARY in the theorem above

YES !!

EXAMPLE 2

$\mathbb{M}_\infty(\mathbb{R}) = \cup_{n=1}^{\infty} \mathbb{M}_n(\mathbb{R})$ algebra of infinite matrices with a finite number of nonzero entries

$L := \mathfrak{sl}_\infty(\mathbb{R}) = \{x \in \mathbb{M}_\infty(\mathbb{R}) \mid \mathrm{tr}(x) = 0\}$

simple Lie algebra of countable dimension

L is a Jordan 3-graded Lie algebra by doing L_{-1} = eLf, L_1 = fLe and L_0 = { eXe + fXf : X∈ L}, where e:=e_{11} and f:= diag(0, 1, …)

$Q_m(L) \cong \mathrm{Der}(L)$ is 3-graded but it is NOT JORDAN 3-GRADED since ad e ∈ $\mathrm{Der}(L)_0$ and ad e ∉ $[\mathrm{Der}(L)_{-1}, \mathrm{Der}(L)_1]$

EXAMPLE 3

$L := \mathfrak{sl}_2(F) = \{x \in \mathbb{M}_2(F) \mid \mathrm{tr}(x) = 0\}$ is a Jordan 3-graded Lie algebra with the grading L_{-1}= Fe_{21}, L_1 = Fe_{12} and L_0= $F(e_{11}- e_{22})$

The semisimplicity of L implies that $Q_m(L) \cong L$

There are algebras satisfying this CONDITION

3 The maximal Lie algebra of quotients of an essential ideal

QUESTION !!

Is $Q_m(I)=Q_m(L)$ for every essential ideal I of L?

This question only makes sense if we assume that **I** itself is a **semiprime Lie algebra.** Similar questions have been studied also in the associative context (see e.g. [1, Proposition 2.1.10])

Theorem. *Let I be an essential ideal of an strongly semiprime Lie algebra L. Then $Q_m(I) \cong Q_m(L)$.*

Corollary. *Let A be a semiprime algebra. Then:*

$$Q_m([A, A]/Z_{[A,A]}) \cong Q_m(A^-/Z).$$

Corollary. *Let A be a prime algebra. If $\mathrm{Der}(A)$ is strongly prime then*

$$Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A)).$$

When is Der (A) STRONGLY PRIME

Theorem. *Let A be a prime algebra. Then the following conditions are equivalent:*

(i) *$\mathrm{Der}(A)$ is strongly prime.*

(i) *Every nonzero ideal of A contains a nonzero ideal of A invariant under every element of $\mathrm{Der}(A)$.*

Moreover, if these conditions hold, then

$$Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A)).$$

Corollary. *Let A be a simple algebra. Then*

$$Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A)).$$

4 The maximal Lie algebra of quotients of A^-/Z

GOAL !!

Give a description of the maximal algebra of quotients **Q_m(A^-/Z)**, where **A** is a **prime associative** algebra.

To this end, we introduce a new Lie algebra!

CONSTRUCTION

Two pairs (δ, I), (μ, J) where I, J are essential ideals of A and $\delta : I \to A$, $\mu : J \to A$ are derivations are *equivalent* if $\delta = \mu$ on some essential ideal of A contained in $I \cap J$.

This is an equivalence relation.

$\mathrm{Der_m}(A)$ = the set of all equivalence classes. One can prove that it is a **Lie algebra.**

If A is prime, then: $\mathrm{Der}(A) \subseteq \mathrm{Der_m}(A) \subseteq \mathrm{Der}(Q_s(A))$

IT SATISFIES

Theorem. *Let A be a prime algebra such that either $\deg(A) \neq 3$ or $\mathrm{char}(A) \neq 3$. Then, $Der(A) \cong Q_m(A^-/Z)$. Moreover, the map $\varphi : \mathrm{Der_m}(A) \to Q_m(A^-/Z)$ defined by $\delta_I \mapsto \bar{\delta}_{\bar{I}}$, where $\bar{\delta} : \bar{I} \to A^-/Z$ maps $\bar{y}$ into $\overline{\delta(y)}$, is an isomorphism of Lie algebras.*

CONSEQUENCES Let A be a prime algebra such that either deg (A)≠3 or char (A) ≠3

1. *If $A = Q_s(A)$, then $Q_m(A^-/Z) \cong \mathrm{Der}(A)$.*
2. *If A is simple, then $Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A)) \cong \mathrm{Der}(A)$.*
3. *If A is affine and satisfies $Q_s(A) = AZ^{-1}$, then $Q_m(A^-/Z) \cong \mathrm{Der}(Q_s(A))$.*

NOTE !!

