Three make a dynamic smile
unspanned skewness and interacting volatility components in option valuation

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Introduction

Three questions

- ☐ How many **sources of dynamic risk** can we identify in index options?
  What is the empirical evidence?
- ☐ How can we **conveniently model** the multiple risk sources in an affine framework?
  And thus account for the empirical evidence?
- ☐ How can we **improve** on existing benchmark models?
  And put the model to an empirical test?

DPS-type affine models

Duffie, Pan, Singleton (DPS, 2000): Transform analysis and asset pricing for affine jump-diffusions
Bates (2000): Post-'87 Crash Fears in the S&P 500 Futures Option Market
Christoffersen et. al (2009): The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work so Well

Affine models with matrix jump diffusions

Leippold, Trojani (wp, 2008): Asset pricing with Matrix Jump Diffusions
Cuchiero, Filipovic, Mayerhofer, Teichmann (2010): Finite Processes on Positive Semidefinite Matrices
Gourieroux, Sufana (wp 2004): Derivative Pricing with Multivariate Stochastic Volatility: Application to Credit Risk
da Fonseca, Grasselli, Tebaldi (2008): A Multifactor Volatility Heston Model

Alternative (affine) multifactor models

Muhle-Karb, Pfaffel, Stelzer (wp, 2010): Option pricing in multivariate stochastic volatility models of OU type
Carr, Wu (wp, 2009): Leverage Effect, Volatility Feedback, and Self-Exciting Market Disruptions: Disentangling the Multi-dimensional Variations in S&P500 Index Options

Empirical evidence

Sample	calls only
Time frame	1996-Sept/2008
Sampling interval	daily
Trading days	3205
Total number of observations	546’971
Average time to maturity	145 days [10d $\sim$ 1yr]
Average moneyness (S/K)	1.05
Data processing	Bakshi(1997), **no cuts**

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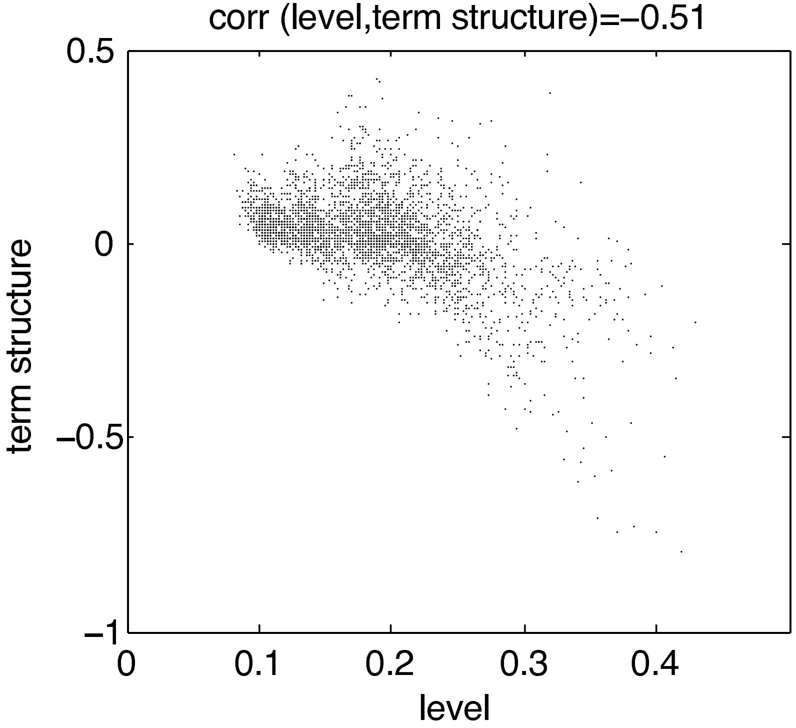
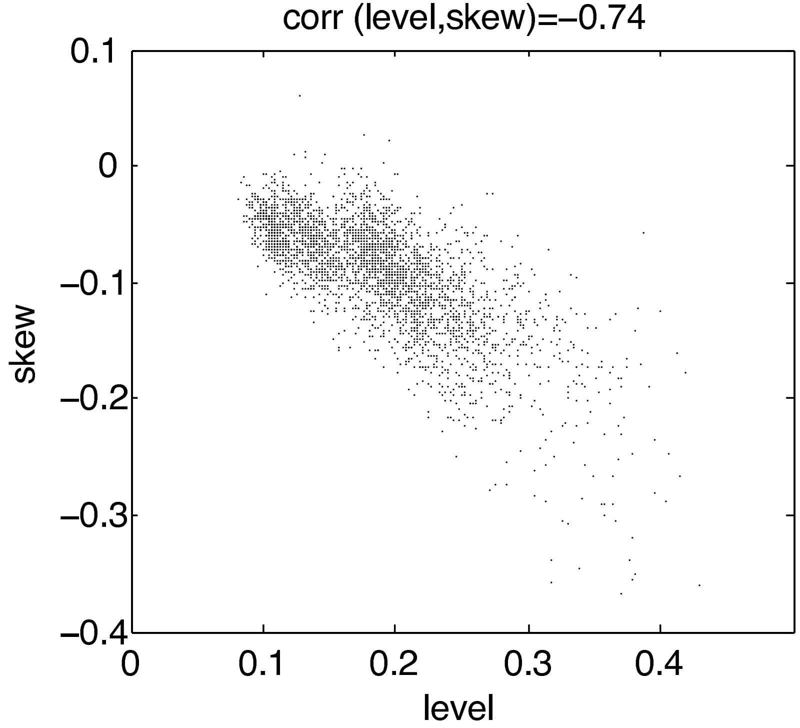
Analytical framework: economically significant factors

level	V_t	$IV(ATM, \tau = 30d)$	
skew	$\mathcal{S}_t$	$[IV(\Delta = 0.4) - IV(\Delta = 0.6)] \cdot \frac{1}{(0.4-0.6)}$	$\tau = 30d$
term struct.	$\mathcal{M}_t$	$[IV(\tau = 90d) - IV(\tau = 30d)] \cdot \frac{360}{(90-30)}$	ATM

