The British Russian Option

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Outline of Talk

1. Setting the scene
2. The British option definition
3. The British Russian option
4. Financial analysis
5. Conclusions

Setting the scene

Consider the standard Black-Scholes-Merton option pricing framework:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^{\mathrm{P}} \quad (S_0 = s) \qquad \text{risky stock}$$
$$dB_t = rB_t dt \quad (B_0 = 1) \qquad \text{riskless bond}$$

where $\mu \in \boldsymbol{R}$ is the drift, $\sigma > 0$ is the volatility coefficient, $W^{\mathrm{P}} = (W_t^{\mathrm{P}})_{t \geq 0}$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, and $r > 0$ is the interest rate.

Standard hedging arguments based on self-financing portfolios leads to the arbitrage-free price of a European option

$$V = \mathbb{E}^{\mathrm{Q}}\left[e^{-rT} h(S_T)\right],$$

where Q is the (risk-neutral) equivalent martingale measure and $h(\cdot)$ is the payoff functional of the contingent claim.

Setting the scene (cont.)

Let us consider the perspective of an option holder who has no ability or desire to sell or hedge his option position, a so-called true buyer.

We ask ourselves:

Why do such investors buy options?

Setting the scene (cont.)

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Why do such investors buy options?

An intuitive answer might be:

...because they are under the belief that the *real-world drift* μ of the underlying asset will differ from the risk free rate r.

Whilst the actual drift of the underlying stock price is irrelevant in determining the arbitrage-free price, to a (true) buyer it is crucial.

Setting the scene (cont.)

The terminal stock price can be written as

$$S_T = S_T(\mu) = s \exp\left(\sigma W_T^{\mathsf{P}} + \left(\mu - \frac{1}{2}\sigma^2\right) T\right)$$

and thus the true buyer's expected value of his payoff from exercising is

$$P = \mathbb{E}^{\mathsf{P}}\left[e^{-rT} h\big(S_T(\mu)\big)\right],$$

whereas the (arbitrage-free) price he will pay for the option is V,

$$V = \mathbb{E}^{\mathsf{Q}}\left[e^{-rT} h\big(S_T(r)\big)\right].$$

Hence the 'rational' true buyer will purchase the option only if $P > V$.

Setting the scene (cont.)

Consider the put option payoff as an example:

$$h(S_T) = \big(K - S_T(\mu)\big)^+.$$

Note that $\mu \mapsto S_T(\mu)$ is increasing so that $\mu \mapsto h\big(S_T(\mu)\big)$ is decreasing and hence

$$\mu \mapsto \mathbb{E}^{\mathrm{P}}\left[e^{-rT} h\big(S_T(\mu)\big)\right] = P(\mu)$$

is also decreasing. Therefore we can see that:

- if $\mu = r$ then the return is fair for the buyer: $V = P$,
- if $\mu < r$ then the return is favourable for the buyer: $V < P$,
- if $\mu > r$ then the return is unfavourable for the buyer: $V > P$.

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However... it's well known that the drift is difficult to measure!

The British option definition

The British option is a new class of early-exercise option that attempts to utilise the idea of optimal prediction in order to provide option holders (true buyers) with an inherent protection mechanism should the holder's beliefs on the future price movements (i.e. μ) not transpire.

Specifically, at any time τ during the term of the contract, the investor can choose to exercise the option, upon which he receives (payable immediately) the best prediction of the option payoff $h(S_T)$, given all the information up to the stopping time τ.

The best prediction is under the assumption that the drift of the underlying S for the remaining term of the contract is μ_c, the so-called contract drift which is specified at the start of the contract.

The British option definition (cont.)

Hence the payoff function of the early-exercise British option is given by

$$\text{payoff} = \mathbb{E}^{\mathrm{R}}\left[h(S_T)|\mathcal{F}_\tau\right],$$

where the expectation is taken with respect to a new probability measure R, under which the underlying asset evolves according to

$$dS_t = \mu_c S_t dt + \sigma S_t dW_t^{\mathrm{R}}.$$

The value of the contract drift μ_c is chosen by the holder to represent the level of protection (from adverse realised drifts) that the holder requires.

In essence, the effect of exercising is to substitute the true (unknown) drift of the stock price for the contract drift for the remaining term of the contract.

