Rational Term Structure Models with Geometric Lévy Martingales

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Talk based on:

- Brody, D. C., Hughston, L. P. & Mackie, E. (2010) "Rational term structure models with geometric Lévy martingales" Imperial College Working Paper.

Related work:

- Flesaker, B. & Hughston, L. P. 1996 Positive interest. *Risk Magazine* **9**, 46–49; reprinted in *Vasicek and Beyond*, L.P. Hughston (ed), London: Risk Publications (1996).
- Brody, D. C. & Hughston, L. P. 2001 Interest rates and information geometry. *Proc. Roy. Soc. London* A**457**, 1343–1364.
- Brody, D. C. & Hughston, L. P. 2002 Entropy and information in the interest rate term structure. *Quantitative Finance* **2**, 70-80.

Term structure density approach

In 1996 Flesaker and Hughston made the observation that for a positive nominal interest-rate system, the price $\{P_{tT}\}_{0\leq t\leq T}$ of a T-maturity discount bond admits the rational representation

$$P_{tT} = \frac{\int_T^\infty(-\partial_u P_{0u})M_{tu}\,\mathrm{d}u}{\int_t^\infty(-\partial_u P_{0u})M_{tu}\,\mathrm{d}u}, \tag{1}$$

where $\{M_{tu}\}_{0\leq t\leq u}$ is a family of positive unit-initialised martingales.

To model the interest rate system we thus need to specify the initial term structure together with a family of positive martingales.

The expression appearing in the integrand of (1), namely,

$$\rho(T) \equiv -\partial_T P_{0T}, \tag{2}$$

defines a probability density function over $\mathbb{R}_+$ associated with an abstract random variable Z associated with the bond maturity.

In fact, each positive term structure admits can be represented as a density

$$\rho_t(z) = -\partial_z P_{t,t+z}. \tag{3}$$

Modelling $\rho_t(z)$ directly is referred to as the Brody -Hughston term-structure density approach.

One of the advantages of the term-structure density approach over the more traditional HJM or market approaches is that the positivity of nominal interest rate, or equivalently the arbitrage freeness, is automatically ensured.

Another advantage of the term-structure density approach has been pointed out more recently by Filipović *et al.* (2009).

Filipović *et al.* showed that it is "less delicate" to add jumps to the Brody-Hughston term-structure density framework than to the familiar HJM framework in the Musiela representation.

In this spirit we shall consider a range of geometric Lévy martingales $\{M_{tu}\}$ in the rational representation (1).

Geometric Lévy martingales

Our goal now is to construct a class of interest rate models based on various Lévy processes.

Let $\{L_t\}_{t\geq 0}$ be a Lévy process with $L_0 = 0$.

For a suitable function suitable $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ we define a martingale family $\{M_{tx}\}_{0\leq t\leq x}$ by setting

$$M_{tx} = \frac{\mathrm{e}^{\phi(x)L_t}}{\mathbb{E}[\mathrm{e}^{\phi(x)L_t}]}. \tag{4}$$

Note that $\{M_{tx}\}$ satisfies $M_{tx} > 0$ and $M_{0x} = 1$.

Then by taking various choices for the underlying Lévy process we are able to generate a variety of interest rate models, each with some functional freedom.

Geometric Brownian motion family

For a standard Brownian motion $\{B_t\}_{t\geq 0}$, we obtain a bond price of the form

$$P_{tT} = \frac{\int_T^\infty \rho(x) \mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t} \mathrm{d}x}{\int_t^\infty \rho(x) \mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t} \mathrm{d}x}, \tag{5}$$

and a corresponding short rate of the form

$$r_t = \frac{\rho(t) \mathrm{e}^{\phi(t)B_t - \frac{1}{2}\phi(t)^2 t} \mathrm{d}x}{\int_t^\infty \rho(x) \mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t} \mathrm{d}x}. \tag{6}$$

Using Ito's lemma we deduce the dynamics of the bond price system is given by

$$\frac{\mathrm{d}P_{tT}}{P_{tT}} = (r_t + \Phi_{tt}(\Phi_{tt} - \Phi_{tT}))\mathrm{d}t + (\Phi_{tT} - \Phi_{tt})\mathrm{d}B_t, \tag{7}$$

where

$$\Phi_{tT} = \frac{\int_T^\infty \phi(x)\rho(x) \mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t} \mathrm{d}x}{\int_T^\infty \rho(x) \mathrm{e}^{\phi(x)B_t - \frac{1}{2}\phi(x)^2 t} \mathrm{d}x}, \tag{8}$$

Positive risk premium implies that $|\phi(x)|$ is decreasing in x.