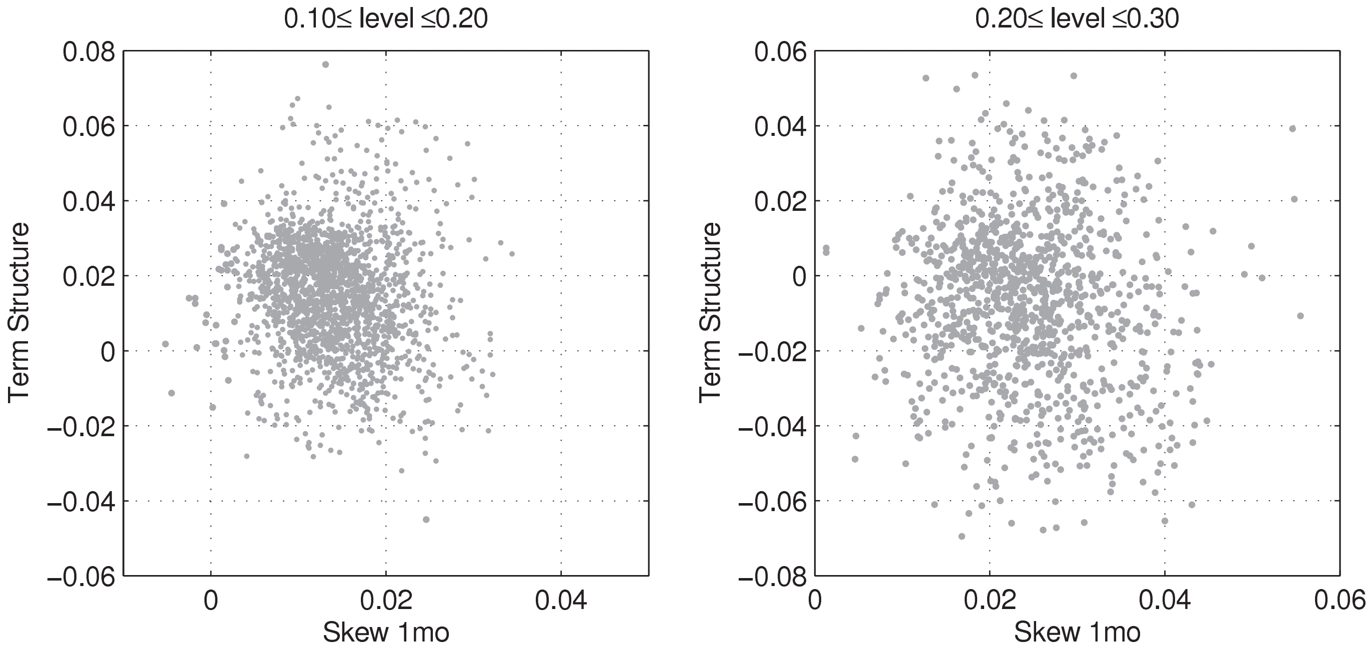
Level effects are dominant





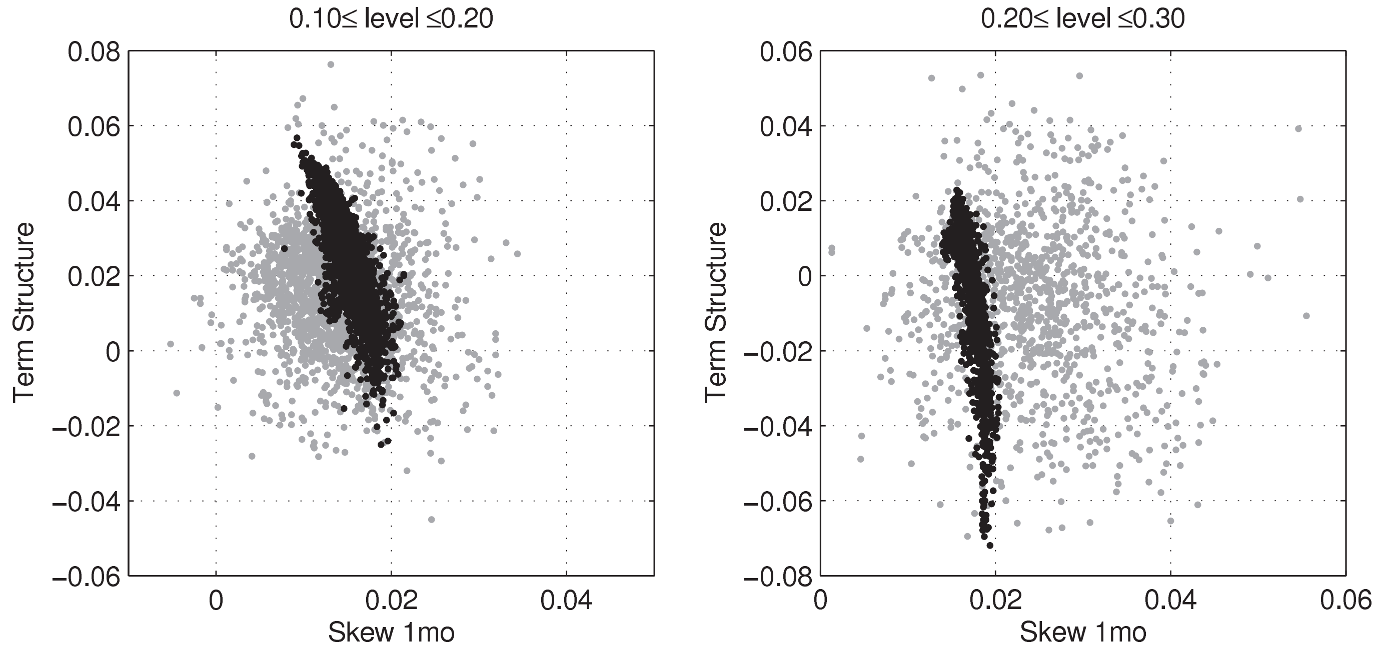
Skewness and term structure **unconditionally** highly correlated to level → level masks more nuanced effects.

Empirical evidence – unspanning



Considerable variation in skewness and term structure that is not spanned by the volatility level.

Empirical evidence – unspanning (2)



Standard two-factor affine models cannot capture **both** unspanned skewness and term structure components

Empirical evidence – factors

Principal component analysis (% of variance explained)

	l	PC 1	PC 2	PC 3	PC 4	
Unconditional	2	96.8	1.9	0.9	0.1	$T = 3206$

Principal component analysis (% of variance explained)

	l	PC 1	PC 2	PC 3	PC 4	
Unconditional	2	96.8	1.9	0.9	0.1	$T = 3206$
$0.08 < V_t \leq 0.13$	3	84.5	6.9	5.6	0.8	$T = 641$
$0.13 < V_t \leq 0.17$	3	84.8	7.1	5.9	0.7	$T = 641$
$0.17 < V_t \leq 0.2$	3	75.4	12.3	8.4	1.3	$T = 641$
$0.20 < V_t \leq 0.23$	3	74.7	12.0	8.6	1.77	$T = 641$
$0.23 < V_t \leq 0.54$	3	87.2	8.7	2.6	0.6	$T = 641$

l = significant components according to mean eigenvalue criterion.
($N = 56$, threshold=$\frac{1}{56}$=1.79%)

Model

Bates-like independent factors – $SV(J)_{3,0}$

$$\frac{dS_t}{S_t} = (r - q - \lambda_t \overline{k})dt + \sqrt{v_{1t}}dz_{1t} + \sqrt{v_{2t}}dz_{2t} + \sqrt{\mathbf{v_{3t}}}\mathbf{dz_{3t}} + kdN_t \quad (1)$$

$$dv_{it} = (\alpha_i - \beta_i v_{it})\,dt + \sigma_i \sqrt{v_{it}} dw_{it} \quad i = 1, 2, 3 \quad (2)$$

Affine Matrix Jump Diffusion – $SV(J)_{3,1}$

$$\frac{dS_t}{S_t} = (r - q - \lambda_t \overline{k})dt + \mathbf{tr}(\sqrt{\mathbf{X_t}}\mathbf{dZ_t}) + kdN_t \quad (3)$$

$$dX_t = [\Omega\Omega' + MX_t + X_t M']dt + \sqrt{X_t}dB_t Q + Q'dB_t'\sqrt{X_t} \quad (4)$$

- X_t is a (2×2) symmetric, pos.def. matrix-valued process
- Interactions for M, Q *not* diagonal

Properties (diffusive part)

stoch. volatility	$V_t := var(\frac{dS_t}{S_t}) = tr[X_t] = X_{11} + X_{22}$
stoch. leverage effect	$cov(\frac{dS_t}{S_t}, dV_t) = 2tr[R'QX_t]$
stoch. persistence	$\frac{1}{dt}E[dV_t] = tr[\Omega\Omega'] + 2tr[MX_t]$

Natural mapping to observable, economically important quantities (Karoui, Durrleman wp 2007)

level	$\sqrt{V_t}$	$=$	$\sqrt{tr[X_t]}$
skew	$\mathcal{S}_t$	$=$	$\frac{1}{2}\frac{tr[RQX_t]}{tr[X_t]^{3/2}}$
term struct	$\mathcal{M}_t$	$\approx$	$\frac{1}{2}\frac{tr[MX_t]}{tr[X_t]^{1/2}}$

Spectral state decomposition

Aim: **Separate** volatility effect from unspanned skewness/term structure

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{12} & X_{22} \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \mathcal{V}_{1,t} & 0 \\ 0 & \mathcal{V}_{2,t} \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

$$(X_{11}, X_{12}, X_{22}) \longrightarrow (V_t, \xi_t, \alpha_t) \qquad V_t = tr[X_t] = \mathcal{V}_{1,t} + \mathcal{V}_{2,t}; \quad \xi = \frac{\mathcal{V}_{1,t}}{\mathcal{V}_{1,t} + \mathcal{V}_{2,t}}$$

Bounded: $\xi[0,1]; \alpha[0,\pi]$

Decompose expressions of the type $tr[AX_t]$:

$$Tr[AX_t] = \frac{V_t}{2}\left[Tr(A) + \underbrace{(2\xi_t - 1)}_{\text{scale}} \underbrace{\Big(\cos(2\alpha_t)(A_{11} - A_{22}) + \sin(2\alpha_t)(A_{12} + A_{21})\Big)}_{\text{direction}}\right]$$

Application: Illustrate unspanned skewness/term structure components via an approximation of the short term volatility surface