The British option definition (cont.)

Analogous with the American option, the no-arbitrage price of the British option is given by

$$V(t,s) = \sup_{t \le \tau \le T} \mathbb{E}^{\mathrm{Q}}_{t,s}\left[e^{-r(\tau - t)} \mathbb{E}^{\mathrm{R}}\left[h(S_T) | \mathcal{F}_\tau \right] \right],$$

i.e. the supremum over all stopping times τ (adapted to the filtration $\mathcal{F}_t$ generated by the process S_t) of the expected discounted future payoff.

In contrast with a standard American option, here the payoff function is now time-dependent (a consequence of optimal prediction).

The British option feature can be seen as a payoff generating mechanism.

The British put option

As a first example we consider briefly the British version of the put option. Its no-arbitrage price is given by

$$V(t,s) = \sup_{t \le \tau \le T} \mathbb{E}^{\mathrm{Q}}_{t,s} \left[e^{-r(\tau - t)} \mathbb{E}^{\mathrm{R}} \left[(K - S_T)^+ | \mathcal{F}_\tau \right] \right].$$

Stationary independent increments imply that

$$\mathbb{E}^{\mathrm{R}} \left[(K - S_T)^+ | \mathcal{F}_t \right] = K\Phi\left(\frac{\log(K/S_t) - (\mu_c - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) - S_t e^{\mu_c (T-t)} \Phi\left(\frac{\log(K/S_t) - (\mu_c + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) =: G(t, S_t),$$

hence the price of the British put option thus becomes

$$V(t,s) = \sup_{t \le \tau \le T} \mathbb{E}^{\mathrm{Q}}_{t,s} \left[e^{-r(\tau - t)} G(\tau, S_\tau) \right].$$

Path-dependent options

Here we introduce and examine the British payoff mechanism in the context of path-dependent options. More specifically lookback (Russian) options.

To retain relative tractability we start by investigating the simple case of a pure maximum lookback option with no strike (referred to as a Russian option).

Payoff functional

$$h(S_T) = \max_{0 \le v \le T} S_v = M_T \qquad \text{(Russian)}$$

The British Russian option

The payoff of the British Russian option at a given stopping time τ can be written as

$$\mathbb{E}^{\mathrm{R}}\left[M_T|\mathcal{F}_\tau\right].$$

Setting $M_t = \max_{0\leq v\leq t} S_v$ for $t \in [0, T]$ and using stationary and independent increments of W governing S we find that

$$\begin{aligned}
\mathbb{E}^{\mathrm{R}}\left[M_T|\mathcal{F}_t\right] &= \mathbb{E}^{\mathrm{R}}\left[S_t\left(\frac{M_t}{S_t} \vee \max_{t\leq v\leq T} \frac{S_v}{S_t}\right)|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathrm{R}}\left[S_t\left(\frac{M_t}{S_t} \vee M_{T-t}\right)|\mathcal{F}_t\right] \qquad \text{with } M_0 = 1 \\
&= S_t G^R\left(t, \frac{M_t}{S_t}\right)
\end{aligned}$$

where $G^R(t,x) = \mathbb{E}^{\mathrm{R}}\left[x \vee M_{T-t}\right]$ for $t \in [0, T]$ and $x \in [1, \infty)$.

The British Russian option (cont.)

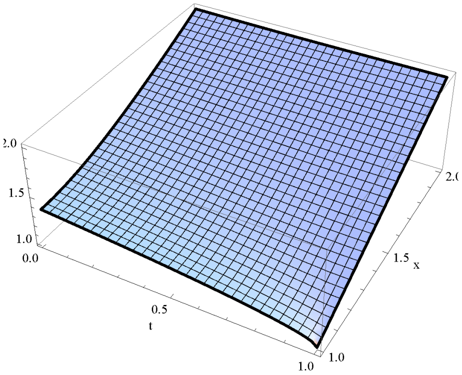
A lengthy calculation based on the known law of M_{T-t} under R shows that

$$\begin{aligned} G^R(t,x) =& x\Phi\left(\frac{\log x - (\mu_c - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &- \frac{\sigma^2}{2\mu_c} x^{2\mu_c/\sigma^2} \Phi\left(-\frac{\log x + (\mu_c - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &+ \left(1 + \frac{\sigma^2}{2\mu_c}\right) e^{\mu_c(T-t)} \Phi\left(-\frac{\log x - (\mu_c + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

for $t \in [0, T)$ and $x \in [1, \infty)$ where Φ is the standard normal distribution function given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy.$$

The British Russian option (cont.)



