Representations Of Lie Tori Of Type A_ℓ Coordinated By Cyclotomic Quantum Tori

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Definitions

The Quantum Torus C_q : Let $q = (q_{ij})$ be a $r \times r$ matrix of non-zero complex numbers satisfying the relation : $q_{ii} = 1$, $q_{ij} = q_{ji}^{-1}$, for all $1 \leq i,j \leq r$.

Let J_q be the ideal of the non-commutative Laurent polynomial ring $S_{[r]} = \mathbb{C}[t_1^{\pm} \ldots t_r^{\pm}]_{n.c}$ generated by the elements, $t_i t_j - q_{ij} t_j t_i$, $1 \leq i, j \leq r$.

The algebra $C_q := S_{[r]}/J_q$ is called the quantum torus of rank r associated to q.

C_q is said to be cyclotomic if q_{ij} is a complex roots of unity for all i.j.

The Lie tori $\mathfrak{sl}_{\ell+1}(C_q)$: Given $M_{\ell+1}(C_q) = M_{\ell+1}(C) \otimes C_q$, the Lie algebra $\mathfrak{sl}_{\ell+1}(C_q)$ is defined as : $\mathfrak{sl}_{\ell+1}(C_q) = \{ X = (x_{ij}) \in M_{\ell+1}(C_q) : \mathrm{Trace}(X) \in [C_q, C_q] \}$ with commutator relations:

$$[x \otimes a, y \otimes b] = B(x, y)I([a, b]) + [x, y] \otimes (a \circ b)/2 + (x \circ y) \otimes [a, b]/2 \quad [b1]$$

$$[I([a, b]), I([c, d])] = I([[a, b], [c, d]]), \quad [b2]$$

$$[I([a, b]), x \otimes c] = x \otimes [[a, b], c], \quad [b3]$$

where $x, y \in \mathfrak{sl}_{\ell+1}(C)$, $a, b, c, d \in C_q$,

$[x, y] = xy - yx$, $x \circ y = xy + yx - 2/(\ell + 1)\mathrm{Tr}(xy)I(1)$,

$[a, b] = ab - ba$, $a \circ b = ab + ba$, and $B(x, y) = 1/(\ell + 1)\mathrm{Tr}(xy)$.

Let Q_+ = positive integer root lattice of $\mathfrak{sl}_{\ell+1}(C)$; $Q_- = -Q_+$, and $Q = Q_+ + Q_-$

$\mathfrak{sl}_{\ell+1}(C_q)$ has a decomposition given by:

$$\mathfrak{sl}_{\ell+1}(C_q) = \Big(\bigoplus_{(\alpha, \mathfrak{m}) \in Q \times \mathbf{Z}^r} \mathfrak{sl}_{\ell+1}(C_q)_\alpha^{\mathfrak{m}} \Big) \oplus \Big(\bigoplus_{\mathfrak{m} \in \mathbf{Z}^r} \mathfrak{sl}_{\ell+1}(C_q)_0^{\mathfrak{m}} \Big)$$

Set $\mathrm{supp}\, \mathfrak{sl}_{\ell+1}(C_q) = \{(\alpha, \mathfrak{m}) \in Q \times \mathbf{Z}^r : \mathfrak{sl}_{\ell+1}(C_q)_\alpha^{\mathfrak{m}} \neq 0 \}$ and $H(C_q) = \oplus\, \mathfrak{sl}_{\ell+1}(C_q)_0^{\mathfrak{m}}$.

KNOWN RESULT : It has been shown in [1], [17] that a rank r cyclotomic torus C_q is isomorphic to a tensor product: $C_q \cong \mathcal{Q}(d_1) \otimes \ldots \otimes \mathcal{Q}(d_s) \otimes C[z_1^{\pm 1} \ldots z_k^{\pm 1}]$, where $\mathcal{Q}(d_i)$ is a rank 2 quantum torus associated to the matrix $q(i) = (q_{kl}[i])$ with $q_{12}[i] = \zeta_i = (q_{21}[i])^{-1}$, where ζ_i is a d_i^{th} root of unity for $1 \leq i \leq s$.

OUR OBSERVATIONS :

Let $\mathfrak{B}(q)$:= Set of maximal commutative subalgebras of C_q and let $\mathcal{Z}(q)$ be the center of C_q.

For $s_q \in \mathfrak{B}(q)$, set $\Gamma(s_q) = \{ \mathfrak{m} = (m_1, \ldots, m_r) \in \mathbf{Z}^r : t^{m_1} \ldots t^{m_r} = t^{\mathfrak{m}} \in s_q \}$ and let $\mathfrak{s}_q \in \mathfrak{B}(q)$ be such that $s_q \cap \mathfrak{s}_q = \mathcal{Z}(q)$.