$$\mathcal{S}_t \propto [RQX_t]$$
$$\mathcal{M}_t \propto [MX_t]$$

Illustration of state decomposition

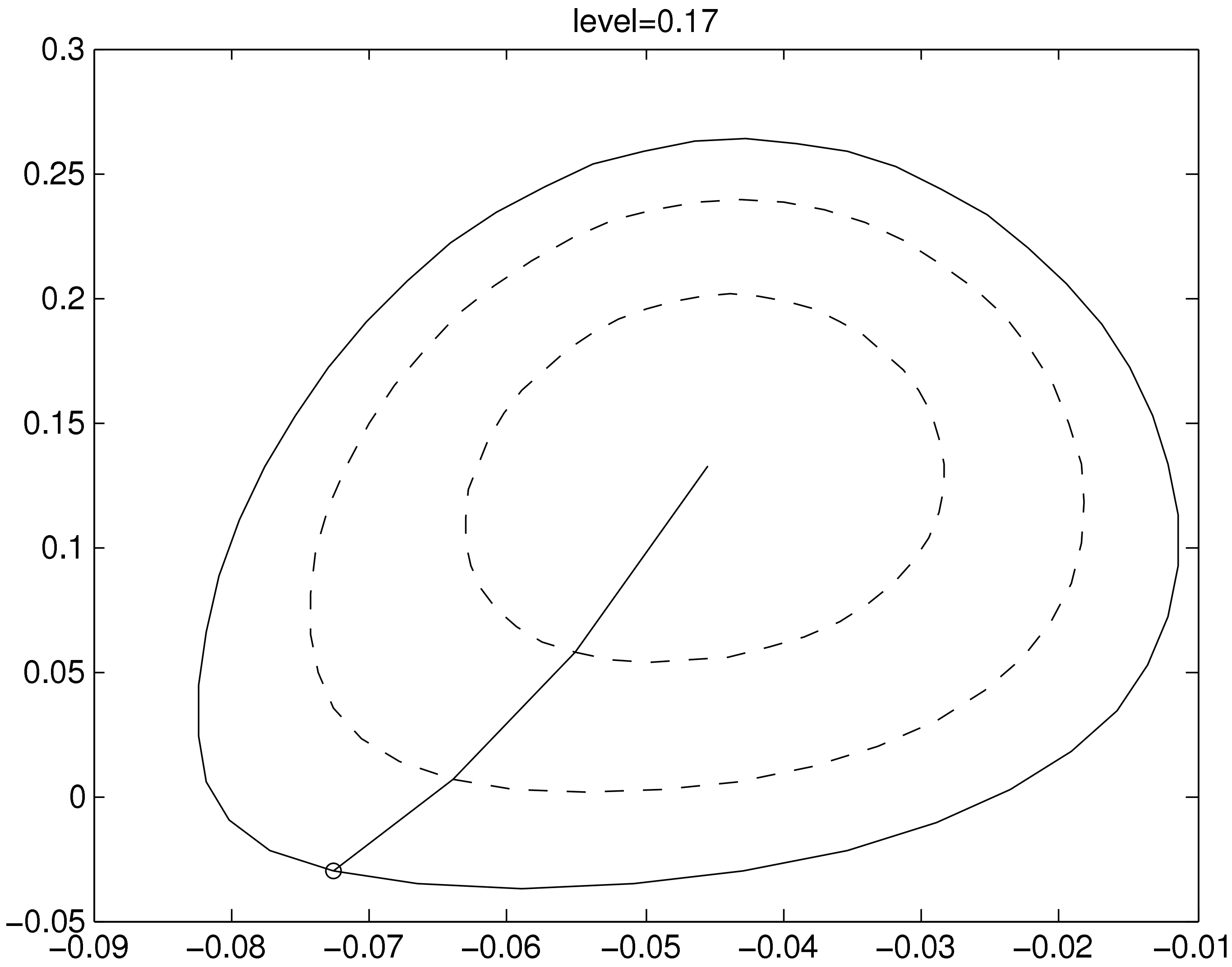


Illustration of state decomposition

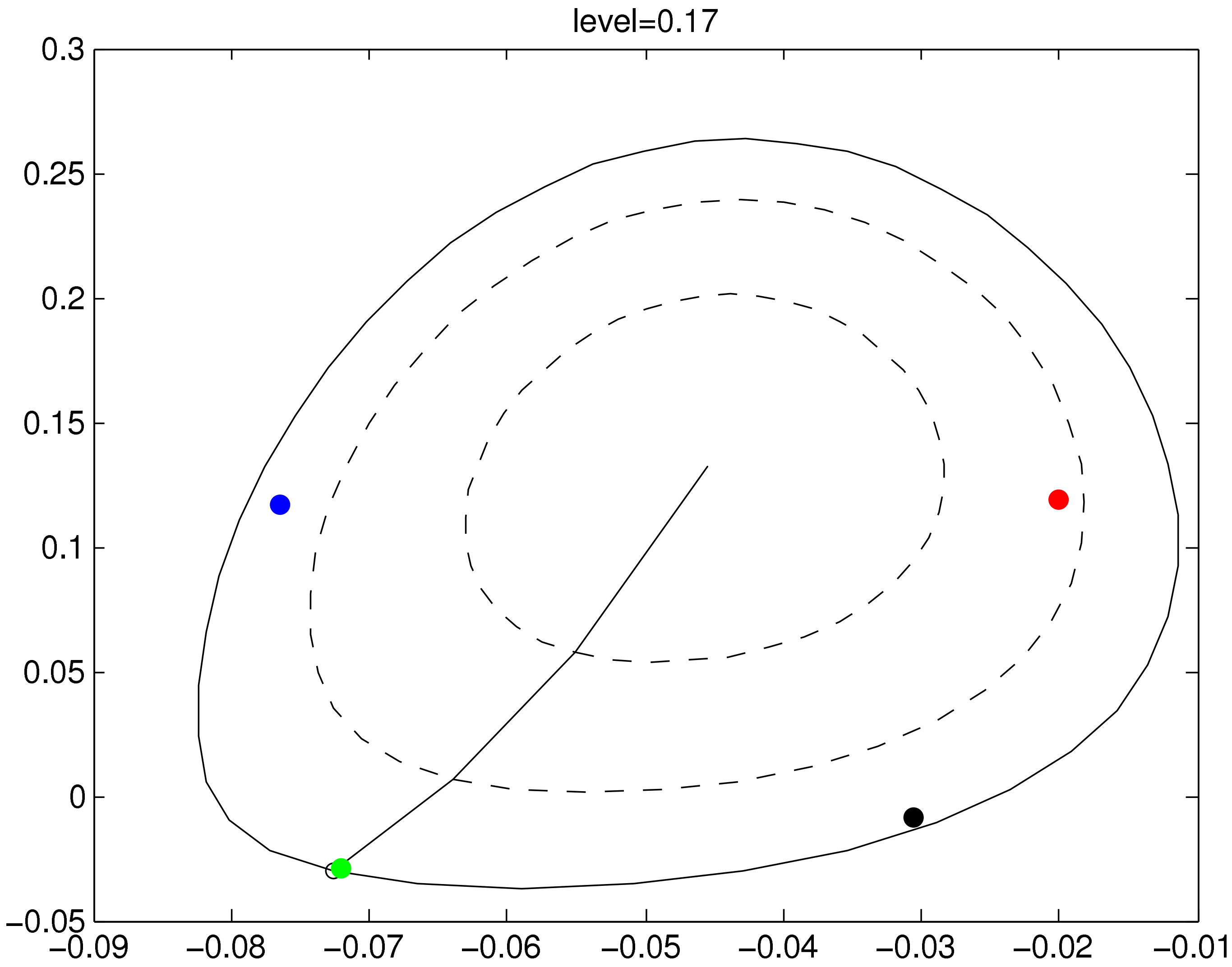
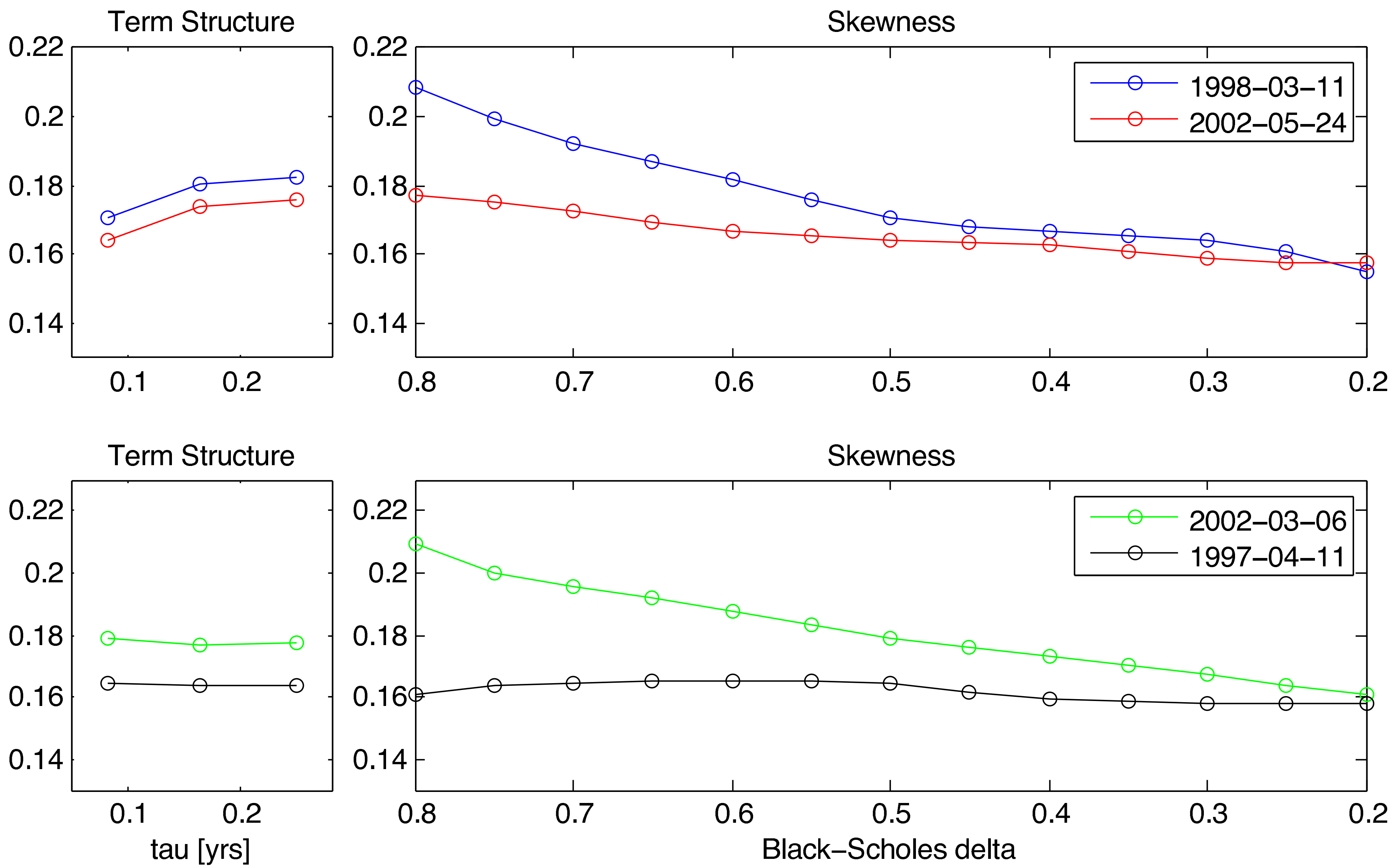


