Pricing algorithm for swing options based on Fourier Cosine Expansions

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Outline

- Details of the swing option
  - Contract details
  - Pricing details
- Fourier Cosine algorithm for swing options
  - Recovery time–the penalty time between two consecutive exercises–plays an important role.
- Numerical results of option contracts varying in recovery time and upper bound of exercise.

swing options

- Swing options give contract holders the right to modify amounts of future delivery of certain commodities, such as electricity or gas.
- We deal with an American style swing options which can be exercised at any time before expiry and more than once, with the following restrictions
  - Recovery time between two consecutive exercises (τ). With exercise amount D we have $\tau_D = C$ or $\tau_D = f(D)$
  - Upper bound of exercise amount $|D|$: $|D| < L$

Payoff of swing option

Payoff of a swing option with varying S and D reads

$$g(S,T,D) = \quad D \cdot (\max(S-K_a,0) - \max(S-S_{max},0) + \max(K_d - S, 0) - \max(S_{min} - S, 0)),$$

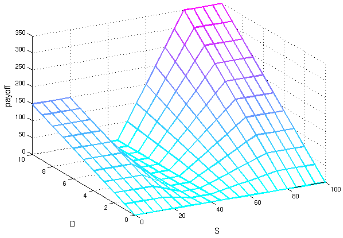


Figure: Example of a payoff of a swing option with $S_{min} = 20$, $K_d = 35$, $K_a = 45$, and $S_{max} = 80$, and S and D varying.

Pricing details

Recovery time $\tau_R(D)$ is assumed to be an increasing function of exercise amount D. The shortest recovery time is when we only exercise one amount of the swing option.

- If $T - t < \tau_R(1)$, it is impossible to exercise more than once before expiry. If profitable to exercise, then exercise at $D_{max} = L$ amount. Therefore we are dealing with an American type option which reads at each step

$$v(s,t) = max(g(s,t,L), c(s,t))$$

Pricing details

- If $T - t \geq \tau_R(1)$, there exists multiple exercise opportunities before expiry. Apart from the optimal exercise time we also need to find the optimal exercise amount at each time step:

$$v(s,t) = \max_D(max(g(s,t,D) + \phi_D^{'}(s,t), c(s,t)))$$

Here $g(s,t,D)$ is the instantaneous profit obtained from the exercise of a swing option and $\phi_D^{'}$ is the continuation value from $t + \tau_R(D)$.

Option pricing based on Fourier Cosine expansions

Truncating the infinite integration range of the Risk-Neutral formula

$$v(x, t_0) = e^{-r\Delta t} \int_a^b v(y, T) f(y|x) dy$$

The conditional density function of the underlying is approximated as follows:

$$f(y|x) \approx \frac{2}{b-a} {\sum_{k=0}^{N-1}}' Re(\varphi(\frac{k\pi}{b-a}; x) \exp(-i\frac{ak\pi}{b-a})) \cos(k\pi\frac{y-a}{b-a}),$$

Replacing $f(y|x)$ by its approximation and interchanging integration and summation, we obtain the COS algorithm for option pricing

$$v(x, t_0) = e^{-r\Delta t} {\sum_{k=0}^{N-1}}' Re(\phi(\frac{k\pi}{b-a}; x) e^{-ik\pi\frac{a}{b-a}}) V_k$$

where V_k is the Fourier Cosine coefficient of option value $v(y, T)$.

Pricing outline for $t : T - t < \tau_R(1)$

The swing option is equivalent to an American option, which is can be obtained from Bermudan option values with diffent numbers of exercise dates, i.e. a 4–point repeated Richardson extrapolation.

Pricing algorithm

- Initialization: Compute $V_k(t_M)$ at $t_M = T$.
- Backward recursion: For $m = M - 1, \cdots, 1$, recover $V_k(t_m) = \frac{2}{b-a} \int_a^b v(x, t_m) cos(k\pi \frac{x-a}{b-a}) dx$ from $V_k(t_{m+1})$, where $v(x, t_m) = max(g(x, t_m), c(x, t_m))$.
- Last step: $v(x, t_0) = e^{-r(t_1 - t_0)} {\sum'}_{k=0}^{N-1} Re(\phi(\frac{k\pi}{b-a}; x) e^{-ik\pi \frac{a}{b-a}}) V_k(t_1)$

In our implementation we set $x = log(s)$.

At expiry

At $t_{\mathcal{M}} = T$ option value v equals the payoff g and we have for the Fourier cosine coefficients of the swing option value:

$$V_k(t_{\mathcal{M}}) \quad = \quad G_k(a, ln(K_d), L) + G_k(ln(K_a), b, L),$$

where

$$G_k(x_1, x_2, L) = \frac{2}{b-a} \int_{x_1}^{x_2} g(x, t_{\mathcal{M}}, L) \cos(k\pi \frac{x-a}{b-a}) dx$$

is the Fourier cosine coefficient of the swing option payoff which has analytic solution.

