Explicit Representation of Cost Efficient Strategies

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Motivation / Context

- Starting point: work on popular US retail investment products. How to explain the demand for complex path-dependent contracts?
- Met with Phil Dybvig at the NFA in Sept. 2008.
- Path-dependent contracts are not "efficient" (JoB 1988, "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market" in RFS 1988).

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Some Assumptions

- Consider an arbitrage-free and complete market.
- Given a strategy with payoff X_T at time T. There exists Q, such that its price at 0 is

$$P_X = E_Q[e^{-rT} X_T]$$

- P ("physical measure") and Q ("risk-neutral measure") are two equivalent probability measures:

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T, \quad P_X = E_Q[e^{-rT} X_T] = E_P[\xi_T X_T].$$

Motivation: Traditional Approach to Portfolio Selection

Investors have a strategy that will give them a final wealth X_T.
This strategy depends on the financial market and is random.

- They want to maximize the **expected utility** of their final wealth X_T

$$\max_{X_T} \left(E_P[U(X_T)]\right)$$

U: utility (increasing because individuals prefer more to less).

- They want to minimize the **cost of the strategy**

$$cost\ at\ 0 = E_Q[e^{-rT}X_T] = E_P[\xi_T X_T]$$

Find optimal payoff X_T $\Rightarrow$ Optimal cdf F of X_T

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Our Approach

- Given the cdf F that the investor would like for his final wealth
- We give an explicit representation of the payoff X_T such that
  - $X_T \sim F$ in the real world
  - X_T corresponds to the cheapest strategy

Outline of the presentation

- What is cost-efficiency?
- Path-dependent strategies/payoffs are not cost-efficient.
- **Explicit** construction of **efficient** strategies.
- Investors (with a fixed horizon and law-invariant preferences) should prefer to invest in path-independent payoffs:
  path-dependent exotic derivatives are usually not optimal!

Efficiency Cost

Dybvig (RFS 1988) explains how to compare two strategies by analyzing their respective efficiency cost.

What is the "efficiency cost"?

It is a criteria for evaluating payoffs independent of the agents' preferences.

Efficiency Cost

- Given a strategy with payoff X_T at time T, and initial price at time 0
$$P_X = E_P\left[\xi_T X_T\right]$$
- F : X_T's distribution under the **physical measure** P.

The distributional price is defined as

$$PD(F) = \min_{\{Y_T \mid Y_T \sim F\}} \{E_P\left[\xi_T Y_T\right]\}$$

The "loss of efficiency" or "efficiency cost" is equal to:

$$P_X - PD(F)$$

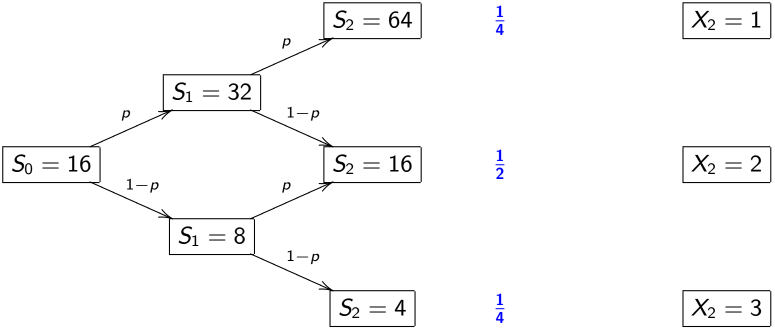
A Simple Illustration

Let's illustrate what the "efficiency cost" is with a simple example. Consider :

- A market with 2 assets: a bond and a stock S.
- A discrete 2-period binomial model for the stock S.
- A strategy with payoff X_T at the end of the two periods.
- An expected utility maximizer with utility function U.