Illustration of state decomposition (2)



Option pricing with (affine) Laplace transform

$$\Psi(\tau;\gamma) := E_t\left[\exp\left(\gamma Y_T\right)\right] = \exp\Big(\gamma Y_t + tr\big[A(\tau)X_t\big] + B(\tau)\Big) \qquad (5)$$

where $A(\tau) = C_{22}(\tau)^{-1}C_{21}(\tau)$ with the 2×2 matrices $C_{ij}(\tau)$:

$$\begin{pmatrix} C_{11}(\tau) & C_{12}(\tau) \\ C_{21}(\tau) & C_{22}(\tau) \end{pmatrix} = \exp\left[\tau \begin{pmatrix} M + \gamma Q'R & -2Q'Q \\ C_0(\gamma) & -(M' + \gamma R'Q) \end{pmatrix}\right] \qquad (6)$$

$$C_0(\gamma) = \frac{\gamma(\gamma-1)}{2} I_2 + \Lambda\left[(1+\overline{k})^{\gamma}\exp\left(\gamma(\gamma-1)\frac{\delta^2}{2}\right) - 1 - \gamma\overline{k}\right] \qquad (7)$$

$$\begin{aligned} B(\tau) \quad = \quad & \left\{r - q + \lambda_0\left[(1+\overline{k})^{\gamma}\exp\left(\gamma(\gamma-1)\frac{\delta^2}{2}\right) - 1 - \gamma\overline{k}\right]\right\}\tau \\ & -\frac{\beta}{2} tr[\log C_{22}(\tau) - \tau(M' + R'Q)] \end{aligned} \qquad (8)$$

See Leippold/Trojani wp 2008

Estimation strategy

- ☐ Sub-sample: 59 monthly observations (2000-2004)
  Challenging, avoid over-fitting
- ☐ Cross-section only; risk-neutral pricing
- ☐ Nested optimum
- ☐ Max. likelihood like Bates(2000), correct for heteroskedasticity

Parameter estimate

$$\widehat{\theta} = \arg\max_{\theta} \; -\frac{1}{2} \sum_t \left(\ln|\Omega_t| + \mathbf{e}_t' \, \Omega_t^{-1} \, \mathbf{e}_t \right) \tag{9}$$

$\mathbf{e}_t(\theta, X_t^*(\theta))$=relative pricing error, Ω_t = conditional cov. matrix of $e_{i,t}$

Implied state by NLS

$$X_t^*(\theta) = \arg\min_{\{X_t\}} \left(\widehat{C}_i(\theta, X_t) - C_i \right)^2 \tag{10}$$

Performance

Pricing

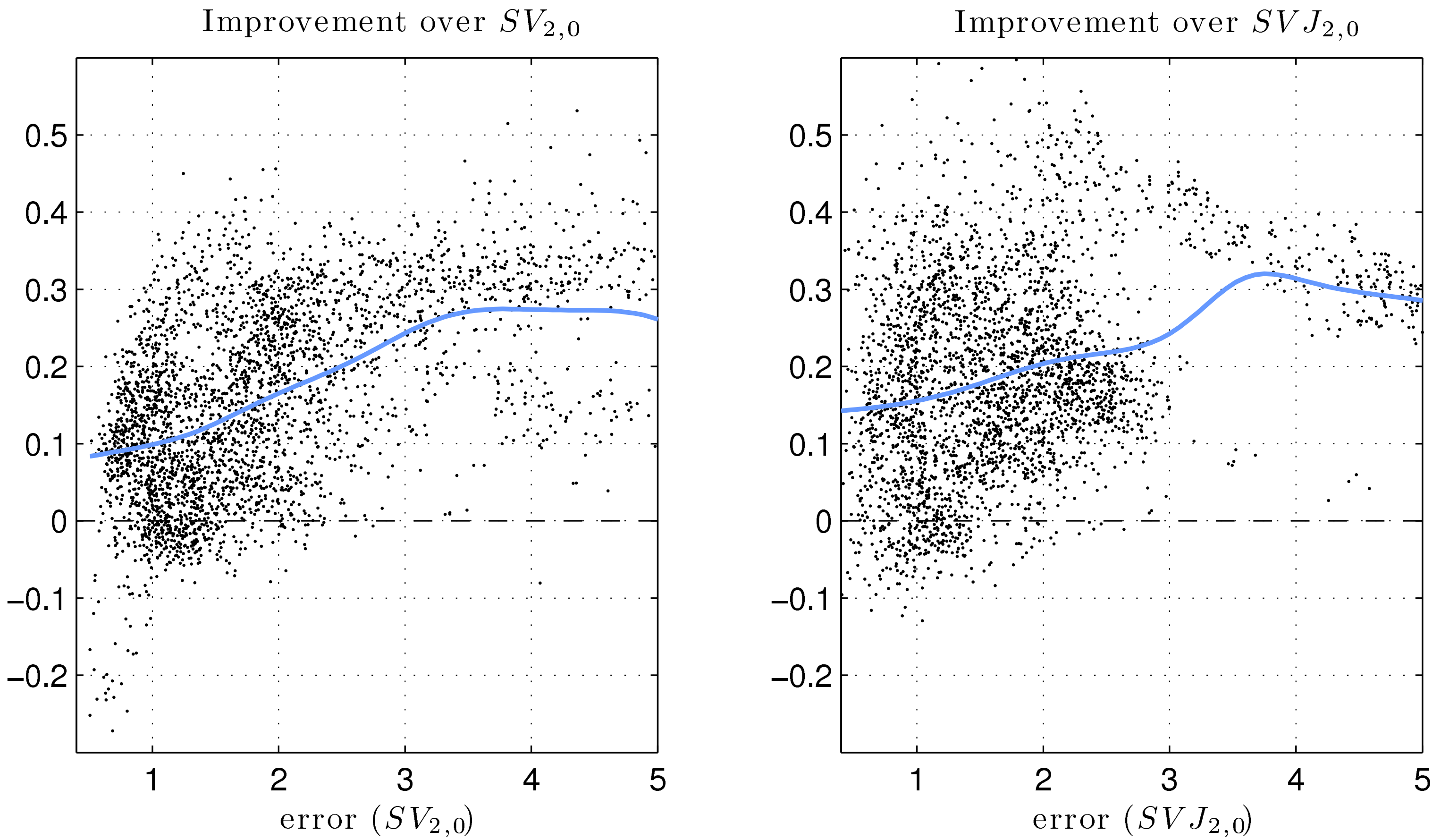
In sample (2000-2004, monthly)

	$SV_{2,0}$	$SV_{3,0}$	$SV_{3,1}$	$SVJ_{2,0}$	$SVJ_{3,1}$
State variables	2	3	3	2	3
$rms\$E$	1.180	1.127	1.048	1.115	0.913
(stdv)	(0.370)	(0.348)	(0.285)	(0.446)	(0.324)
Within bid-ask	0.603	0.617	0.640	0.635	0.633

Full sample (1996-09/2008)

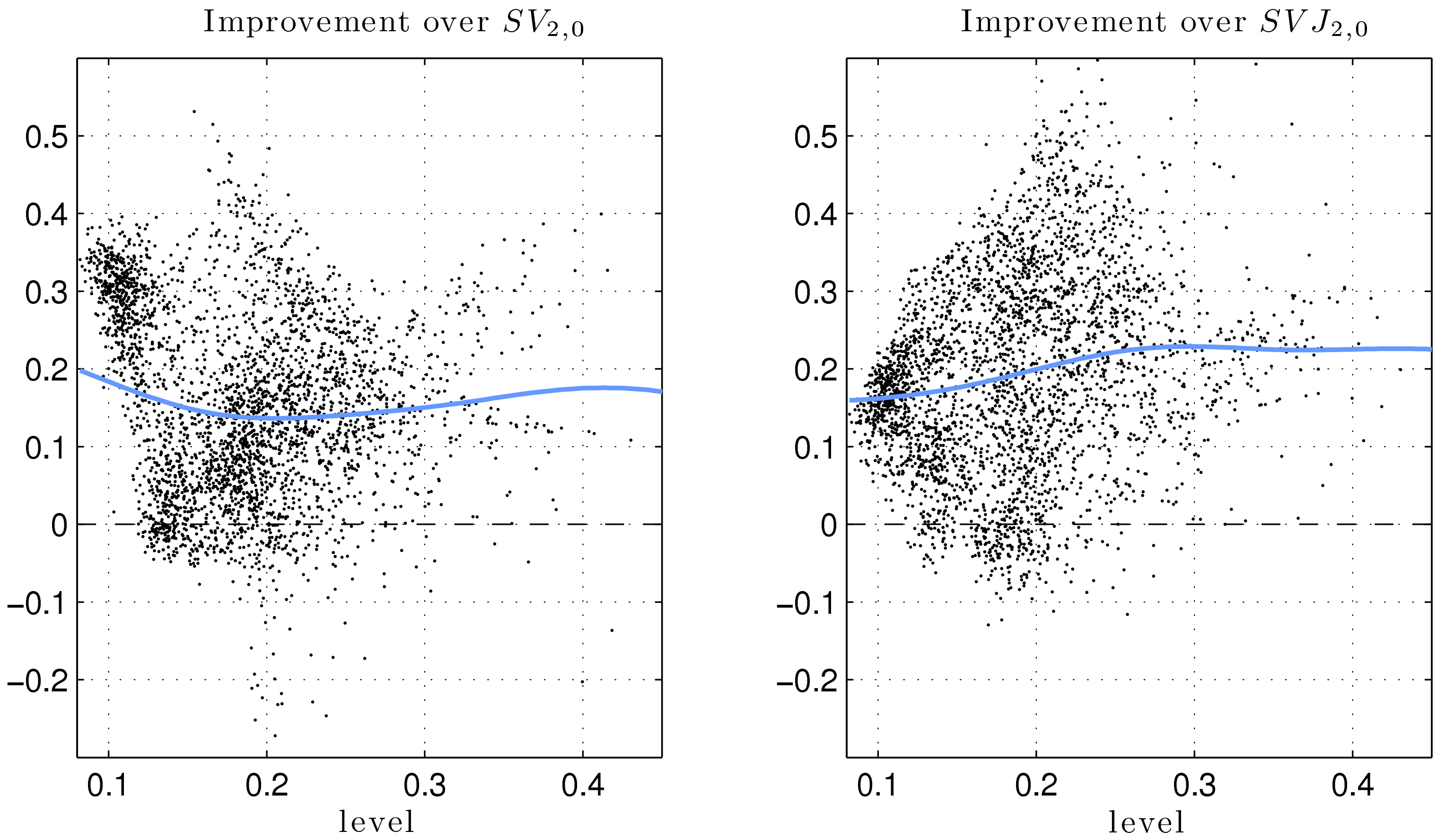
	$SV_{2,0}$	$SV_{3,0}$	$SV_{3,1}$	$SVJ_{2,0}$	$SVJ_{3,1}$
State variables	2	3	3	2	3
$rms\$E$	1.937	2.057	1.570	1.862	1.457
(stdv)	(1.101)	(1.727)	(0.808)	(1.129)	(0.809)
$rmsIVE$	2.69	2.61	2.60	3.18	2.36
Within bid-ask	0.437	0.461	0.540	0.452	0.527