The price today of a call option expiring at time t with strike price K, on a discount bond maturing at time T is given by

$$C_{0t} = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-\int_0^t r_s \mathrm{d}s}(P_{tT} - K)^+]. \tag{9}$$

The option price in the geometric Brownian motion example turns out to be

$$\begin{aligned} C_{0t} \; &= \; \int_T^{\infty} \rho(x)\, N\left(\pm\frac{\xi^*}{\sqrt{t}} \mp \phi(x)\sqrt{t}\right) \mathrm{d}x \\ &\quad -K \int_t^{\infty} \rho(x)\, N\left(\pm\frac{\xi^*}{\sqrt{t}} \mp \phi(x)\sqrt{t}\right) \mathrm{d}x, \end{aligned} \tag{10}$$

where ξ^* is a critical value on the boundary of positive payoffs.

Here the $(\pm, \mp)$ signs corresponds to the combination $(+, -)$ if $\phi(x)$ is increasing in x, and $(-, +)$ if $\phi(x)$ is decreasing in x.

Geometric Brownian motion family: bond price

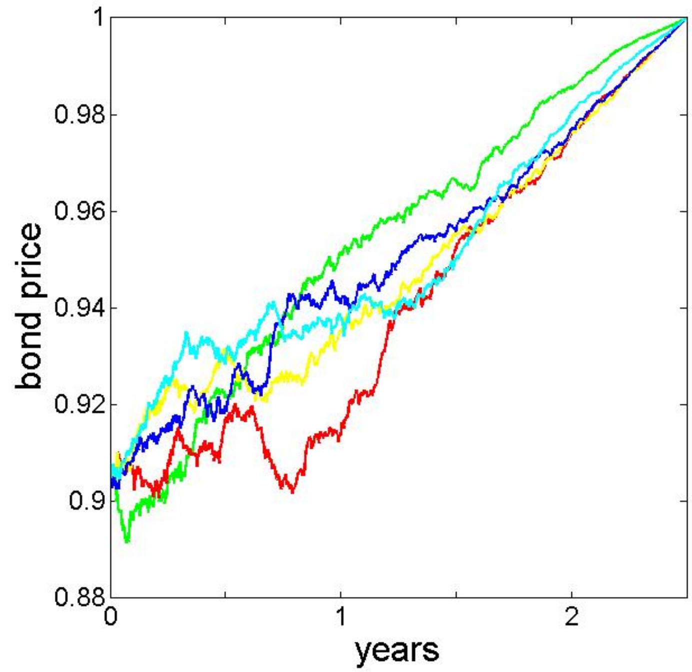

Figure 1: Simulation of the bond price in a term-structure density model with a parametric martingale family based on a geometric Brownian motion. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$.

Geometric Brownian motion family: short rate

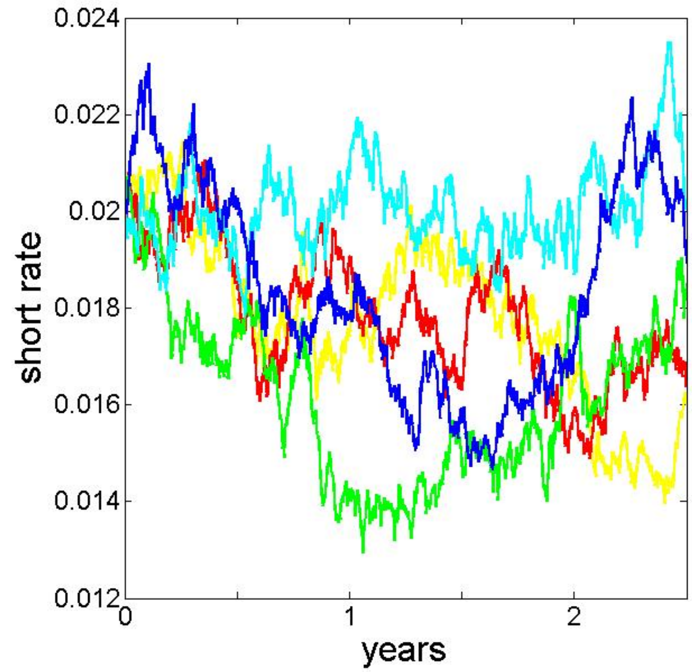

Figure 2: Simulation of the short rate in a term-structure density model with a parametric martingale family based on a geometric Brownian motion. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$.