A simple illustration for X_2, a payoff at $T = 2$

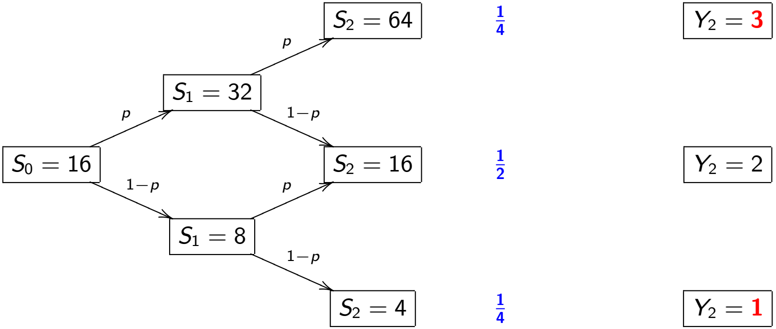
Real-world probabilities$=p = \frac{1}{2}$



$$E[U(X_2)] = \frac{U(1) + U(3)}{4} + \frac{U(2)}{2}$$

Y_2, a payoff at $T = 2$ distributed as X_2

Real-world probabilities$=p = \frac{1}{2}$



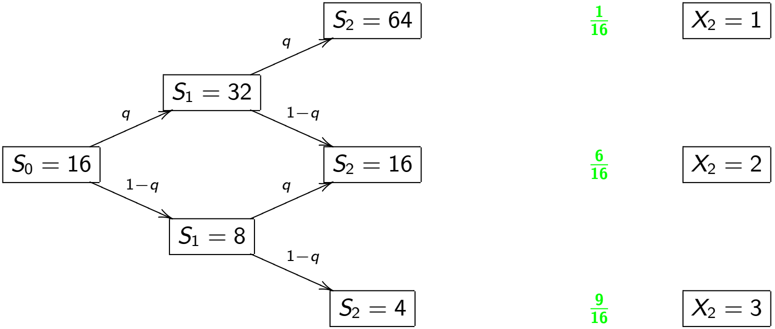
$$E[U(Y_2)] = \frac{U(3) + U(1)}{4} + \frac{U(2)}{2}$$

(X and Y have the same distribution under the physical measure and thus the same utility)

X_2, a payoff at $T=2$

risk neutral

probabilities$=q=\frac{1}{4}$.

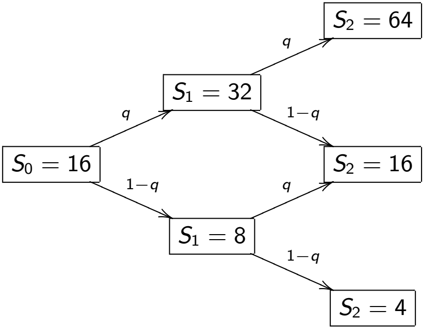


$$P_{X_2} = \text{Price of } X_2 = \left(\frac{1}{16}+\frac{6}{16}2+\frac{9}{16}3\right)=\frac{5}{2}$$

Y_2, a payoff at $T = 2$

risk neutral

probabilities$=q=\frac{1}{4}$.



$\frac{1}{16}$	$\mathbf{Y_2 = 3}$
$\frac{6}{16}$	$Y_2 = 2$
$\frac{9}{16}$	$\mathbf{Y_2 = 1}$

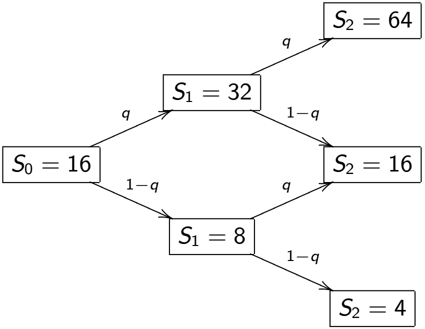
$$P_{Y_2} = \left(\frac{1}{16}3 + \frac{6}{16}2 + \frac{9}{16}1\right) = \frac{3}{2}$$

$$P_{X_2} = \text{Price of } X_2 = \left(\frac{1}{16} + \frac{6}{16}2 + \frac{9}{16}3\right) = \frac{5}{2} \quad ,$$

A simple illustration for X_2, a payoff at $T = 2$

risk neutral

probabilities$=q = \frac{1}{4}$.