- One can associate with each subalgebra $s_q \in \mathfrak{B}(q)$, of a normalized cyclotomic quantum torus C_q, an abelian group $G(s_q) = \mathbf{Z}^r / \Gamma(s_q)$ of rank $d_1 \ldots d_s$.
- Any Borel subalgebra of $\mathfrak{sl}_{\ell+1}(C_q)$ is of the form $(\mathfrak{n}_q^+ \oplus H(s_q))$ or $(\mathfrak{n}_q^- \oplus H(s_q))$, where $\mathfrak{n}_q^\pm$ is the subalgebra of $\mathfrak{sl}_{\ell+1}(C_q)$ generated by the elements of $\mathfrak{sl}_{\ell+1}(C_q)_\alpha^{\mathfrak{m}}$ for $(\alpha, \mathfrak{m}) \in \mathrm{supp}\, \mathfrak{sl}_{\ell+1}(C_q)$, with $\alpha \in Q_\pm$ and $H(s_q)$ is the subalgebra of $\mathfrak{sl}_{\ell+1}(C_q)$ generated by the elements of $\mathfrak{sl}_{\ell+1}(C_q)_0^{\mathfrak{m}}$, for $\mathfrak{m} \in \Gamma(s_q)$.
- The multiloop Lie algebra $\mathfrak{sl}_{\ell+1}(C) \otimes s_q$ is a subalgebra of $\mathfrak{sl}_{\ell+1}(C_q)$ for all $s_q \in \mathfrak{B}(q)$.
- Let V be an irreducible $\mathfrak{sl}_{\ell+1}(C_q)$-module with finite dimensional weight spaces. Then there exists non-zero vector $v \in V$ such that $U_q(\mathfrak{n}_q^+) v = 0$ and $V = U_q(\mathfrak{sl}_{\ell+1}(C_q)) v$, where $U_q(\mathfrak{a})$ denotes the universal enveloping algebra of any subalgebra $\mathfrak{a}$ of $\mathfrak{sl}_{\ell+1}(C_q)$.

Finite Dimensional Irreducible Representations Of $\mathfrak{sl}_{\ell+1}(C_q)$

The Idea !!

Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_{\ell+1}(C_q)$ generated by a vector v. Then there exists a positive Borel subalgebra $\mathfrak{b}(s_q) = (\mathfrak{n}_q^+ \oplus H(s_q))$ of $\mathfrak{sl}_{\ell+1}(C_q)$ such that

$$U_q(\mathfrak{n}_q^+ \oplus H(s_q)). v \in \mathbf{C}v .$$

It follows from the representation theory of multiloop Lie algebra that there exists a finitely supported functions $f : (C^\times)^r \to P^+$ such that :

$$h \otimes t^{\mathfrak{m}} . v = \sum_{a \in \mathrm{supp}\, f} f(a)(h)\, ev_a (t^{\mathfrak{m}})\, v, \quad \text{for all } \mathfrak{m} \in \Gamma(s_q),$$

where P^+ is the positive integral weight lattice and $ev_a : s_q \to \mathbf{C}$ denotes the evaluation map at the point $a \in (C^\times)^r$. This implies that the finite-dimensional irreducible $\mathfrak{sl}_{\ell+1}(C_q)$–modules are tensor products of $\mathfrak{sl}_{\ell+1}(C_q)$-modules which are analogous to the evaluation modules defined for the multiloop Lie algebras.

Analogs of Evaluation Modules for $\mathfrak{sl}_{\ell+1}(C_q)$

Suppose that $\chi_a : (C^\times)^r \to P^+$ is a function supported at a point $a \in (C^\times)^r$. Let v be a non-zero vector of an irreducible finite-dimensional $\mathfrak{sl}_{\ell+1}(C_q)$–module V such that :

$$\mathfrak{n}_q^+ . v = 0 \quad \text{and} \quad h \otimes t^{\mathfrak{m}} . v = \chi_a(a)(h)\, ev_a (t^{\mathfrak{m}})\, v. \quad \text{for all } \mathfrak{m} \in \Gamma(s_q), \text{ for some } s_q \in \mathfrak{B}(q).$$

$H(C_q)$ is not a commutative algebra, hence if V is a non-trivial $\mathfrak{sl}_{\ell+1}(C_q)$-module, then

$$\dim U_q(H(C_q)) > 1, \text{ implying,} \quad h \otimes t^{\mathfrak{s}} . v \notin \mathbf{C}v \text{ for } \mathfrak{s} \in \mathbf{Z}^r \setminus \Gamma(s_q).$$