Improvements by pricing error of the 2-factor model



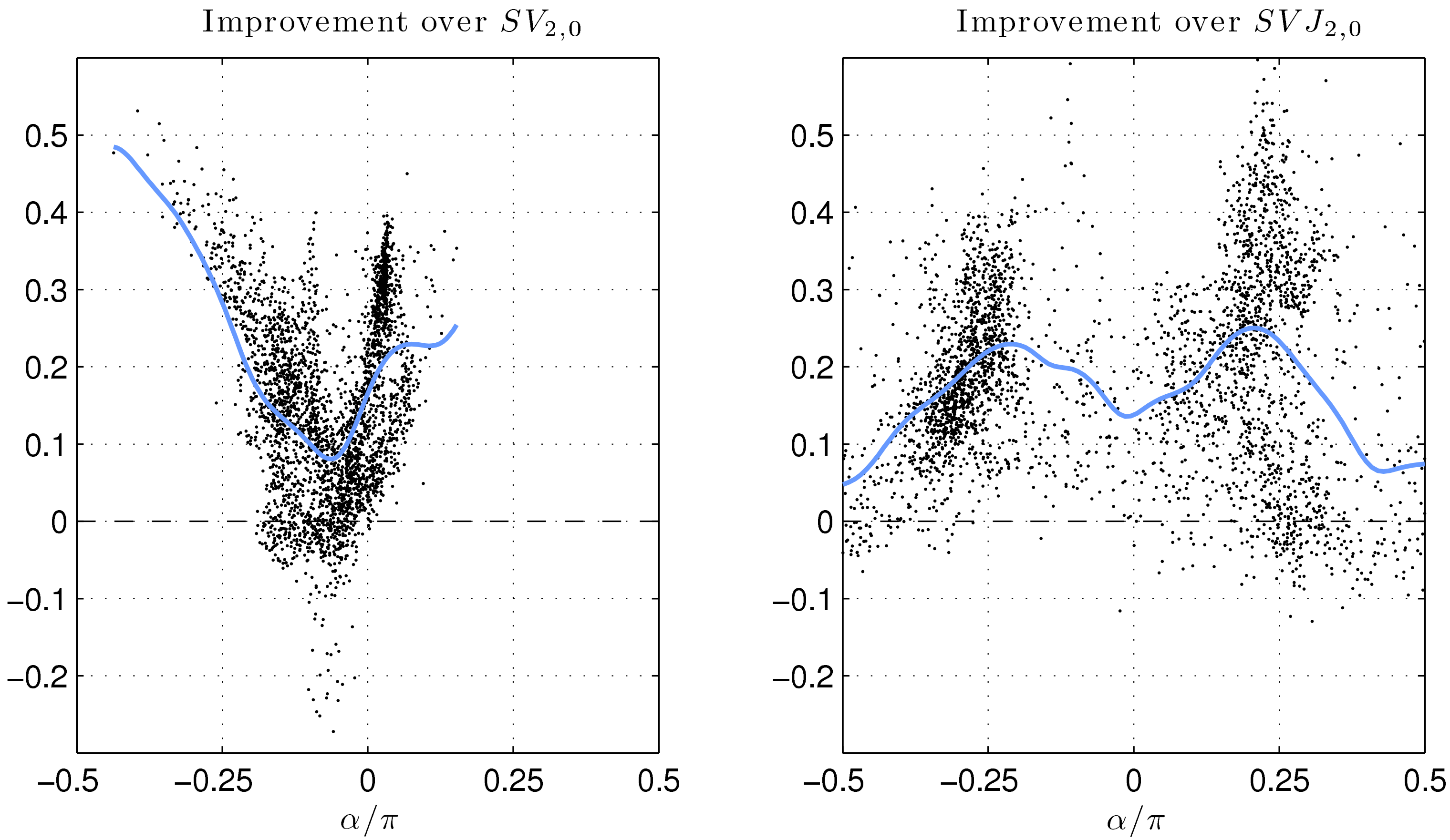
$$\text{Improvement} = \frac{\epsilon_{SV2,0} - \epsilon_{SV3,1}}{\epsilon_{SV2,0}}$$

Improvements by volatility level



$$\text{Improvement} = \frac{\epsilon_{SV(J)2,0} - \epsilon_{SV(J)3,1}}{\epsilon_{SV(J)2,0}}$$

Improvements by model-implied α



$$\text{Improvement} = \frac{\epsilon_{SV(J)2,0} - \epsilon_{SV(J)3,1}}{\epsilon_{SV(J)2,0}}$$