$\frac{1}{16}$ $\boxed{X_2 = 1}$

$\frac{6}{16}$ $\boxed{X_2 = 2}$

$\frac{9}{16}$ $\boxed{X_2 = 3}$

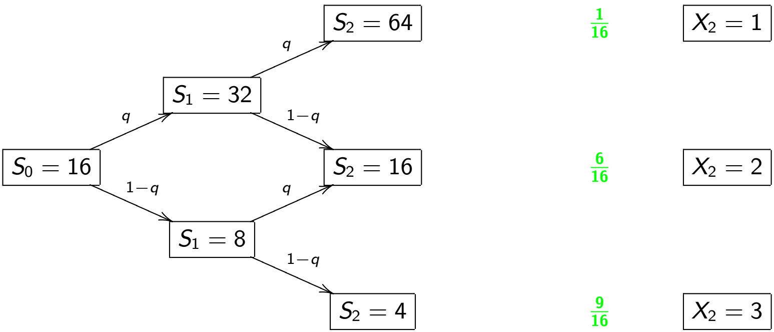
$$P_D = Cheapest = \frac{3}{2}$$

$$P_{X_2} \;=\; \text{Price of } X_2 = \frac{5}{2} \quad ,$$

A simple illustration for X_2, a payoff at $T = 2$

risk neutral

probabilities$=q = \frac{1}{4}$.

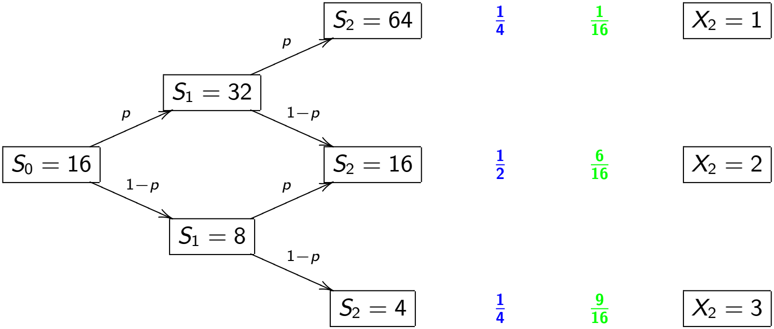


$$P_D = Cheapest = \frac{3}{2}$$

$$P_{X_2} \; = \; \text{Price of } X_2 = \frac{5}{2} \quad , \quad \text{Efficiency cost} = P_{X_2} - P_D$$

A simple illustration for X_2, a payoff at $T=2$

Real-world probabilities$=p=\frac{1}{2}$ and **risk neutral** probabilities$=q=\frac{1}{4}$.



$$E[U(X_2)]=\frac{U(1)+U(3)}{4}+\frac{U(2)}{2} \quad , \quad P_D = Cheapest = \frac{3}{2}$$

$$P_{X_2} \;=\; \text{Price of } X_2=\frac{5}{2} \quad , \quad \text{Efficiency cost} = P_{X_2}-P_D$$

Cost-Efficiency

- The **cost** of the payoff X_T is $c(X_T) = E[\xi_T X_T]$.
- The **"distributional price"** of a cdf F is defined as

$$PD(F) = \min_{\{Y \mid Y \sim F\}} \{c(Y)\}$$

We want to find the strategy Y that realizes this minimum.

Given a payoff X_T with cdf F. We define its inverse F^{-1} as follows:

$$F^{-1}(y) = \min\{x \ / \ F(x) \geq y\}.$$

Theorem

Define

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

then $X_T^\star \sim F$ and $X_T^\star$ is a.s. unique such that

$$PD(F) = c(X_T^\star)$$

Path-dependent payoffs are inefficient

Corollary

To be cost-efficient, the payoff of the derivative has to be of the following form:

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right)$$

It becomes a European derivative written on S_T as soon as the state-price process ξ_T can be expressed as a function of S_T. Thus path-dependent derivatives are in general not cost-efficient.

Corollary

Consider a derivative with a payoff X_T which could be written as

$$X_T = h(\xi_T)$$

Then X_T is cost efficient if and only if h is non-increasing.