However $\mathfrak{n}_q^+ . v = 0$, implies $\mathfrak{n}_q^+ . h \otimes t^{\mathfrak{s}} . v = 0$, for all $\mathfrak{s} \in \mathbf{Z}^r$. In particular,

$$h \otimes t^{\mathfrak{s}} . v \text{ is a highest weight vector of V for } \mathfrak{s} \in \mathbf{Z}^r \setminus \Gamma(s_q).$$

Hence there exists a positive Borel subalgebra $\mathfrak{b}(c_q)$ with $c_q \in \mathfrak{B}(q)$ such that :

$$\mathfrak{b}(c_q).\, h \otimes t^{\mathfrak{s}} . v \in C.\, h \otimes t^{\mathfrak{s}} . v, \quad \text{for all } \mathfrak{s} \in \mathbf{Z}^r \setminus \Gamma(s_q).$$

Irreducibility of the module V and the fact that the center of the algebra $H(C_q)$ acts on all the highest weight vectors by the same scalar, imply that there exists $\zeta \in G(\mathfrak{s}_q)$ such that :

$$h \otimes t^{\mathfrak{m}} .\, h \otimes t^{\mathfrak{s}} . v = \chi_a(a.\zeta^{\mathfrak{s}})(h)\, ev_{a.\zeta^{\mathfrak{s}}} (t^{\mathfrak{m}})\, h \otimes t^{\mathfrak{s}} . v, \quad \text{for } \mathfrak{s} \in \mathbf{Z}^r \setminus \Gamma(s_q), \mathfrak{m} \in \Gamma(s_q),$$

where $\chi_a(a.\zeta^{\mathfrak{s}}) = \chi_a(a)$, for all $\mathfrak{s} \in \mathbf{Z}^r \setminus \Gamma(s_q)$. Further owing to the bracket operation [b1] in $H(C_q)$, it is seen that the module generated by v is an irreducible $\mathfrak{sl}_{\ell+1}(C_q)$-module if and only if

$$\chi_a(a) \text{ is a miniscule weight of } \mathfrak{sl}_{\ell+1}(C).$$

Let $F(s_q)$ be the set of all finitely supported functions $f : (C^\times)^r \to P^+$ such that $f(a)$ is a miniscule weight for all $a \in$ support of f, and let $F(s_q, \mathcal{Z}(q))$ be the subset of $F(s_q)$ consisting of all functions f such that

$$ev_a(t^{\mathfrak{d}}) \neq ev_b(t^{\mathfrak{d}}), \text{ for } a,b \text{ supp } f \text{ and } t^{\mathfrak{d}} \in \mathcal{Z}(q),$$

where $\mathfrak{d} \in \mathbf{Z}^r$ denotes a multi-index element .

Given $\chi_a \in F(s_q, \mathcal{Z}(q))$ and $\zeta \in G(\mathfrak{s}_q)$, let (s_q, χ_a, ζ) denote the set of all finitely supported functions $g \in F(s_q, \mathcal{Z}(q))$ for which $\mathrm{supp}\, g = \zeta^{\mathfrak{s}} \cdot a$ for $\mathfrak{s} \in G(s_q)$.

Then the analogs of the evaluation modules for $\mathfrak{sl}_{\ell+1}(C_q)$ is given by $\mathcal{V}(s_q, \chi_a, \zeta)$ which is a module generated by a highest weight vector v on which $\mathfrak{h} \otimes s_q$ acts by the function χ_a and $\mathfrak{h} \otimes s_q$ acts on any other highest weight vector of $\mathcal{V}(s_q, \chi_a, \zeta)$ by a function of the form $\zeta^{\mathfrak{s}}.\chi_a$, for $\mathfrak{s} \in G(s_q)$.

Given $s_q \in \mathfrak{B}(q)$, $f = \sum_{i=1}^{r} \chi_{a_i} \in \mathbf{F}(s_q, \mathcal{Z}(q))$, $\zeta \in \mathbf{G}(\mathfrak{s}_q)^r$,

set : $\mathcal{V}(s_q, f, \zeta) = \bigotimes_{a \in \mathrm{supp}\, f} \mathcal{V}(s_q, \chi_a, \zeta)$.