The case of the Lie algebra **K/Z_K** that arises from an associative algebra with involution * is analogous to the case of A^-/Z. The only difference is that we have to deal only with derivations δ preserving * (in the sense $\delta(x^*) = \delta(x)^*$).

5 Max-closed algebras

REMARK. This question makes sense since $Q_m(L)$ is semiprime (see [6, Proposition 2.7 (ii)])

WHEN IS TAKING THE MAXIMAL ALGEBRA OF QUOTIENTS A CLOSURE OPERATION?
Is **$Q_m(Q_m(L))=Q_m(L)$** for every semiprime Lie algebra?

Let L be a **SEMIPRIME** Lie algebra. We say that **L** is ***MAX-CLOSED*** if it satisfies **$Q_m(Q_m(L))=Q_m(L)$**

EXAMPLES of max-closed algebras

Let **L** be a **simple** Lie algebra. Then **Q_m(L)≅Der(L)** is an strongly prime Lie algebra and L is max-closed.

Let **A** be a **prime** algebra such that either deg(A)≠3 or char(A)≠3. Then **A^-/Z** is max-closed.

Let **A** be a **prime affine PI** algebra such that either deg(A)≠3 or char(A)≠3, and let **J** be a **noncentral Lie ideal** of A. Then the Lie algebra **J/(J∩Z)** is max-closed.

QUESTION !!

IS EVERY LIE ALGEBRA MAX-CLOSED?

NO !!

This A we shall deal with is the one that Passman used in [4] to show that Q_s(.) is not a closure operation.

EXAMPLE of a Lie algebra which is not max-closed

Let K be a field and set $A := K[t][x, y \mid xy = tyx]$. Then we have:

(i) A is a domain with center $Z = K[t]$.

(ii) $Q_s(A) = K(t)[x, y \mid xy = tyx]$.

(iii) $Q_s(Q_s(A)) = K(t)[x^{-1}, x, y^{-1}, y \mid xy = tyx]$.

Theorem. *Let K be a field and $A = K[t][x, y \mid xy = tyx]$. Then:*

(i) $\mathrm{Der}(Q_s(A)) \subsetneq Q_m(\mathrm{Der}(Q_s(A)))$.

(ii) The algebra $L = A^-/Z$ is not max-closed.

References

[1] K. I. BEIDAR, W. S. MARTINDALE III, A. V. MIKHALEV, *Rings with generalized identities*, Marcel Dekker, New York, 1996.

[2] E. GARCÍA, M. GÓMEZ LOZANO, Jordan systems of Martindale-like quotients, *J. Pure Appl. Algebra* 194 (2004), 127-145.

[3] C. MARTÍNEZ, The Ring of Franctions of a Jordan Algebra, *J. Algebra* 237 (2001), 798-812.

[4] D. S. PASSMAN, Computing the symmetric ring of quotients, *J. Algebra* 105 (1987), 207-235.

[5] J. S. O., M. SILES MOLINA, Algebras of quotients of graded Lie algebras (Preprint).

[6] M. SILES MOLINA, Algebras of quotients of Lie algebras, *J. Pure and Applied Algebra* 188 (2004), 175-188.

[7] Y. UTUMI, On quotients rings, *Osaka J. Math.* 8 (1956), 1-18.

Jointly presented by

Matej Brešar
Departament of Mathematics.
University of Maribor
FNM, Koroška 160. 2000 Maribor. Slovenia.
bresar@uni-mb.si

Francesc Perera
Departament de Matemàtiques.
Universitat Autònoma de Barcelona.
08193 Bellaterra, Barcelona. Spain.
perera@mat.uab.cat

Mercedes Siles Molina
Dpto. Algebra, Geometría y Topología.
Facultad de Ciencias.
Universidad de Málaga. 29071 Málaga. Spain.
mercedes@agt.cie.uma.es

Summer School and Conference in Geometric Representation Theory and Extended Affine Lie Algebras

University of Ottawa, Ontario, Canada
Summer School: June 15-27, 2009
Conference: June 29 - July 3, 2009