Black and Scholes Model

Under the physical measure P,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t^P$$

Under the risk neutral measure Q,

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^Q$$

S_t has a lognormal distribution.

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T = e^{-rT}a\left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = \exp\left(\frac{1}{2}Tb(r+\mu-\sigma^2) - rT\right)$ $b = \frac{\mu-r}{\sigma^2}$.

Black and Scholes Model

Any path-dependent financial derivative is inefficient. Indeed

$$\xi_T = e^{-rT}\left(\frac{dQ}{dP}\right)_T = e^{-rT} a \left(\frac{S_T}{S_0}\right)^{-b}$$

where $a = \exp\left(\frac{1}{2}Tb(r+\mu-\sigma^2) - rT\right)$ $b = \frac{\mu-r}{\sigma^2}$.
To be cost-efficient, the payoff has to be written as

$$X^\star = F^{-1}\left(1 - F_\xi\left(a\left(\frac{S_T}{S_0}\right)^{-b}\right)\right)$$

It is a European derivative written on the stock S_T (**and the payoff is increasing with** S_T when $\mu > r$).

The Least Efficient Payoff

Theorem

Let F be a cdf such that $F(0) = 0$. Consider the following optimization problem:

$$\max_{\{Z \mid Z \sim F\}} \{c(Z)\}$$

The strategy $Z_T^\star$ that generates the same distribution as F with the highest cost can be described as follows:

$$Z_T^\star = F^{-1}\left(F_\xi\left(\xi_T\right)\right)$$

Consider a strategy with payoff X_T distributed as F. The cost of this strategy satisfies

$$P_D(F) \leqslant c(X_T) \leqslant E[\xi_T F^{-1}(F_\xi(\xi_T))] = \int_0^1 F_\xi^{-1}(v) F^{-1}(v) dv$$

Put option in Black and Scholes model

Assume a strike K. The payoff of the put is given by

$$L_T = (K - S_T)^+ .$$

The payoff that has the **lowest** cost and is distributed such as the put option is given by

$$Y_T^\star = F_L^{-1}\left(1 - F_\xi\left(\xi_T\right)\right).$$

Put option in Black and Scholes model

Assume a strike K. The payoff of the put is given by

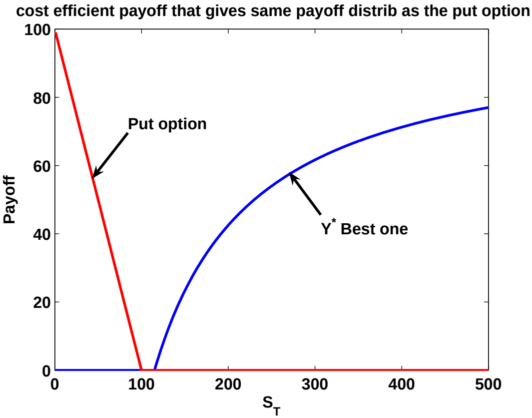
$$L_T = (K - S_T)^+ .$$

The cost-efficient payoff that will give the same distribution as a put option is

$$Y_T^\star = \left(K - \frac{S_0^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)T}}{S_T} \right)^+ .$$

This type of power option “dominates” the put option.

Cost-efficient payoff of a put



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100$.

Distributional price of the put = 3.14

Price of the put = 5.57

Efficiency loss for the put = 5.57-3.14= 2.43

Geometric Asian contract in Black and Scholes model

Assume a strike K. The payoff of the Gemoetric Asian call is given by

$$G_T = \left(e^{\frac{1}{T}\int_0^T \ln(S_t)dt} - K\right)^+$$

which corresponds in the discrete case to $\left(\left(\prod_{k=1}^n S_{\frac{kT}{n}}\right)^{\frac{1}{n}} - K\right)^+$.

