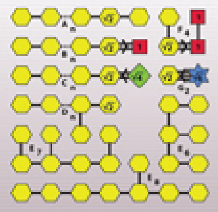
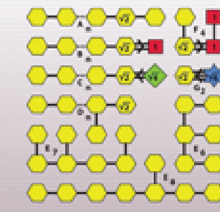


Combinatorial bases of Feigin-Stoyanovsky's type subspaces of standard modules for $D_4^{(1)}$



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Abstract

Let $\tilde{\mathfrak{g}}$ be an affine Lie algebra of type $D_4^{(1)}$ and $L(\Lambda)$ its standard module with a highest weight vector v_Λ. For a given $\mathbb{Z}$-gradation $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1$, we define Feigin-Stoyanovsky's type subspace as

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda.$$

Following the ideas from [G], we reduce the Ponicaré-Brikhoff-Witt spanning set of $W(\Lambda)$ to a basis and prove its linear independence by using Dong-Lepowsky intertwining operators (cf. [T]).

Setting

- $\mathfrak{g}$ simple Lie algebra of type D_ℓ, $\mathfrak{h}$ Cartan subalgebra
- R corresponding root system, $\alpha_1, \ldots, \alpha_\ell$ fixed simple roots, x_α, $\alpha \in R$ fixed root vectors
- $Q = Q(R)$ and $P = P(R)$ root and weight lattices
- $\omega_1, \ldots, \omega_\ell$ fundamental weights (with $\omega_0 = 0$)

We define the affine Lie algebra $\tilde{\mathfrak{g}}$ associated to $\mathfrak{g}$:

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

c being the canonical central element and d the degree operator, with Lie product given in the usual way (cf. [K]). Let $x(n) = x \otimes t^n$ for $x \in \mathfrak{g}$, $n \in \mathbb{Z}$ and set $x(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}$. Denote by $\Lambda_0, \ldots, \Lambda_\ell$ the corresponding fundamental weights of $\tilde{\mathfrak{g}}$.

For given integral dominant weight $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell$, denote by $L(\Lambda)$ the standard $\tilde{\mathfrak{g}}$-module with highest weight Λ, and let v_Λ be a highest weight vector of $L(\Lambda)$. Then $k = \Lambda(c) = k_0 + k_1 + 2k_2 + \cdots + 2k_{\ell-2} + k_{\ell-1} + k_\ell$ is the level of $L(\Lambda)$.

Definition of Feigin-Stoyanovsky's type subspaces

For fixed minuscule weight $\omega = \omega_1$ we define (cf. [FS], [P])

$$\Gamma = \{\alpha \in R \mid \langle \alpha, \omega \rangle = 1\}$$
$$= \{\gamma_2, \ldots, \gamma_\ell, \gamma_{\underline{\ell}}, \ldots, \gamma_{\underline{2}} \mid \gamma_i = \epsilon_1 + \epsilon_i,\ \gamma_{\underline{i}} = \epsilon_1 - \epsilon_i,\ i = 2, \ldots, \ell\}.$$

This gives us a $\mathbb{Z}$-grading:

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1,$$

with $\mathfrak{g}_0 = \mathfrak{h} + \sum_{\langle \alpha, \omega \rangle = 0} \mathfrak{g}_\alpha$, $\mathfrak{g}_{\pm 1} = \sum_{\alpha \in \pm\Gamma} \mathfrak{g}_\alpha$, and the corresponding $\mathbb{Z}$-grading

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1.$$

Then

$$\tilde{\mathfrak{g}}_1 = \operatorname{span}\{x_\gamma(n) \mid \gamma \in \Gamma, n \in \mathbb{Z}\}$$

is a commutative subalgebra and a $\tilde{\mathfrak{g}}_0$-module.

Definition 1 *For a standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$, Feigin-Stoyanovsky's type subspace of $L(\Lambda)$ is*

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda, \tag{1}$$

where $U(\tilde{\mathfrak{g}}_1)$ is the universal enveloping algebra of $\tilde{\mathfrak{g}}_1$.

By Poincaré-Birkhoff-Witt theorem, the spanning set of $W(\Lambda)$ consists of monomial vectors

$$\{x_{\delta_1}(-n_1) x_{\delta_2}(-n_2) \ldots x_{\delta_r}(-n_r) v_\Lambda \mid r \in \mathbb{Z}_+,\ n_i \in \mathbb{N},\ \delta_i \in \Gamma\}. \tag{2}$$

Elements of this spanning set can be identified with the monomials from $U(\tilde{\mathfrak{g}}_1) \cong S(\tilde{\mathfrak{g}}_1)$ so we refer to the elements of the form $x_\delta(-j)$, $\delta \in \Gamma$, $n \in \mathbb{N}$, as the variables, elements or factors of the monomial. To reduce the spanning set (2) of $W(\Lambda)$ to a basis, we establish a linear order on the set of monomials. This order has na importan property, it is namely compatible with multiplication in $U(\tilde{\mathfrak{g}}_1)$. We also define that $x(\pi)$ is a submonomial of $x(\mu)$ ($x(\pi) \prec x(\mu)$) if $x(\mu) = x(\pi')x(\pi)$.

