Term structure models driven by Wiener processes and Poisson measures: Existence and positivity

Stefan Tappe
ETH Zürich
stefan.tappe@math.ethz.ch

(joint work with Damir Filipović and Josef Teichmann)

6th World Congress of the Bachelier Finance Society

June 25th, 2010 (Toronto)

Introduction

- Zero Coupon Bonds $P(t,T)$.
- The Heath-Jarrow-Morton-Musiela (HJMM) equation:

$$dr_t = \left(\frac{d}{d\xi} r_t + \alpha_{\mathrm{HJM}}(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt).$$

- Establish *existence and positivity*.
- The Brody-Hughston equation:

$$d\rho_t = \left(\frac{d}{d\xi} \rho_t + \rho_t(0) \rho_t \right) dt + \sigma(\rho_t) dW_t + \int_E \gamma(\rho_{t-}, x)(\mu(dt, dx) - F(dx)dt).$$

Zero Coupon Bonds

- Zero Coupon Bonds $P(t,T)$.
- Financial assets paying the holder one unit of cash at T.

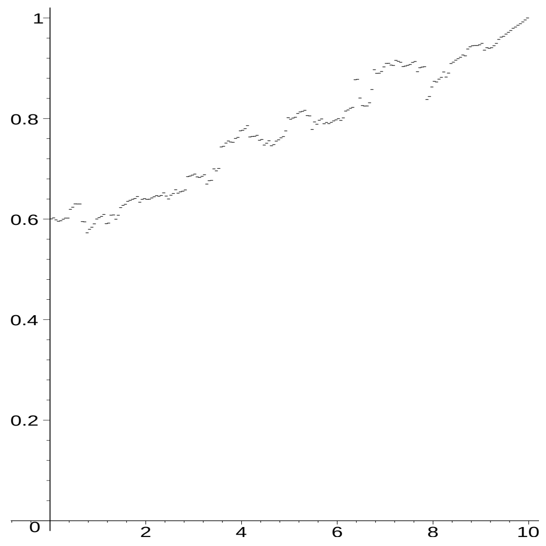

Figure 1: Price process of a T-bond with date $T = 10$.

The HJM model with jumps

- Björk, Kabanov, Runggaldier, Di Masi 1997 [1]: For $T \geq 0$ we have

$$f(t,T) = f^*(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW_s + \int_0^t \int_E \gamma(s,x,T)(\mu(ds,dx) - F(dx)ds), \quad t \in [0,T].$$

- Implied bond market:

$$P(t,T) = \exp\Big(- \int_t^T f(t,s)ds \Big).$$

From HJM to Stochastic Equations

- Drift and volatilities depend on the current forward curve:

$$\alpha(t,T,\omega) = \alpha(t,T,f(t,\cdot,\omega)),$$
$$\sigma(t,T,\omega) = \sigma(t,T,f(t,\cdot,\omega)),$$
$$\gamma(t,x,T,\omega) = \gamma(t,x,T,f(t,\cdot,\omega)).$$

- Infinite dimensional stochastic equation:

$$\begin{cases} df(t,T) &= \alpha(t,T,f(t,\cdot))dt + \sigma(t,T,f(t,\cdot))dW_t \\ & \quad + \int_E \gamma(t,x,T,f(t,\cdot))(\mu(dt,dx) - F(dx)dt) \\ f(0,T) &= f^*(0,T). \end{cases}$$

The transformed equation

- *Musiela parametrization* of forward rates:

$$r_t(\xi) := f(t, t+\xi), \quad \xi \geq 0.$$

- Making the transformation $f(t,T) \rightsquigarrow r_t(\xi)$ we obtain

$$\begin{aligned} r_t = S_t h_0 &+ \int_0^t S_{t-s}\alpha(r_s)ds + \int_0^t S_{t-s}\sigma(r_s)dW_s \\ &+ \int_0^t \int_E S_{t-s}\gamma(r_{s-},x)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0 \end{aligned}$$

- where $(S_t)_{t\geq 0}$ denotes the shift-semigroup $S_t h := h(t+\cdot)$ on H.