The British Russian *gain function* $G^R(t, x)$ for $\mu_c = -0.01$, $r = 0.1$, $\sigma = 0.4$ and $T = 1$.

The British Russian option (cont.)

Hence the no-arbitrage price of the British Russian option becomes

$$V(t, M_t, S_t) = \sup_{t \le \tau \le T} \mathbb{E}^{\mathrm{Q}} \left[e^{-r(\tau - t)} S_\tau G^R\big(\tau, \tfrac{M_\tau}{S_\tau}\big) \right].$$

The underlying Markov process in the optimal stopping problem above equals (t, M_t, S_t) thus making it three dimensional.

Due to the absence of a strike, we are able to reduce the dimensionality by performing an appropriate measure change and introducing the process

$$X_t = \frac{M_t}{S_t},$$

the ratio of the current maximum to the current price.

The British Russian option (cont.)

Hence the no-arbitrage price of the British Russian option becomes

$$V(t, M_t, S_t) = S_t \sup_{t \leq \tau \leq T} \mathbb{E}^{\hat{Q}} \left[G^R(\tau, X_\tau) \right] =: S_t V^R(t, X_t),$$

where Itô's formula gives

$$dX_t = -rX_t dt + \sigma X_t dW_t^{\hat{Q}} + dZ_t \quad (X_0 = x)$$

with $x \in [1, \infty)$, where $W_t^{\hat{Q}} = \sigma t - W_t^{Q}$ and $Z_t = \int_0^t I(X_v = 1) \frac{dM_v}{S_v}$. Note that 1 is an instantaneously reflecting boundary point.

Note that (from a PDE point of view) we are effectively making a symmetry reduction $V(t, M_t, S_t) = S_t V^R(t, \frac{M_t}{S_t}) = S_t V^R(t, X_t)$ where we now want to solve for $V^R(t, X_t)$.

A free-boundary problem representation

General optimal stopping theory can now be applied to this problem and analogous with the American option problem we have that

$$\mathcal{C} = \{(t,x) : V^R(t,x) > G^R(t,x)\} \quad \text{(continuation set)},$$
$$\mathcal{D} = \{(t,x) : V^R(t,x) = G^R(t,x)\} \quad \text{(stopping set)},$$

with the optimal stopping time defined as

$$\tau_* = \inf\{t \in [0,T] : X_t \in \mathcal{D}\},$$

i.e. the first time that the process X enters the stopping region. It can be shown that the stopping and continuation regions are separated by a smooth function $b^R(t)$, the early-exercise boundary, and hence $\mathcal{C} = \{(t,x) : x \in (1, b^R(t))\}$.

A free-boundary problem representation (cont.)

Applying standard optimal stopping and Markovian arguments, again analogous to the American put option, the problem can be conveniently expressed as the following free-boundary value problem:

$$\begin{cases} V^R_t + \frac{1}{2}\sigma^2 x^2 V^R_{xx} - rxV^R_x = 0 \text{ for } x \in (1, b^R(t)) \text{ and } t \in [0, T), \\ V^R(t, b^R(t)) = G^R(t, b^R(t)) \text{ for } t \in [0, T] \text{ (instantaneous stopping)}, \\ V^R_x(t, b^R(t)) = G^R_x(t, b^R(t)) \text{ for } t \in [0, T) \text{ (smooth fit)}, \\ V^R_x(t, 1+) = 0 \text{ for } t \in [0, T) \text{ (normal reflection)}, \end{cases}$$

where subscripts denote partial derivatives and the gain function $G^R(t, x)$ is as given previously.