Vertex operator construction of level one modules

We use the well known Frenkel-Kac-Segal vertex operator algebra construction of the standard $\tilde{\mathfrak{g}}$-modules of level 1, see [FK], [S]. We use the notation from [LL] and the details can be found in [FLM], [DL] or [LL].

Let $M(1)$ denote the Fock space for the homogeneous Heisenberg subalgebra and let $\mathbb{C}[P]$ be the group algebra of the weight lattice P with a basis e^λ, $\lambda \in P$. Set $V_Q = M(1) \otimes \mathbb{C}[Q]$ and $V_P = M(1) \otimes \mathbb{C}[P]$. The action of Heisenberg subalgebra on the tensor product V_P extends to the action of Lie algebra $\tilde{\mathfrak{g}}$ and then, as a $\tilde{\mathfrak{g}}$-module

$$M(1) \otimes \mathbb{C}[P] = L(\Lambda_0) + L(\Lambda_1) + L(\Lambda_{\ell-1}) + L(\Lambda_\ell)$$

with highest weight vectors $v_{\Lambda_0} = e^{\omega_0} = \mathbf{1}$, $v_{\Lambda_1} = e^{\omega_1}$, $v_{\Lambda_{\ell-1}} = e^{\omega_{\ell-1}}$ and $v_{\Lambda_\ell} = e^{\omega_\ell}$ respectively.

We shall use the intertwining operators:

$$\mathcal{Y} : V_P \longrightarrow (\operatorname{End} V_P)\{z\},$$

$$v \mapsto \mathcal{Y}(v, z),$$

defined for $v = v^* \otimes e^\mu \in V_P$ as:

$$\mathcal{Y}(v, z) = Y(v, z) e^{i\pi\mu} c(\cdot, \mu). \tag{3}$$

Operators $Y(e^\lambda, z_1)$ and $\mathcal{Y}(e^\mu, z_2)$, $\mu, \lambda \in P$, satisfy the ordinary Jacobi identity and restrictions of $\mathcal{Y}(e^\mu, z)$ are in fact maps

$$\mathcal{Y}(e^\mu, z) : L(\Lambda_i) \longrightarrow L(\Lambda_j) \tag{4}$$

if $\mu + \omega_i \equiv \omega_j \bmod Q$. These restrictions give us the intertwining operators between standard modules of level 1 (cf. [DL]) which commute with the action of $\tilde{\mathfrak{g}}_1$ if and only if $\langle \gamma, \mu \rangle \geq 0$ for all $\gamma \in \Gamma$.

Simple current operator

For $\lambda \in P$ we denote by e^λ the multiplication operator $1 \otimes e^\lambda$ on $V_P = M(1) \otimes \mathbb{C}[P]$. Let

$$e(\lambda) : V_P \longrightarrow V_P$$
$$e(\lambda) = e^\lambda \epsilon(\cdot, \lambda).$$

Then $e(\lambda)$ is obviously a linear bijection and we have the commutation relation:

$$x_\gamma(n) e(\omega) = e(\omega) x_\gamma(n+1).$$

We will call $e(\omega)$ a simple current operator (cf. [DLM]). The restrictions of this operator give the bijections between the level 1 modules.

We also use the operator

$$e(n\omega) \cong e(\omega)^n, \qquad \text{for } n \in \mathbb{N}$$

(here $e(n\omega) \cong e(\omega)^n$ means that vectors $e(n\omega)v_\Lambda$ and $e^{n\omega}v_\Lambda$ are proportional).

Difference and initial conditions

We say that a monomial $x(\pi)$ satisfies the difference conditions (or DC in short) for level k if:

1) $b_2 + a_{\underline{2}} + a_{\underline{3}} + a_{\underline{4}} + a_4 + a_3 \leq k$,
2) $b_2 + a_{\underline{3}} + a_{\underline{4}} + a_4 + a_3 + a_2 \leq k$,
3) $b_3 + b_2 + a_{\underline{2}} + a_{\underline{3}} + a_{\underline{4}} + a_4 \leq k$,
4) $b_3 + b_2 + a_{\underline{2}} + a_{\underline{4}} + a_4 + a_3 \leq k$,
5) $b_4 + b_3 + b_2 + a_{\underline{2}} + a_{\underline{3}} + a_4 \leq k$,
6) $b_{\underline{4}} + b_3 + b_2 + a_{\underline{2}} + a_{\underline{3}} + a_{\underline{4}} \leq k$,
7) $b_{\underline{4}} + b_4 + b_3 + b_2 + a_{\underline{2}} + a_{\underline{3}} \leq k$,
8) $b_{\underline{3}} + b_{\underline{4}} + b_4 + b_2 + a_{\underline{2}} + a_{\underline{3}} \leq k$,
9) $b_{\underline{3}} + b_{\underline{4}} + b_4 + b_3 + b_2 + a_{\underline{2}} \leq k$,
10) $b_{\underline{2}} + b_{\underline{3}} + b_{\underline{4}} + b_4 + b_3 + a_{\underline{2}} \leq k$.