Geometric gamma family

Let $\{\gamma_t\}_{t\geq 0}$ be a gamma process whose increments $\gamma_t - \gamma_s$, for $0 \leq s \leq t < \infty$, have a probability density $f(x)$ such that

$$f(x) = g_\gamma(x;(t-s)m,\kappa) = \frac{x^{m(t-s)-1}\mathrm{e}^{-x/\kappa}}{\Gamma(m(t-s))\kappa^{m(t-s)}}. \tag{11}$$

Here, m is the rate parameter and κ is the scale parameter of $\{\gamma_t\}_{t\geq 0}$.

We are able to show that the bond price is of the form

$$P_{tT} = \frac{\int_T^\infty \rho(x)(1-\phi(x)\kappa)^{-mt}\mathrm{e}^{\phi(x)\gamma_t}\mathrm{d}x}{\int_t^\infty \rho(x)(1-\phi(x)\kappa)^{-mt}\mathrm{e}^{\phi(x)\gamma_t}\mathrm{d}x}, \tag{12}$$

and that the associated short rate is given by

$$r_t = \frac{\rho(t)(1-\phi(t)\kappa)^{-mt}\mathrm{e}^{\phi(t)\gamma_t}\mathrm{d}x}{\int_t^\infty \rho(x)(1-\phi(x)\kappa)^{-mt}\mathrm{e}^{\phi(x)\gamma_t}\mathrm{d}x}. \tag{13}$$

For increasing $\phi(x)$ we deduce that the required option price in the example of the geometric gamma process is of the following form:

$$C_{0t} = \int_T^\infty \rho(x)\Gamma\left(mt, \gamma^*\left(\frac{1}{\kappa} - \phi(x)\right)\right) \mathrm{d}x$$
$$-K \int_t^\infty \rho(x)\Gamma\left(mt, \gamma^*\left(\frac{1}{\kappa} - \phi(x)\right)\right) \mathrm{d}x, \tag{14}$$

where γ^* is a critical value on the boundary of positive payoffs.

Here

$$\Gamma(a,x) = \int_x^\infty \frac{t^{a-1}\mathrm{e}^{-t}}{\Gamma(a)} dt \tag{15}$$

is the "upper" incomplete gamma function.

Geometric gamma process family: bond price

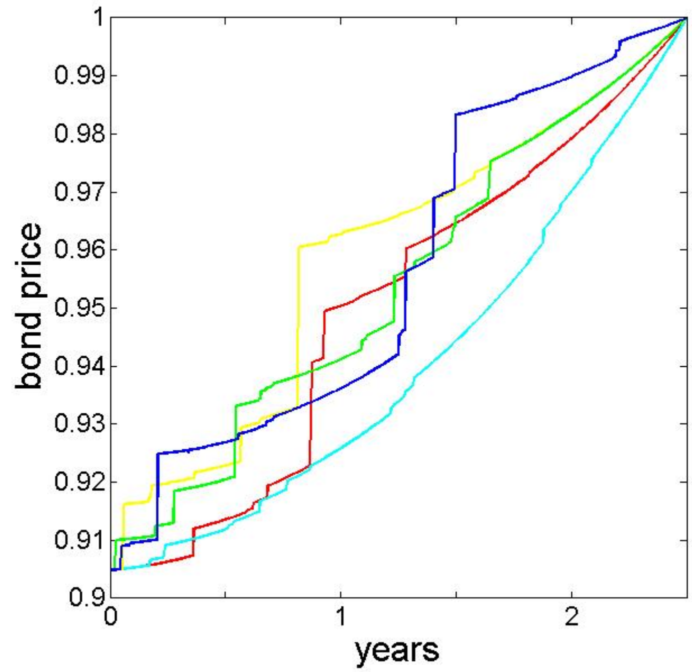

Figure 3: Simulation of the bond price in a term-structure density model with a parametric martingale family based on a geometric gamma martingale. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$, $m = 1$, $\kappa = 0.5$.