The efficient payoff that is distributed as the payoff G_T is given by

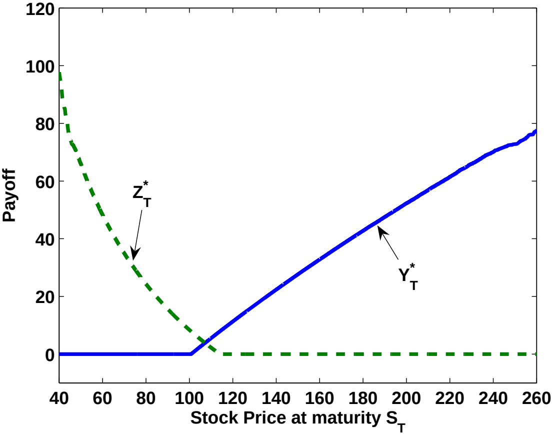
$$G_T^\star = d\left(S_T^{1/\sqrt{3}} - \frac{K}{d}\right)^+$$

where $d := S_0^{1-\frac{1}{\sqrt{3}}} e^{\left(\frac{1}{2}-\sqrt{\frac{1}{3}}\right)\left(\mu-\frac{\sigma^2}{2}\right)T}$.

This payoff $G_T^\star$ is a power call option. If $\sigma = 20\%, \mu = 9\%$, $r = 5\%$, $S_0 = 100$. The price of this geometric Asian option is 5.94. The payoff $G_T^\star$ costs only 5.77.

Similar result in the discrete case.

Example: the discrete Geometric option



With $\sigma = 20\%, \mu = 9\%, r = 5\%, S_0 = 100,\ T = 1$ year, $K = 100,\ n = 12$.

Price of the geometric Asian option $= 5.94$. The distributional price is 5.77.

The least-efficient payoff $Z_T^\star$ costs 9.03.

Utility Independent Criteria

Denote by

- X_T the final wealth of the investor,
- $V(X_T)$ the objective function of the agent,

Assumptions

1. **Agents' preferences depend only on the probability distribution of terminal wealth:** "law-invariant" preferences. (if $X_T \sim Z_T$ then: $V(X_T) = V(Z_T)$.)
2. **Agents prefer "more to less"**: if c is a non-negative random variable $V(X_T + c) \geqslant V(X_T)$.
3. The market is perfectly liquid, no taxes, no transaction costs, no trading constraints (in particular short-selling is allowed).
4. The market is **arbitrage-free** and **complete**.

For any inefficient payoff, there exists another strategy that these agents will prefer.

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Link with First Stochastic Dominance

Theorem

Consider a payoff X_T with cdf F,

1. *Taking into account the initial cost of the derivative, the cost-efficient payoff $X_T^\star$ of the payoff X_T dominates X_T in the first order stochastic dominance sense :*

$$X_T - c(X_T)e^{rT} \prec_{fsd} X_T^\star - P_D(F)e^{rT}$$

2. *The dominance is strict unless X_T is a non-increasing function of ξ_T.*

Thus the result is true for any preferences that respect first stochastic dominance.

Explaining the Demand for Inefficient Payoffs

1. State-dependent needs
   - **Background risk:**
     - Hedging a long position in the market index S_T (background risk) by purchasing a put option P_T,
     - the background risk can be path-dependent.
   - **Stochastic benchmark or other constraints:** If the investor wants to outperform a given (stochastic) benchmark Γ such that:
     $$P\left\{\omega \in \Omega \,/\, W_T(\omega) > \Gamma(\omega)\right\} \geqslant \alpha.$$
   - **Intermediary consumption.**
2. **Other sources of uncertainty:** the state-price process is not always a monotonic function of S_T (non-Markovian interest rates for instance)
3. **Transaction costs, frictions:** Preference for an available inefficient contract rather than a cost-efficient payoff that one needs to replicate.

Conclusion

- A preference free framework for ranking different investment strategies.
- For a given investment strategy, we derive an explicit analytical expression
  1. for the cheapest strategy that has the same payoff distribution.
  2. for the most expensive strategy that has the same payoff distribution.
- There are strong connections between this approach and stochastic dominance rankings.

This may be useful for improving the design of financial products.