Dynamic properties

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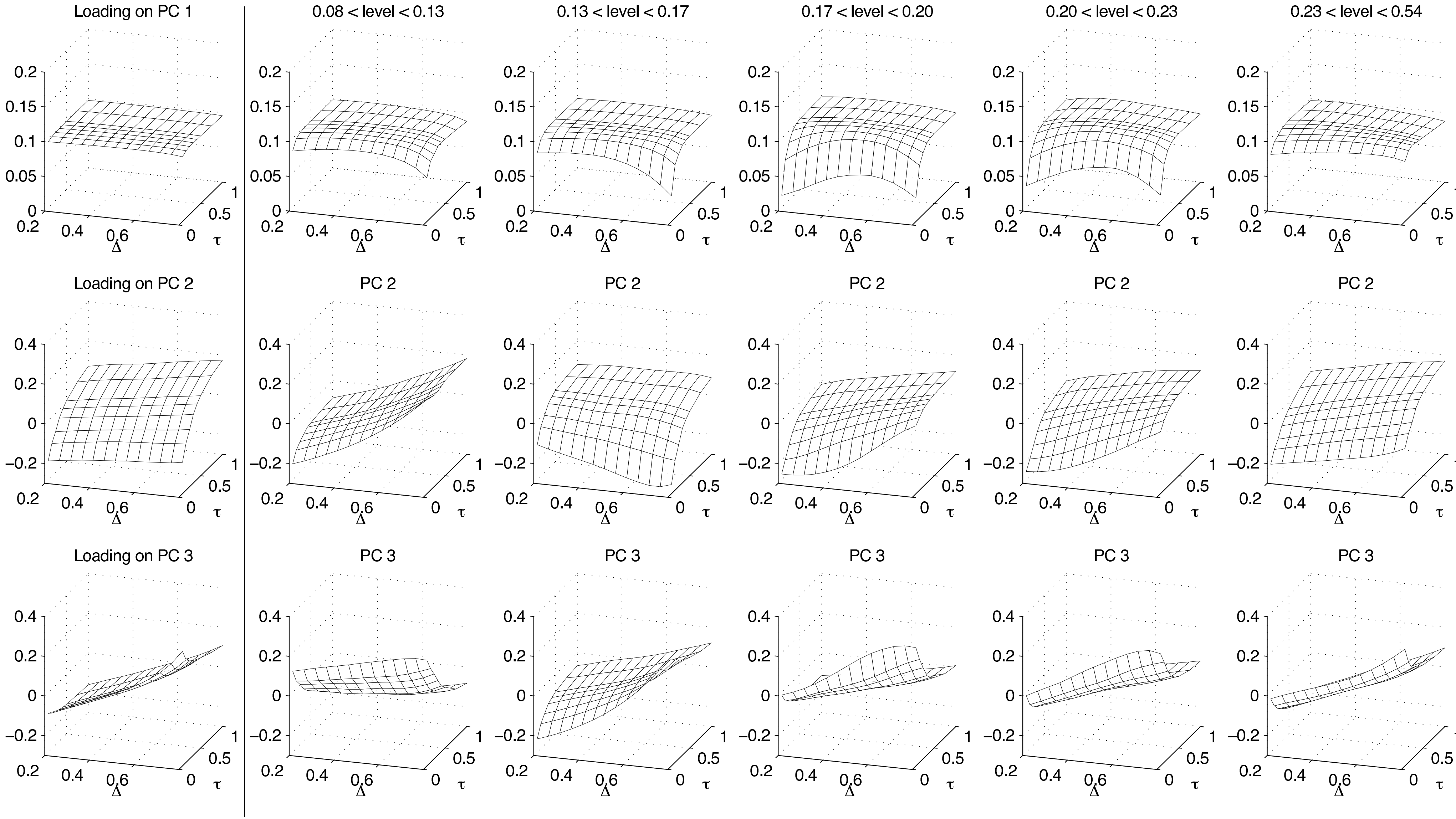
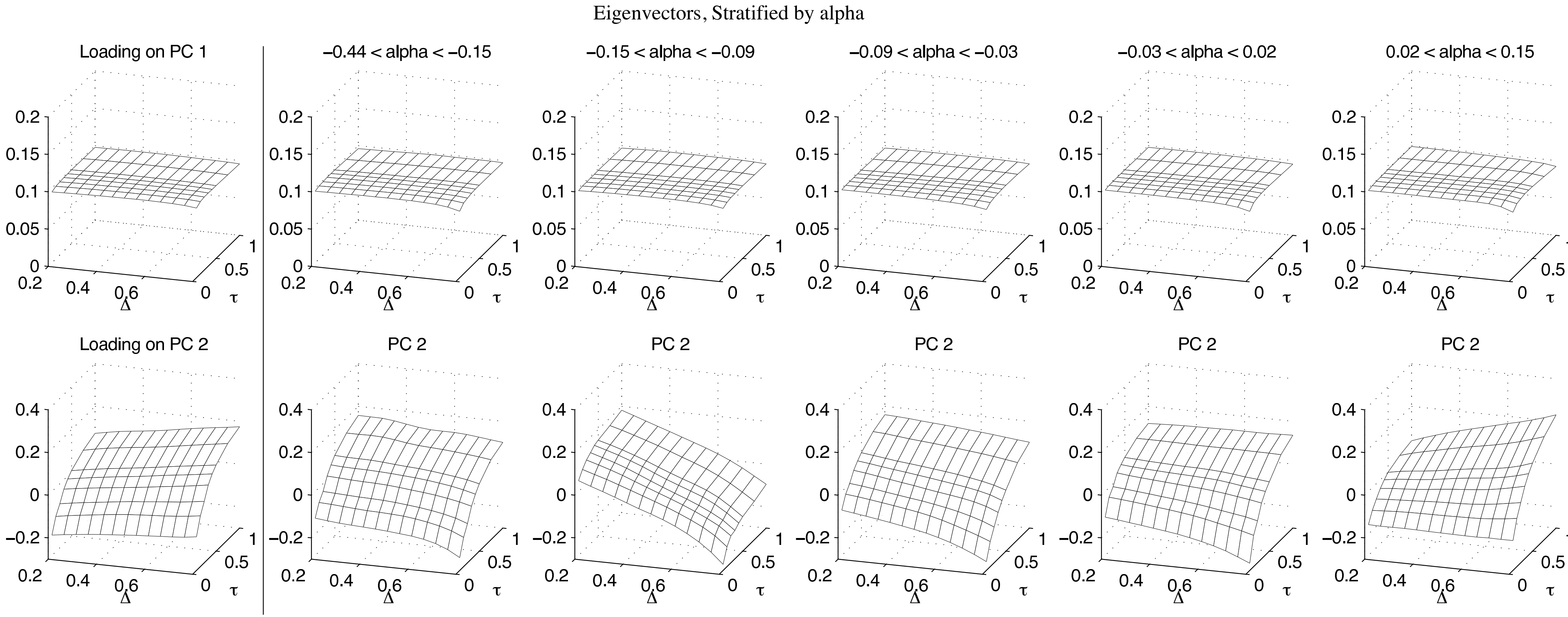
	l	PC 1	PC 2	PC 3	PC 4	
Unconditional	2	97.0	1.9	0.8	0.1	$T = 3206$
$-0.44 < \alpha/\pi \leq -0.15$	2	96.0	2.1	1.0	0.3	$T = 641$
$-0.15 < \alpha/\pi \leq -0.09$	1	97.1	1.3	1.0	0.2	$T = 641$
$-0.09 < \alpha/\pi \leq -0.03$	1	97.1	1.7	0.7	0.2	$T = 641$
$-0.03 < \alpha/\pi \leq 0.02$	1	97.0	1.7	0.9	0.1	$T = 641$
$0.02 < \alpha/\pi \leq 0.15$	2	96.2	1.9	1.3	0.2	$T = 641$

$l =$ significant components according to mean eigenvalue criterion.
($N = 56$, threshold=$\frac{1}{56}$=1.79%)

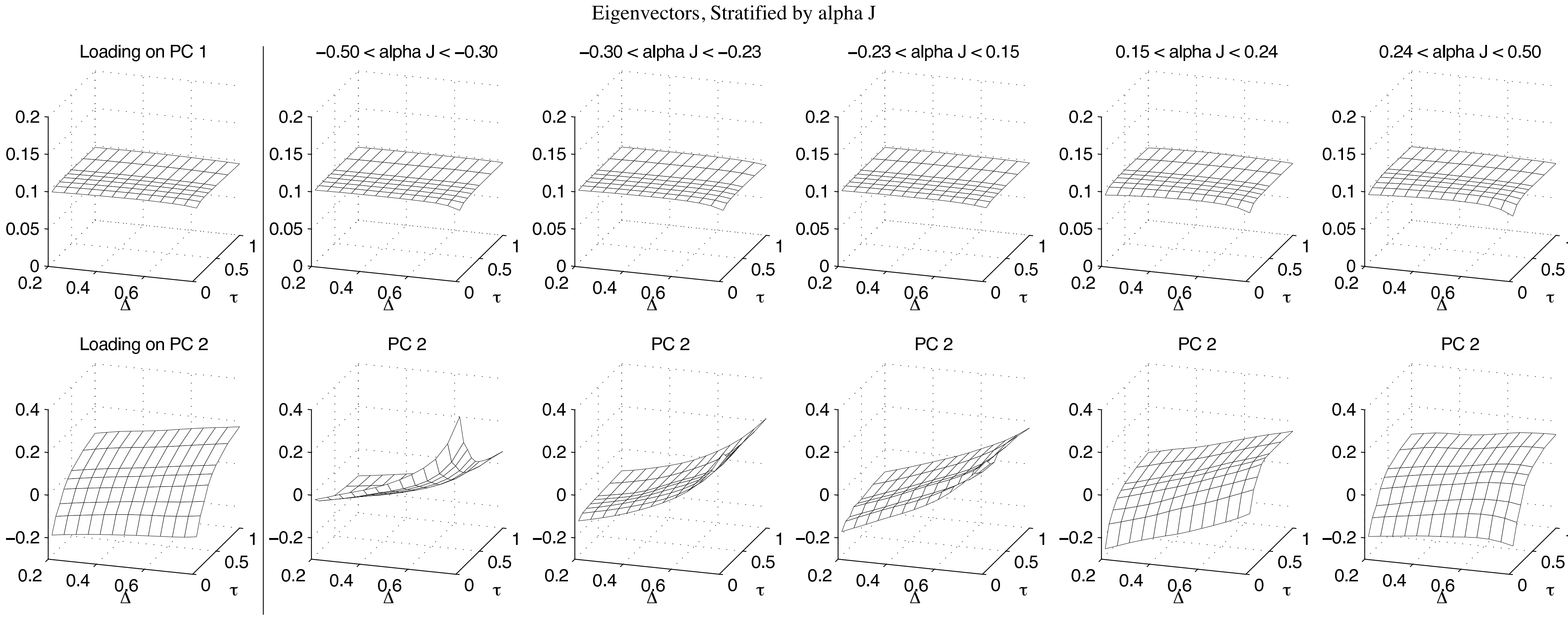
Eigenvectors/level



Eigenvectors/alpha ($SV_{3,1}$)



Eigenvectors/alpha ($SVJ_{3,1}$)



Stochastic Coefficient Interpretation

Stochastic Coefficients

Interpret eigenvalues of X_t as two volatility factors:

$$d\mathcal{V}_{1t} = \left(\beta(\tilde{Q}_t'\tilde{Q}_t)^{11} + 2(\tilde{M}_t)^{11}\mathcal{V}_{1t} + \frac{\mathcal{V}_{1t}(\tilde{Q}_t'\tilde{Q}_t)^{22} + \mathcal{V}_{2t}(\tilde{Q}_t'\tilde{Q}_t)^{11}}{\mathcal{V}_{1t} - \mathcal{V}_{2t}}\right)dt + 2\sqrt{\mathcal{V}_{1t}(\tilde{Q}_t'\tilde{Q}_t)^{11}}d\nu_{1t}$$