From HJMM to SPDEs

- Thus, $(r_t)_{t \geq 0}$ is a *mild solution* of the SPDE

$$dr_t = \left(\frac{d}{d\xi} r_t + \alpha(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt),$$

- with given vector fields

$$\alpha : H \to H, \quad \sigma : H \to L_2^0(H), \quad \gamma : H \times E \to H,$$

- where $\frac{d}{d\xi}$ is the infinitesimal generator of $(S_t)_{t \geq 0}$.

The HJMM equation

- The bond market $P(t,T)$ should be free of arbitrage.
- Under a martingale measure $\mathbb{Q} \sim \mathbb{P}$ we have

$$\alpha_{\mathrm{HJM}}(h) = \sum_j \sigma^j(h) \int_0^{\bullet} \sigma^j(h)(\eta)d\eta - \int_E \gamma(h,x) \left(e^{-\int_0^{\bullet} \gamma(h,x)(\eta)d\eta} - 1 \right) F(dx).$$

- This leads to the *HJMM equation*

$$dr_t = \left(\frac{d}{d\xi} r_t + \alpha_{\mathrm{HJM}}(r_t) \right) dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-},x)(\mu(dt,dx) - F(dx)dt).$$

Stochastic partial differential equations

- Consider the SPDE

$$\begin{cases} dr_t &= (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt) \\ r_0 &= h_0, \end{cases}$$

- with given vector fields

$$\alpha : H \to H, \quad \sigma : H \to L_2^0(H), \quad \gamma : H \times E \to H,$$

- where $A : \mathcal{D}(A) \subset H \to H$ is the generator of a C_0-semigroup on H.

Assumptions for the existence result

- *Lipschitz continuity:* For all $h_1, h_2 \in H$ we have

$$\|\alpha(h_1) - \alpha(h_2)\| + \|\sigma(h_1) - \sigma(h_2)\|_{L_2^0(H)} + \left(\int_E \|\gamma(h_1, x) - \gamma(h_2, x)\|^2 F(dx) \right)^{1/2} \leq L\|h_1 - h_2\|.$$

- *Linear growth:* We have $\int_E \|\gamma(0, x)\|^2 F(dx) < \infty$.

- We assume that $(S_t)_{t \geq 0}$ is *pseudo-contractive*, that is

$$\|S_t\| \leq e^{\omega t}, \quad t \geq 0.$$

Existence- and uniqueness result

- Unique mild solutions for the SPDE

$$\begin{cases} dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt) \\ r_0 = h_0, \end{cases}$$

- i.e., the "Variation of constants formula" is satisfied:

$$\begin{aligned} r_t = S_t h_0 &+ \int_0^t S_{t-s}\alpha(r_s)ds + \int_0^t S_{t-s}\sigma(r_s)dW_s \\ &+ \int_0^t \int_E S_{t-s}\gamma(r_{s-}, x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0. \end{aligned}$$

The HJMM equation

- The HJMM equation is an SPDE

$$dr_t = (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt),$$

- for which we have

$$H = H_\beta, \quad A = \frac{d}{d\xi}, \quad \alpha = \alpha_{\mathrm{HJM}},$$

- where α_{HJM} is given by

$$\alpha_{\mathrm{HJM}}(h) = \sum_j \sigma^j(h) \int_0^\bullet \sigma^j(h)(\eta)d\eta - \int_E \gamma(h,x) \left(e^{-\int_0^\bullet \gamma(h,x)(\eta)d\eta} - 1 \right) F(dx).$$

The space of forward curves

- For $\beta > 0$ we define the space

$$H_\beta := \{h : \mathbb{R}_+ \to \mathbb{R} : h \text{ is absolutely continuous with } \|h\|_\beta < \infty\},$$

- where the norm is defined by

$$\|h\|_\beta := \left(|h(0)|^2 + \int_{\mathbb{R}_+} |h'(\xi)|^2 e^{\beta \xi} d\xi \right)^{1/2}.$$

- The shift semigroup $(S_t)_{t \geq 0}$ on H_β has the generator

$$A = \frac{d}{d\xi}, \quad \mathcal{D}\left(\frac{d}{d\xi}\right) = \{h \in H_\beta : h' \in H_\beta\}.$$