Using relations between vertex operators we show that monomials $x(\pi)$ which don't satisfy the inequalities above can be excluded from the spanning set (2). To find the necessary relations between vertex operators we first use the relations we have on vacuum vector such as:

$$x_{\gamma_4}^{a_4+a_3}(-1) x_{\gamma_{\underline{3}}}^{a_{\underline{4}}}(-1) x_{\gamma_{\underline{2}}}^{a_2+a_3+b_2}(-1)\mathbf{1} = 0. \tag{5}$$

This gives us the relation between fields:

$$x_{\gamma_4}^{a_4+a_3}(z) x_{\gamma_{\underline{3}}}^{a_{\underline{4}}}(z) x_{\gamma_{\underline{2}}}^{a_2+a_3+b_2}(z) = 0. \tag{6}$$

The coefficients of z in such relations give us now the starting equations. We act on those equations with elements from the semisimple part of $\mathfrak{g}_0$. In this manner, we get new relations in which monomials that don't satisfy the DC are minimal so they can be expressed as the sum of the greater ones.

We say that a monomial $x(\pi)$ satisfies the initial conditions for $W(\Lambda)$ (or IC in short) if

$$\begin{aligned}
&1)\quad b_2 \leq k - k_1 - k_3 - k_4 - 2k_2 = k_0,\\
&2)\quad b_3 + b_2 \leq k - k_1 - k_2 - k_3 - k_4 = k_0 + k_2,\\
&3)\quad b_4 + b_3 + b_2 \leq k - k_1 - k_4 - k_2 - = k_0 + k_2 + k_3,\\
&4)\quad b_{\underline{4}} + b_3 + b_2 \leq k - k_1 - k_2 - k_3 = k_0 + k_2 + k_4,\\
&5)\quad b_{\underline{4}} + b_4 + b_3 + b_2 \leq k - k_1 - k_2 = k_0 + k_2 + k_3 + k_4,\\
&6)\quad b_{\underline{3}} + b_{\underline{4}} + b_4 + b_2 \leq k - k_1 - k_2 = k_0 + k_2 + k_3 + k_4,\\
&7)\quad b_{\underline{3}} + b_{\underline{4}} + b_4 + b_3 + b_2 \leq k - k_1 = k_0 + 2k_2 + k_3 + k_4,\\
&8)\quad b_{\underline{2}} + b_{\underline{3}} + b_{\underline{4}} + b_4 + b_3 \leq k - k_1 = k_0 + 2k_2 + k_3 + k_4.
\end{aligned} \tag{7}$$

Now we can express the following important property of the monomials that satisfy both the DC and the IC:

Proposition 2 *Let $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2 + k_3\Lambda_3 + k_4\Lambda_4$ be a weight of level k. Monomial $x(\pi)$ satisfies the difference and initial conditions for $W(\Lambda)$ if and only if $x(\pi)$ has a partition on submonomials $x(\pi_{i,s_i})$, $i = 0, 1, 2, 3, 4$, $s_i = 1, \ldots, k_i$, where each $x_{\pi_{i,s_i}}$ satisfies difference and initial conditions for $W(\Lambda_i)$.*

Intertwining operators and linear independence

The irreducible $\mathfrak{g}$-modules of weight $\omega_{\ell-1}$ and ω_ℓ are on the top of the $\tilde{\mathfrak{g}}$-modules $L(\Lambda_{\ell-1})$ and $L(\Lambda_\ell)$. In the lattice construction their weight vectors are of the form $1 \otimes e^\mu = e^\mu$ where μ denotes the weight, $\mu = \frac{1}{2}\left(\sum_{i \in \Sigma} \epsilon_i - \sum_{j \notin \Sigma} \epsilon_j\right)$. Vertex operators $\mathcal{Y}(1 \otimes e^\mu, z)$ commute with the action of $\tilde{\mathfrak{g}}_1$ if and only if $1 \in \Sigma$. For a monomial $x(\pi)$ and a highest weight vector v_Λ a specific coefficients of this intertwining operators give us the operators with the following properties:

Proposition 3 *Let $L(\Lambda)$ be a level 1 or 2 module and let $x(\pi)$ be a monomial that satisfies both DC and IC for $W(\Lambda)$. Then there exist a partition of monomial $x(\pi)$, $x(\pi) = x(\pi_2)x(\pi_1)$ and an intertwining operator $I_{\pi,\Lambda}$ which commutes with the action of $\tilde{\mathfrak{g}}_1$ so that the following holds:*

a) operator $I_{\pi,\Lambda}$ acts on $x(\pi_1)$ and $I_{\pi,\Lambda} x(\pi_1) v_\Lambda = C \cdot e^{n\omega} v_{\Lambda'} \neq 0$,
b) $I_{\pi,\Lambda} x(\pi_1') v_\Lambda = 0$ for $x(\pi_1') > x(\pi_1)$ such that $x(\pi_1) \not\prec x(\pi_1')$,
c) $x(\pi_2)^{+n}$ satisfies DC and IC for $W(\Lambda')$

(here $x(\pi)^{+n}$ denotes a monomial obtained from $x(\pi)$ by lifting the degree in all the factors by n). We call this operator an intertwining operator as well. Using these operators we can now prove the main result:

Theorem 4 *Let $L(\Lambda)$ be a standard module of level k for the affine Lie algebra of type $D_4^{(1)}$ and let v_Λ be its highest weight vector. The set of monomial vectors*

$$\{\ x(\pi) v_\Lambda \mid x(\pi) \text{ satisfies } DC \text{ and } IC \text{ for } W(\Lambda)\ \} \tag{8}$$

is a basis for the Feigin-Stoyanovsky subspace $W(\Lambda)$.

Sketch of the proof: We know that vectors from (8) span the subspace $W(\Lambda)$. It is left to show their linear independence. We will do this simultaneously for all modules of level k using the intertwining operators. Suppose now

$$\sum_\mu c_\mu x(\mu) v_\Lambda = 0 \tag{9}$$

where all monomials satisfy the DC and IC for $W(\Lambda)$. Assume further on that degree of every monomial from the sum above is greater than or equal to some $n \in \mathbb{N}$. We fix $x(\pi)$ from (9) and we prove that $c_\pi = 0$. Suppose also that $c_\mu = 0$ for $x(\mu) < x(\pi)$. Our fixed monomial $x(\pi)$ has a partition on submonomials $x(\pi^i)$ which satisfy DC and IC for $W(\Lambda_i)$. This implies each of this submonomials $x(\pi^i)$ satisfies the DC and IC either for level $k = 1$ or for level $k = 2$. Then for each of these submonomials $x(\pi^i)$ there exist a partition $x(\pi^i) = x(\pi_2^i) x(\pi_1^i)$, and an intertwining operator $I_{\pi_1^i, \Lambda_i}$ with properties as in Proposition 3. For our fixed monomial $x(\pi)$ we can now take the tensor product of the operators above and we get an intertwining operator $i_{\pi,\Lambda}$ such that

1) $I_{\pi,\Lambda} x(\pi) v_\Lambda = C \cdot e^{n\omega} v_{\Lambda'} \neq 0$ for some n and some $v_{\Lambda'}$,
2) $I_{\pi,\Lambda} x(\mu) v_\Lambda = 0$ for any $x(\mu) > x(\pi)$ such that $x(\pi) \not\prec x(\mu)$,
3) if $x(\mu) = x(\mu')x(\pi)$ then $x(\mu')^{+n}$ satsifies the DC and IC for $W(\Lambda')$.

We now apply $I_{\pi,\Lambda}$ to the sum (9) and get:

$$I_{\pi,\Lambda}\left(\sum_{\substack{x(\mu)>x(\pi)\\ x(\mu)\not\succ x(\pi)}} c_\mu x(\mu)\right) v_\Lambda + I_{\pi,\Lambda}\left(\sum_{\substack{x(\mu)\geq x(\pi)\\ x(\mu)\succ x(\pi)}} c_\mu x(\mu)\right) v_\Lambda + $$
$$+ I_{\pi,\Lambda}\left(\sum_{x(\mu)<x(\pi)} c_\mu x(\mu)\right) v_\Lambda = 0.$$

The coefficients in the third sum are equal to zero by assumption. In the first sum we have $I_{\pi,\Lambda} x(\mu) v_\Lambda = 0$. Therefore only the second term remains and we have:

$$0 = \sum_{\substack{x(\mu)\geq x(\pi)\\ x(\mu)\succ x(\pi)}} c_\mu x(\mu') \cdot e^{n\omega} v_{\Lambda'} = D \cdot e(n\omega) \sum_{\substack{x(\mu)\geq x(\pi)\\ x(\mu)\succ x(\pi)}} c_\mu x(\mu')^{+n} v_{\Lambda'}$$

Since $e(n\omega)$ is an injection we can apply the assuption of the induction and conclude $c_\pi = 0$.

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