Backward Recursion

At each time step $t_m, m = M-1, \cdots, 1$

$$V_k(t_m) = \frac{2}{b-a}\int_a^b v(x,t_m)cos(k\pi\frac{x-a}{b-a})dx$$

where $v(x,t_m) = max(g(x,t_m), c(x,t_m))$. We identify the regions where $v = c$ and those where $v = g$ and split V_k accordingly.
By Newton's method we find the early exercise points where $c = g$. For swing options there are two early exercise points, x_m^d and x_m^a.
$V_k(t_m)$ can be split as

$$V_k(t_m) = G_k(a, x_m^d, D) + C_k(x_m^d, x_m^a, t_m) + G_k(x_m^a, b, D),$$

where C_k and G_k are the Fourier cosine coefficients of the continuation value and swing option payoff.

Calculation of G_k and C_k

At each time step $t_m, m = M-1, \cdots, 1$, we have

- G_k has analytic solution with computation complexity $O(N)$.
- C_k can be rewritten as a matrix-vector product representation:

$$\mathbf{C}(x_1, x_2, t_m) = \frac{e^{-r\Delta t}}{\pi} \mathrm{Im}\left\{(M_c + M_s)\mathbf{u}\right\},$$

For Lévy processes the matrices M_s and M_c have a Toeplitz and Hankel structure, respectively and C_k can be calculated with the help of the Fast Fourier Transform, with computation complexity $O(N log_2 N)$. For other processes, C_k is calculated with computation complexity of $O(N^2)$.

Pricing algorithm for $t : T - t \geq \tau_R(1)$

In the interval $\{t : T - t > \tau_R(1)\}$ the swing option can be exercised more than once before expiry and recovery time plays an important role. In this case we have

$$v(x,t) = \max(\max_D g(x,t,D) + \phi_D^t(x,t), c(x,t))$$

It is an American-style option with recovery time and multiple exercise opportunities.

- Due to recovery time, the payoff also includes $\phi_D^t(x,t)$, the continuation value from $t + \tau_R(D)$.
- Due to multiple exercise opportunities, we take the maximum over the resulting payoff for all possible values of D, and the continuation value from the previous time step.

Backward Recursion

With

- A_D, $D = 1, \cdots, L$ is the regions in which exercising the swing option with D commodity units results in the highest profit $g(x, t_m, D) + \phi_D^{t_m}(x, t_m)$.
- A_c is the region in which $c(x, t)$ is the maximum. In other words, with the commodity price in A_c, it is profitable not to exercise the swing option.

Then for $m = M - 1, \cdots, 1$,

$$\begin{aligned} V_k(t_m) &= \frac{2}{b-a}\Big(\int_{A_c} c(x, t_{m+1})\cos(\frac{k\pi(x-a)}{b-a})dx \\ &+ \sum_{D=1}^{L}\int_{A_D} g(x, t_m, D) + \phi_D^{t_m}(x, t_m)\cos(\frac{k\pi(x-a)}{b-a})dx\Big) \end{aligned}$$

And $v(x, t_0) = e^{-r(t_1 - t_0)}{\sum{}'}_{k=0}^{N-1} Re(\phi(\frac{k\pi}{b-a}; x)e^{-ik\pi\frac{a}{b-a}})V_k(t_1)$

Calculation of V_k

At each time step t_m, $m = M-1, \cdots, 1$, $V_k(t_m)$ can be rewritten as:

$$\begin{aligned} V_k(t_m) &= \frac{2}{b-a}\Bigg(\int_{A_c} c(x, t_{m+1}) \cos(\frac{k\pi(x-a)}{b-a})dx \\ &+ \sum_{D=1}^{L}\int_{A_D} g(x, t_m, D) \cos(\frac{k\pi(x-a)}{b-a})dx \\ &+ \sum_{D=1}^{L}\int_{A_D} \phi_D^{t_m}(x, t_m) \cos(\frac{k\pi(x-a)}{b-a})dx\Bigg) \\ &\triangleq V_c + V_g + V_\phi \end{aligned}$$

- A_D, $D = 1, \cdots, L$, and A_c are determined by Newton's method.
- V_c and V_g are calculated the same way as G_k and C_k.
- V_ϕ is calculated similarly as G_c, but from $V_k(t_m + \tau_R(D))$ instead of $V_k(t_{m+1})$. This implies we need to store intermediate values of V_k.