MAIN RESULTS:

* Let V be a finite dimensional modules for the Lie algebra $\mathfrak{sl}_{\ell+1}(C_q)$. Then V is of the form $\mathcal{V}(s_q, f, \zeta)$, where $f \in \mathbf{F}(s_q, \mathcal{Z}(q))$ and $\zeta \in \mathbf{G}(\mathfrak{s}_q)^{|f|}$, where $|f| = \#\, \mathrm{supp}\, f$.

* Let s_q, $c_q \in \mathfrak{B}(q)$ and $f_1 \in \mathbf{F}(s_q, \mathcal{Z}(q))$, $f_2 \in \mathbf{F}(c_q, \mathcal{Z}(q))$ with $|f_j| = r_j$, $j=1,2$ and let $\zeta \in \mathbf{G}(\mathfrak{s}_q)^{r_1}$ and $\eta \in \mathbf{G}(\mathfrak{c}_q)^{r_2}$. Then there exists a $\mathfrak{sl}_{\ell+1}(C_q)$-module isomorphism

$$\gamma : \mathcal{V}(s_q, f_1, \zeta) \longrightarrow \mathcal{V}(c_q, f_2, \eta) \quad \text{if and only if}$$

i. $s_q = c_q$.

ii. For each highest weight vector $v \in \mathcal{V}(s_q, f_1, \zeta)$, $\gamma(v)$ is a highest weight vector of $\mathcal{V}(c_q, f_2, \eta)$ such that upto a scaling factor $f_{(1,v)} \in (s_q, f_1, \zeta)$ is $G(s_q)$ - equivariant to $f_{(2,\gamma(v))} \in (c_q, f_2, \eta)$, where $f_{(i,w)}$ denotes the finitely supported function by which $\mathfrak{h} \otimes s_q$ acts on the highest weight vector w for some $s_q \in \mathfrak{B}(q)$.

References:

1. Allison, B., Berman, S., Faulkner, J.,Pianzola, A., Realization of graded-simple algebras as loop algebras. Forum Math. 20 (2008), no. 3.
2. Allison, B., Berman, S., Faulkner, J., Pianzola, A., Multiloop realization of extended affine Lie algebras and Lie tori, Trans. Amer. Math. Soc. (to appear).
3. Berman, S., Gao, Y., Krylyuk, Y. S., Quantum tori and the structure of elliptic quasi-simple Lie algebras. J. Funct. Anal. 135 (1996), no. 2.
4. Berman, S., Szmigielski, J., Principal realization for the extended affine Lie algebra of type ${\rm sl}\sb 2$ with coordinates in a simple quantum torus with two generators. Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), 39--67, Contemp. Math., 248, Amer. Math. Soc., Providence, RI, 1999.
5. Billig, Y., Lau, M., Thin coverings of modules, J. Algebra 316, (2007), no. 1, 147-173; 0605680v1.
6. Chari, V., Integrable representations of affine Lie-algebras. Invent. Math. 85 (1986), no. 2.
7. Chari, V., Pressley, A., New unitary representations of loop groups. Math. Ann. 275 (1986), no. 1.
8. Chari, V., Pressley, A., Weyl Modules for Classical and Quantum Affine Algebras, Represent. Theory 5 (2001) ; math.QA/0004174.
9. Chari, V., Fourier, G., Khandai, T., A Categorical Approach to Weyl modules, In Preparation.
10. Feigin, B., Loktev, S., Multi-dimensional Weyl modules and symmetric functions. Comm. Math. Phys. 251 (2004), no. 3.
11. Garland, H., The Arithmetic Theory of Loop Algebras, J. Algebra 53 (1978).
12. Gao, Y., Representations of extended affine Lie algebras coordinatized by certain quantum tori. Compositio Math. 123 (2000), no. 1.
13. Gao, Y., Vertex operators arising from the homogeneous realization for $\widehat{{\rm g}\ell_n}$. Comm. Math. Phys. 211(2000).
14. Joseph, A, The admissibility of simple bounded modules for an affine Lie algebra. Algebra. Represent. Theory 3 (2000), no. 2.
15. Kac, Victor G. Infinite-dimensional Lie algebras. Third edition. Cambridge University Press, Cambridge, 1990.
16. Lau, M., Representations of Multiloop Algebras, arXiv:0811. 2011v2 [math.RT]
17. Neeb, K. H., On the classification of rational quantum tori and the structure of their automorphism groups, Canad. Math. Bull. 51 (2008), no. 2.
18. Rao, S. Eswara, A class of integrable modules for the core of EALA coordinatized by quantum tori. J. Algebra 275 (2004), no.1.
19. Rao, S. Eswara, Unitary modules for EALAs co-ordinatized by a quantum torus. Comm. Algebra 31 (2003), no. 5.