A nonlinear integral representation

Theorem

The arbitrage-free price of the British Russian option admits the following early-exercise premium representation

$$V^R(t,x) = e^{-r(T-t)}\, G^R(t,x)|_{\mu_c=r} + \int_t^T J(t,x,v,b^R(v))\, dv$$

for all $(t,x) \in [0,T]\times[0,\infty)$. *Furthermore, the rational-exercise boundary of the British Russian option can be completely characterised as the unique continuous solution* $b^R : [0,T] \to \boldsymbol{R}_+$ *to the nonlinear integral equation*

$$G^R(t,b^R(t)) = e^{-r(T-t)}\, G^R(t,b^R(t))|_{\mu_c=r} + \int_t^T J(t,b^R(t),v,b^R(v))\, dv$$

for all $t \in [0,T]$.

A nonlinear integral representation (cont.)

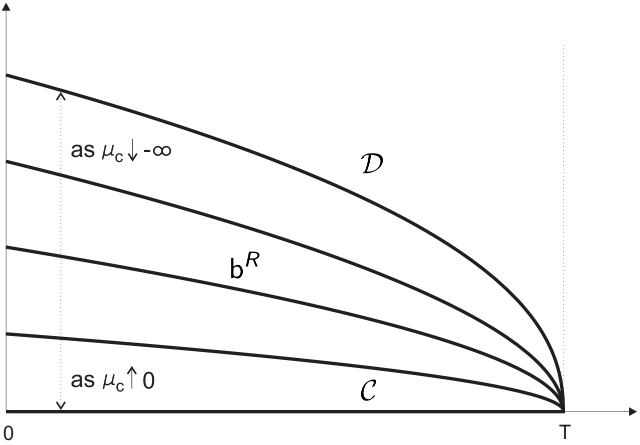
The probability density function of X (started at x at time t and ending at y at time v) under $\hat{\mathrm{Q}}$ is given

$$f^R(t,x,v,y) = \frac{1}{\sigma y\sqrt{v-t}}\Bigg[\varphi\Big(\frac{1}{\sigma\sqrt{v-t}}\Big[\log\frac{x}{y} - (r+\frac{\sigma^2}{2})(v-t)\Big]\Big) + x^{1+2r/\sigma^2}\varphi\Big(\frac{1}{\sigma\sqrt{v-t}}\Big[\log xy + (r+\frac{\sigma^2}{2})(v-t)\Big]\Big)\Bigg] + \frac{1+2r/\sigma^2}{y^{2(1+r/\sigma^2)}}\,\Phi\Big(-\frac{1}{\sigma\sqrt{v-t}}\Big[\log xy - (r+\frac{\sigma^2}{2})(v-t)\Big]\Big)$$

for $y \geq 1$ where φ is the standard normal density function given by $\varphi(x) = (1/\sqrt{2\pi})\,\mathrm{e}^{-x^2/2}$ for $x \in \boldsymbol{R}$.

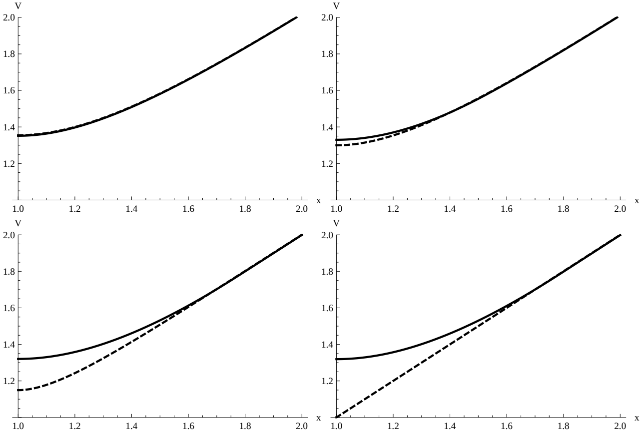
This is a complicated but well behaved, easily computable, function.

The British Russian early-exercise boundary



Note that the limiting case, as $\mu_c \downarrow -\infty$, is the well known (American) Russian early-exercise boundary.

The British Russian value function



The value function (at $t = 0$) of the British Russian option (in x-space) for $\mu_c = -0.01, -0.1, -0.5, -\infty$ with $r = 0.1$, $\sigma = 0.4$ and $T = 1$.

Financial analysis of option returns

We now address the following question:

What would the return on an option be if the underlying process entered a given region at a given time (and we exercised)?