Assumptions on the vector fields

- *Lipschitz continuity:* For all $h_1, h_2 \in H_\beta$ we have

$$\|\sigma(h_1) - \sigma(h_2)\|_{L_2^0(H_\beta)} \leq L\|h_1 - h_2\|_\beta,$$

$$\bigg(\int_E e^{\Phi(x)} \|\gamma(h_1,x) - \gamma(h_2,x)\|_\beta^2 F(dx) \bigg)^{1/2} \leq L\|h_1 - h_2\|_\beta.$$

- *Boundedness:* For all $h \in H_\beta$ we have

$$\|\sigma(h)\|_{L_2^0(H_\beta)} \leq M,$$

$$\int_E e^{\Phi(x)} \big(\|\gamma(h,x)\|_\beta^2 \vee \|\gamma(h,x)\|_\beta^4 \big) F(dx) \leq M.$$

Solution of the HJMM equation

- The HJM drift term $\alpha_{\mathrm{HJM}} : H_\beta \to H_\beta$ is Lipschitz continuous:

$$\|\alpha_{\mathrm{HJM}}(h_1) - \alpha_{\mathrm{HJM}}(h_2)\|_\beta \leq K \|h_1 - h_2\|_\beta.$$

- Unique mild solutions for the HJMM equation

$$\begin{cases} dr_t &= (\frac{d}{d\xi} r_t + \alpha_{\mathrm{HJM}}(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt) \\ r_0 &= h_0. \end{cases}$$

- The interest rates $r_t(\xi)$ should not be negative.

Positivity preserving models

- Let $P \subset H_\beta$ be the convex cone

$$P = \{h \in H_\beta : h \geq 0\} = \bigcap_{\xi \in \mathbb{R}_+} \{h \in H_\beta : h(\xi) \geq 0\}.$$

- The HJMM equation is *positivity preserving* if for all $h_0 \in P$ we have

$$\mathbb{P}(r_t \in P) = 1, \quad t \geq 0.$$

- Stochastic invariance problem.

A general invariance result

- Consider an SPDE on the space H_β of forward curves

$$dr_t = \left(\frac{d}{d\xi} r_t + \alpha(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt),$$

- with given vector fields

$$\alpha : H_\beta \to H_\beta, \quad \sigma : H_\beta \to L_2^0(H_\beta), \quad \gamma : H_\beta \times E \to H_\beta.$$

- This SPDE is positivity preserving if and only if we have (1)–(4).

The volatility and the jumps

- At the boundary, the volatility σ is parallel to the edge:

$$\sigma^j(h)(\xi) = 0, \quad h \geq 0 \text{ with } h(\xi) = 0. \tag{1}$$

- The convex cone P captures all jumps:

$$h + \gamma(h, x) \in P, \quad h \in P \text{ and } F\text{-almost all } x \in E. \tag{2}$$

Small jumps at the boundary

- In general, we have:

$$\int_E \|\gamma(h,x)\|_\beta^2 F(dx) < \infty, \quad \text{but} \quad \int_E \|\gamma(h,x)\|_\beta F(dx) = \infty.$$

- Small jumps, which are not parallel to boundary, are of finite variation:

$$\int_E |\gamma(h,x)(\xi)| F(dx) < \infty, \quad h \geq 0 \text{ with } h(\xi) = 0. \tag{3}$$

The drift vector field

- Subtract the $F(dx)dt$-part of the stochastic integral to the drift:

$$dr_t = \left(\frac{d}{d\xi} r_t + \alpha(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt).$$

- At the boundary, the corrected drift term is inward pointing:

$$\alpha(h)(\xi) - \int_E \gamma(h, x)(\xi) F(dx) \geq 0, \quad h \geq 0 \text{ with } h(\xi) = 0. \tag{4}$$

Remarks concerning the drift vector field

- The convex cone P has particular properties.