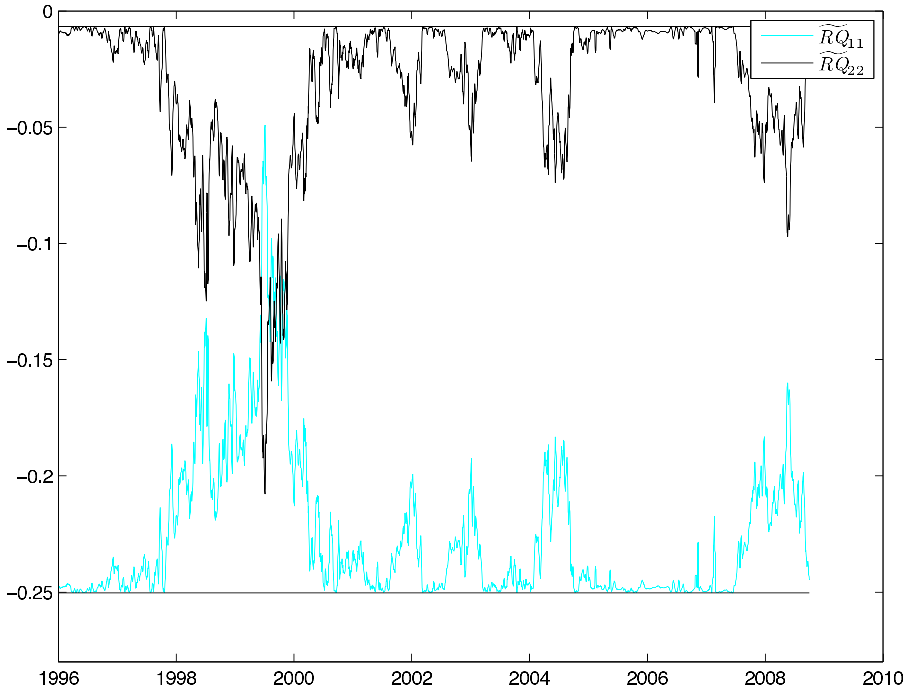
$$d\mathcal{V}_{2t} = \left(\beta(\tilde{Q}_t'\tilde{Q}_t)^{22} + 2(\tilde{M}_t)^{22}\mathcal{V}_{2t} - \frac{\mathcal{V}_{1t}(\tilde{Q}_t'\tilde{Q}_t)^{22} + \mathcal{V}_{2t}(\tilde{Q}_t'\tilde{Q}_t)^{11}}{\mathcal{V}_{1t} - \mathcal{V}_{2t}}\right)dt + 2\sqrt{\mathcal{V}_{2t}(\tilde{Q}_t'\tilde{Q}_t)^{22}}d\nu_{2t}$$

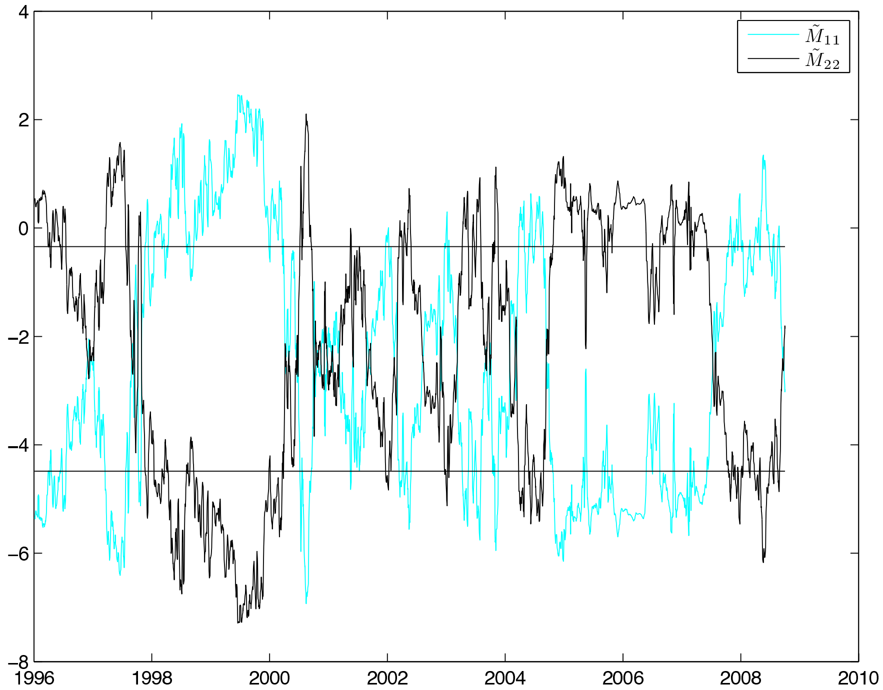
$(\nu_1, \nu_2)'$ standard Brownian motion in $\mathbb{R}^2$

$\tilde{M}_t = \mathcal{O}_t' M \mathcal{O}_t$ and $\tilde{Q}_t = \mathcal{O}_t' Q \mathcal{O}_t$.

$$\mathcal{O}_t = \begin{pmatrix} \cos(\alpha_t) & -\sin(\alpha_t) \\ \sin(\alpha_t) & \cos(\alpha_t) \end{pmatrix}$$

Continuous regime shift





Factors $\mathcal{V}_{1t}$ and $\mathcal{V}_{2t}$ cannot cross (Wishart property),
but mean-reversion and vol-of vol *can*

Conclusion

Conclusion

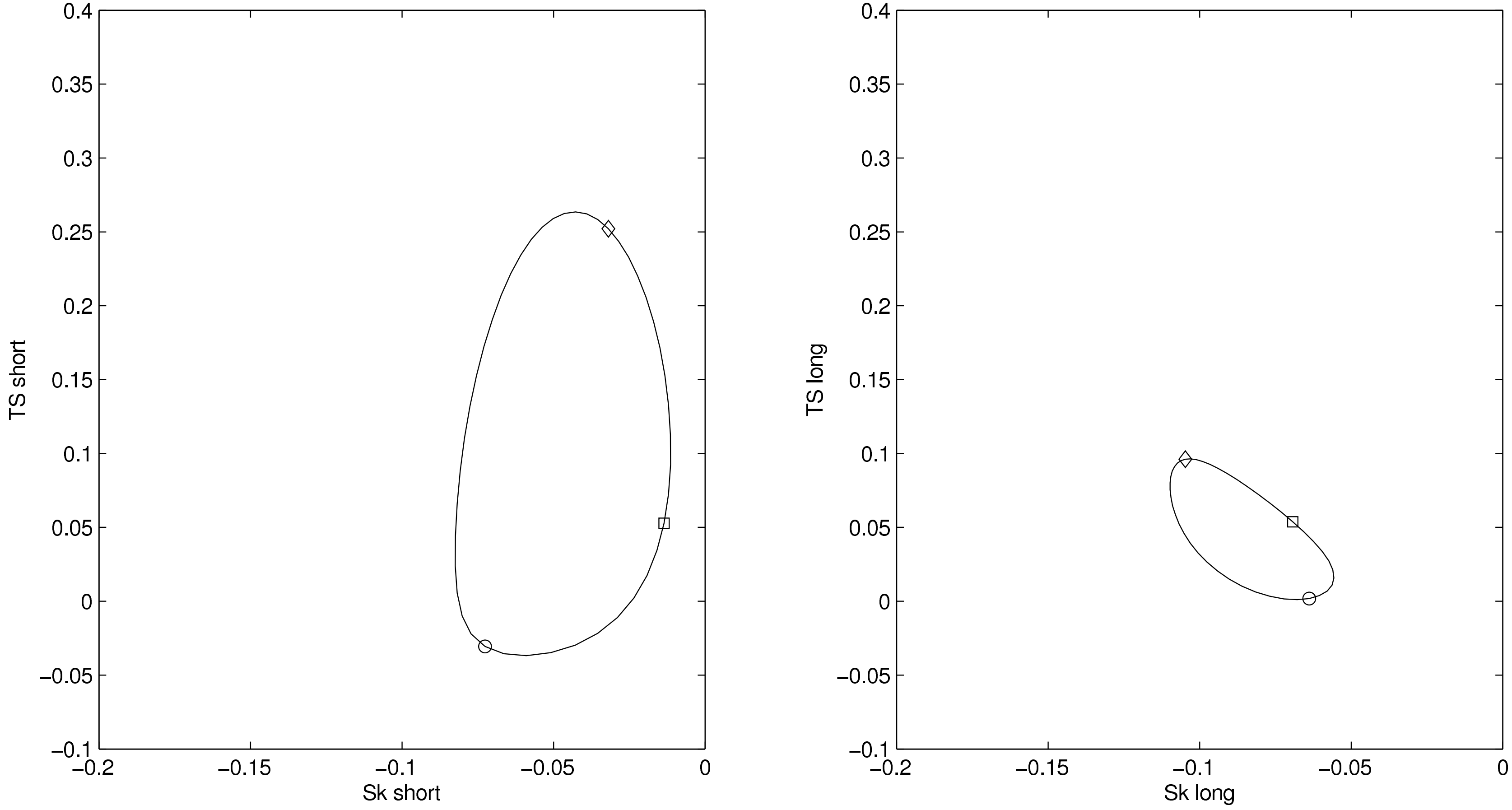
- Identified interacting + unspanned components in the volatility surface of S&P 500 index options.
- Matrix jump diffusion is a convenient framework for modeling interacting + unspanned factors.
- Estimated full matrix jump-diffusion model and nested models
- Find:
  - Three factors are indeed needed
  - Better in and out-of sample fit
  - Third factor should be interaction factor (α)
  - Largest improvements where 2 factor models are weak and $\alpha \neq 0$
- Appropriate conditioning provides evidence for a conditional two-factor structure $\longrightarrow$ stochastic coefficient model.