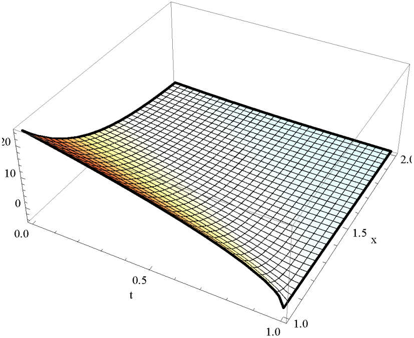
We call this a skeleton analysis of option returns since we do not discuss probabilities or risk associated with such events, these are placed under the subjective assessment of the option holder.

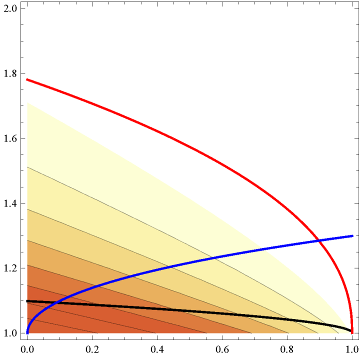
We define the return on an option i as

$$R^i(t,x)/100 = \frac{G^i(t,x)}{V^i(0,x_0)}$$

For the British Russian option, we draw comparisons with the standard (American) Russian option.

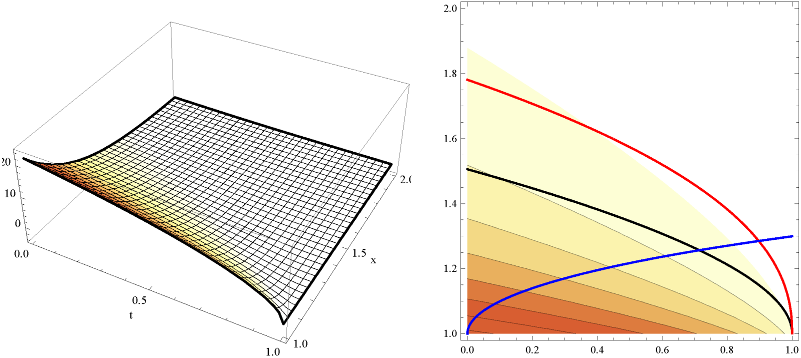
Financial analysis of the British Russian option





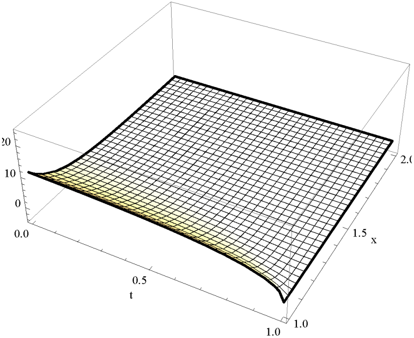
Difference in returns for $\mu_c = -0.01$, $r = 0.1$, $\sigma = 0.4$ and $T = 1$. Note that the British Russian option generally produced higher returns than than the (American) Russian option.

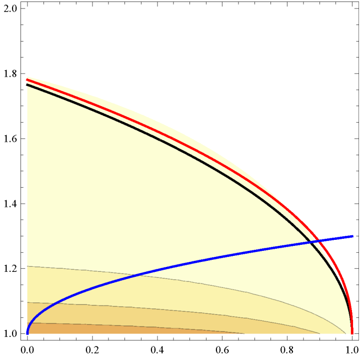
Financial analysis of the British Russian option (cont.)



Difference in returns for $\mu_c = -0.10$, $r = 0.1$, $\sigma = 0.4$ and $T = 1$. Note that the British Russian option generally produced higher returns than than the (American) Russian option.

Financial analysis of the British Russian option (cont.)





Difference in returns for $\mu_c = -0.50$, $r = 0.1$, $\sigma = 0.4$ and $T = 1$. Note that the British Russian option generally produced higher returns than than the (American) Russian option.

Conclusions

We have (*hopefully*):

- **Outlined the motivation** behind the introduction of the British option.
- **Extended** the British payoff mechanism to Path dependent options.
- **Formulated** the British Russian optimal stopping problem (arbitrage-free price).
- **Shown an equivalent** integral representation of the early-exercise boundary.
- **Solved** the associated free-boundary value problem to determine the optimal early-exercise boundary.
- **Provided** some preliminary financial analysis of the British Russian option returns, finding generally high returns.

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Thank you for your attention!