Constant recovery time

In this case additional profit is not connected to an extra penalty. We have either $D = L$ or $D = 0$. Two early-exercise points x_m^d and x_m^a are to be determined, so that

$$c(x_m^d, t_m) = g(x_m^d, t_m, L) + \phi_L^{t_m}(x_m^d, t_m),$$

and

$$c(x_m^a, t_m) = g(x_m^a, t_m, L) + \phi_L^{t_m}(x_m^a, t_m),$$

And $V_k(t_m)$ is split into three parts,

$$V_k(t_m) = G_k(a, x_m^d, L) + C_k(x_m^d, x_m^a, t_m) + G_k(x_m^a, b, L).$$

Numerical results

We discuss two types of recovery time functions:

- Constant recovery time: If $D \neq 0$, we set $\tau_R(D, t) = \frac{1}{4}$. In other words, the option holder needs to wait three months between two consecutive swing actions, independent of the time point of exercise or the size D.
- State-dependent recovery time: We assume $\tau_R(D, t) = \frac{D}{12}$ which implies that if the option holder exercises the swing option with D units, he/she has to wait D months before the option can be exercised again.

In our numerical examples presented here, the underlying follows the CGMY model (exponential Lévy process) with $Y = 1.5$.

Convergence over M and estimation of American option

Two approximation methods are compared:

- Direct approximation: Bermudan-style options with $\mathcal{M} = N/2$.
- Richardson 4-point extrapolation technique.

$n = \log_2 N$	$P(N/2)$		Richardson	
	option value	CPU time	option value	CPU time
7	137.423	0.27	137.395	0.59
8	137.408	0.53	137.390	0.99
9	137.399	2.00	137.390	1.79
10	137.394	8.39	137.390	3.40
11	137.392	39.55	137.390	6.68
12	137.391	203.27	137.390	13.21

Table: Convergence over $\mathcal{M}$ and comparison between two approximation methods for American-style swing option, with $t = T - 0.5$, $\tau_R(D) = 0.25$, $C = 1$, $G = 5$, $M = 5$, $Y = 1.5$

American style swing option value

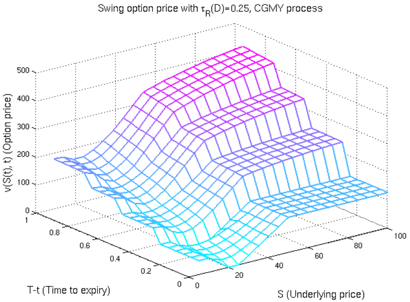
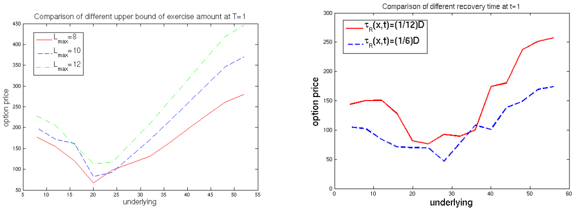


Figure: American-style swing option values under the CGMY processes with constant recovery time, $\tau_R(D) = 0.25$.

Jumps are observed at $T - t = 0.25$, $T - t = 0.5$ and $T - t = 0.75$, where the maximum number of remaining exercise possibilities is reduced by 1.

Swing contracts with varying flexibility



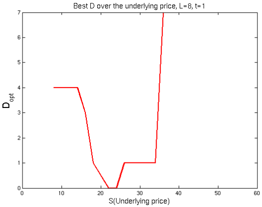
(a) Varying upper bound L (b) Varying recovery time $\tau_R(D)$

Figure: CGMY process, $T - t = 1$; Left: Different values for L, and fixed $\tau_R(D, t) = \frac{1}{12}D$; Right: Different Recovery time, and fixed $L = 5$.

- Higher values of L give rise to higher option values.
- Longer recovery time gives lower option prices

The optimal exercise amount D_{opt}

Below is a figure of D_{opt} over different underlying prices, with $\tau_R(D) = \frac{1}{12}D$.



- As S goes beyond K_d and K_a, D_{opt} tends to increase, because in this region instantaneous profit $g(x,t,D)$ tends to dominate in the payoff $g(x,t,D) + \phi_D^t(x,t)$.
- Between $S = 20$ and $S = 25$, $D_{opt} = 0$, since $g(x,t,D) = 0$ for all $D > 0$ in this interval.

Convergence of the algorithm

With N the number of Fourier Cosine expansion terms, and L the upper bound of exercise amount,

	N	256	512
L=2	option price	136.8724	136.8724
	CPU time	0.1669	0.2466
L=5	option price	150.0041	150.0041
	CPU time	0.6505	1.1660
L=10	option price	199.6870	199.6870
	CPU time	2.4115	4.3819

- With $N = 256$ the swing option algorithm reaches basis point accuracy.
- The algorithm is flexible regarding the variation in parameter L.

Conclusions

- We presented an efficient pricing algorithm for swing options with early-exercise features, based on Fourier Cosine Expansions.
- It performs well for different swing contracts with varying flexibility in upper bounds of exercise amount and different recovery times.
- For Lévy processes the Fast Fourier Transform can be applied in the backward recursion procedure, which gives us Bermudan-style swing option prices accurate to one basis point in milli-seconds for constant recovery time, and in less than one, to three seconds for dynamic recovery time with different values of L.
- The Richardson 4-point extrapolation technique is efficient in pricing American-style swing options.