- The shift semigroup $(S_t)_{t \geq 0}$ leaves P invariant:

$$S_t P \subset P \quad \text{for all } t \geq 0.$$

- No Stratonovich correction term, because

$$\big(D\sigma^j(h)\sigma^j(h) \big)(\xi) = 0, \quad h \geq 0 \text{ with } h(\xi) = 0.$$

Invariance conditions for the HJMM equation

- This SPDE is positivity preserving if and only if we have (1)–(4).
- *Consequence:* The HJMM equation

$$dr_t = \left(\frac{d}{d\xi} r_t + \alpha_{\mathrm{HJM}}(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt)$$

- is positivity preserving if and only if

$$\sigma^j(h)(\xi) = 0, \quad h \geq 0 \text{ with } h(\xi) = 0 \tag{5}$$

$$\gamma(h, x)(\xi) = 0, \quad h \geq 0 \text{ with } h(\xi) = 0 \text{ and } F\text{-almost all } x \in E \tag{6}$$

$$h + \gamma(h, x) \in P, \quad h \in P \text{ and } F\text{-almost all } x \in E. \tag{7}$$

Another approach to bond price markets

- Following Brody, Hughston 2001 [2] we define the bond prices

$$P(t,T) = \int_{T-t}^{\infty} \rho_t(\xi) d\xi,$$

- where $(\rho_t)_{t \geq 0}$ is a process of probability densities on $\mathbb{R}_+$.
- Then we have $P(T,T) = 1$ for all $T \geq 0$
- and $T \mapsto P(t,T)$ is non-increasing with limit 0 for $T \to \infty$.

The Brody-Hughston equation

- Consider the Brody-Hughston equation

$$d\rho_t = \left(\frac{d}{d\xi} \rho_t + \rho_t(0)\rho_t \right) dt + \sigma(\rho_t) dW_t + \int_E \gamma(\rho_{t-}, x)(\mu(dt, dx) - F(dx)dt),$$

- on the state space H_β^0 with vector fields

$$\sigma : H_\beta^0 \to L_2^0(H_\beta^0), \quad \gamma : H_\beta^0 \times E \to H_\beta^0,$$

- where $H_\beta^0 = \{h \in H_\beta : \lim_{\xi \to \infty} h(\xi) = 0\}$.

Stochastic invariance problem

- We observe that $H^0_\beta \subset L^1(\mathbb{R}_+)$.
- Stochastic invariance of the convex set $\mathcal{P} \subset H^0_\beta$ of probability densities

$$\mathcal{P} = \left\{ h \in H^0_\beta : h \geq 0 \text{ and } \int_{\mathbb{R}_+} h(\xi) d\xi = 1 \right\}$$

$$= \underbrace{\{ h \in H^0_\beta : h \geq 0 \}}_{\text{Use our previous results}} \cap \underbrace{\left\{ h \in H^0_\beta : \int_{\mathbb{R}_+} h(\xi) d\xi = 1 \right\}}_{\text{Invariance conditions are known}}$$

- Unique mild solutions for the Brody-Hughston equation.

Conclusion

- Zero Coupon Bonds $P(t,T)$.
- The Heath-Jarrow-Morton-Musiela (HJMM) equation:

$$dr_t = \left(\frac{d}{d\xi} r_t + \alpha_{\mathrm{HJM}}(r_t) \right) dt + \sigma(r_t) dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx)dt).$$

- We have established *existence and positivity*.
- The Brody-Hughston equation:

$$d\rho_t = \left(\frac{d}{d\xi} \rho_t + \rho_t(0)\rho_t \right) dt + \sigma(\rho_t) dW_t + \int_E \gamma(\rho_{t-}, x)(\mu(dt, dx) - F(dx)dt).$$

References

[1] Björk, T., Di Masi, G., Kabanov, Y., Runggaldier, W. (1997): Towards a general theory of bond markets. *Finance and Stochastics* **1**(2), 141–174.

[2] Brody, D. C., Hughston, L. P. (2001): Interest rates and information geometry, *Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences*, **457**, 1343–1363.

[3] Filipović, D., Tappe, S., Teichmann, J. (2010): Term structure models driven by Wiener processes and Poisson measures: Existence and positivity. Forthcoming in *SIAM Journal on Financial Mathematics*.

[4] Heath, D., Jarrow, R., Morton, A. (1992): Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica* **60**(1), 77–105.