Future

- Use insights for more parsimonious models
- Apply to other fields of finance + economics

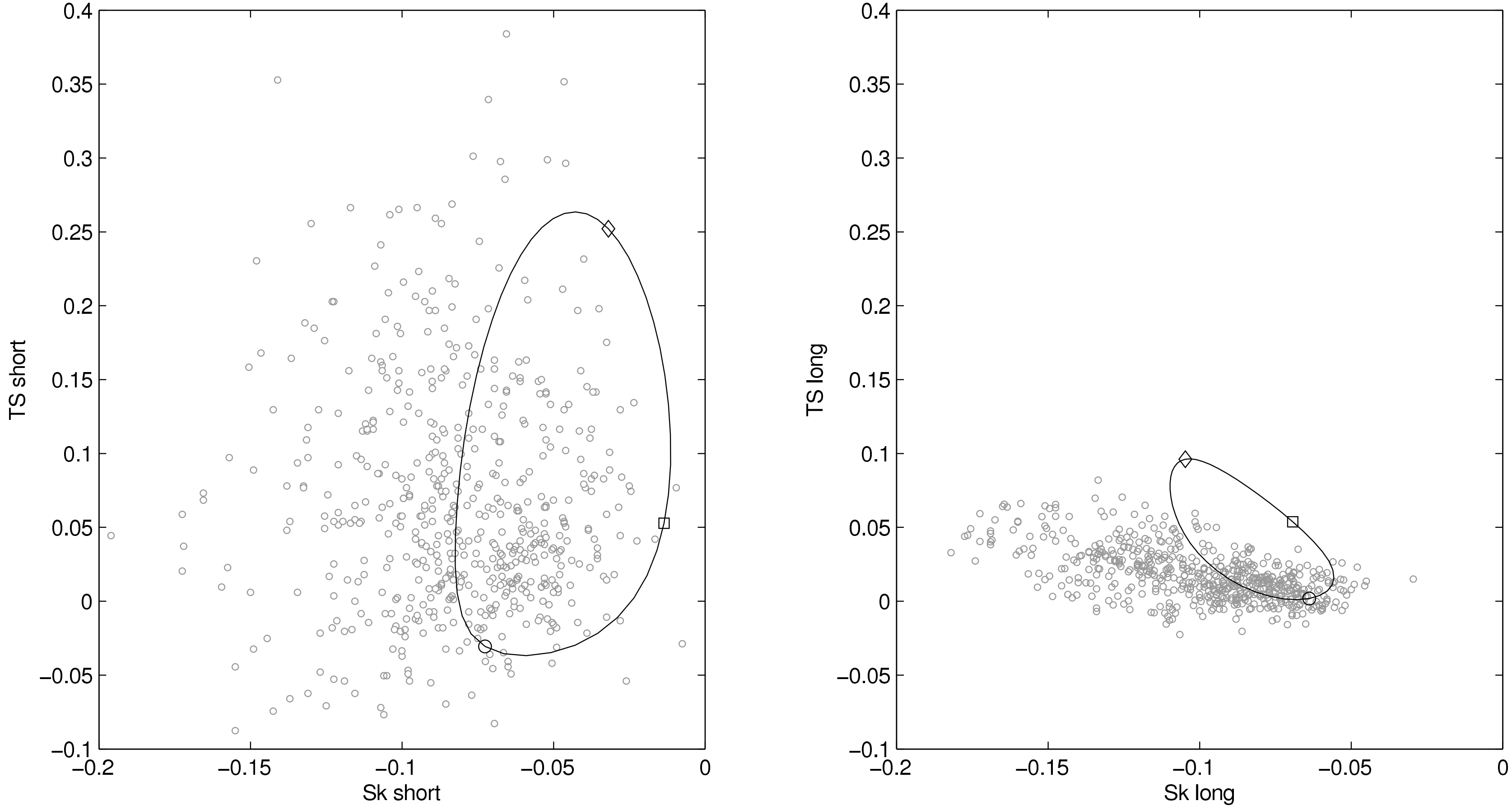
Spare slides

Short and long term expansion



level=0.17

Short and long term expansion



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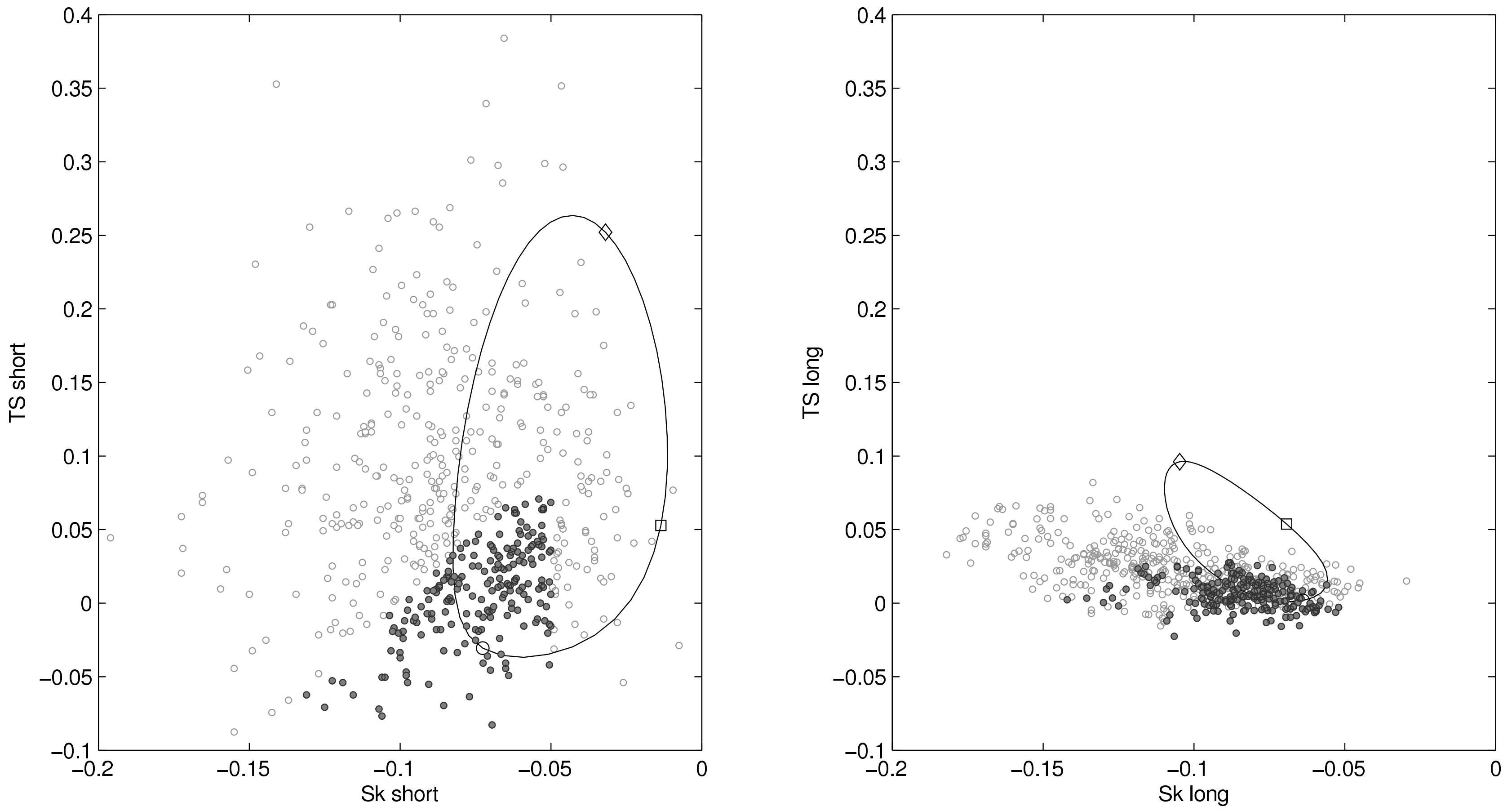
Short and long term expansion



level=0.17

Numerical aspects

- Nested optimization: very heavy computation
- Major load: calculating the Laplace transform (not performing the Fourier inversion)
- Matrix logarithm – use matrix rotation count algorithm

Improve speed

- Optimized MATLAB code on a MATLAB cluster (32 cores)
- Genetic optimization permits parallelizing parameter estimation
- Cos-FFT (250 instead of 4096 evaluations of Laplace transform)
- Separate evaluation of state-dependent and maturity-dependent parts of Laplace transform
- Select a sample with few distinct maturities (monthly data, all Wednesdays)
- Estimation still takes 1 week

Jumps

- Jump size like Bates: iid jumps

$$\ln(1+k) \sim N(\ln(1+\bar{k}) - \frac{\delta^2}{2}, \delta^2)$$

- Jump intensity: extend Bates to matrix case

$$\lambda_t = \lambda_0 + \Lambda_{11} X_{11} + \Lambda_{12} X_{12} + \Lambda_{22} X_{22} = \lambda_0 + tr[\Lambda X_t]$$

- Identification $\rightarrow \Lambda$ upper triangular
- Ensure positive jump intensity:

$$\Lambda_{11} > 0$$
$$\Lambda_{22} > 0$$
$$|\Lambda_{12}| < 2\sqrt{\Lambda_{11}\Lambda_{22}}$$

- Unspanned jump intensity component