Geometric gamma process family: short rate

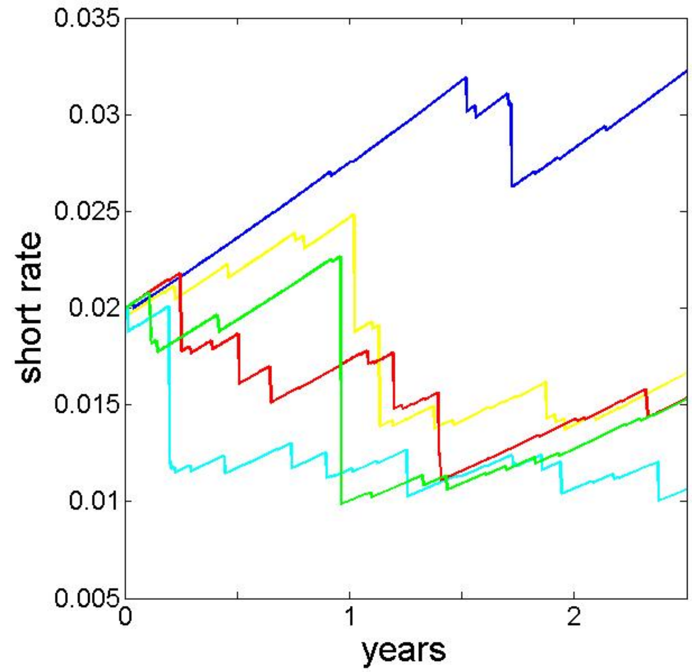

Figure 4: Simulation of the short rate in a term-structure density model with a parametric martingale family based on a geometric gamma martingale. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$, $m = 1$, $\kappa = 0.5$.

Geometric variance-gamma family

Let $\{V_t\}_{t\geq 0}$ be a variance-gamma process.

The increments $V_t - V_s$ of $\{V_t\}$ thus have the same distribution as a Brownian motion with volatility parameter σ and drift μ, time-changed by a gamma subordinator, whose increments have distribution $g(x)$ such that

$$g(x) = g_\gamma(x; (t-s)/\nu, \nu) \tag{16}$$

for $0 \leq s \leq t < \infty$.

The parameters μ, σ and ν control the properties of the variance-gamma distribution.

We are able to show that bond price is of the form

$$P_{tT} = \frac{\int_T^\infty \rho(x)(1-\nu\mu\phi(x)-\frac{1}{2}\nu\sigma^2\phi(x)^2)^{-\frac{t}{\nu}}\mathrm{e}^{\phi(x)V_t}\mathrm{d}x}{\int_t^\infty \rho(x)(1-\nu\mu\phi(x)-\frac{1}{2}\nu\sigma^2\phi(x)^2)^{-\frac{t}{\nu}}\mathrm{e}^{\phi(x)V_t}\mathrm{d}x}, \tag{17}$$

and that the associated short rate is given by

$$r_t = \frac{\rho(t)(1-\nu\mu\phi(t)-\frac{1}{2}\nu\sigma^2\phi(t)^2)^{-\frac{t}{\nu}}\mathrm{e}^{\phi(t)V_t}}{\int_T^\infty \rho(x)(1-\nu\mu\phi(x)-\frac{1}{2}\nu\sigma^2\phi(x)^2)^{-\frac{t}{\nu}}\mathrm{e}^{\phi(x)V_t}\mathrm{d}x}. \tag{18}$$

Using a methodology similar to that of Madan et al. [1998], we derive the option price in the example of the geometric variance-gamma process.

We first condition on the gamma time-change and numerically calculate ξ^*, which is a critical value of the Brownian motion on the boundary of positive payoffs, and then we integrate over the uncertainty of the gamma time-change.

The required option price is of the form

$$\begin{aligned} C_{0t} \; &= \int_T^\infty \rho(x)\Psi\left(\pm\xi^*\alpha(x), \mp\frac{\sigma\phi(x)}{\alpha(x)}, \frac{t}{\nu}\right)\mathrm{d}x \\ &\quad -K\int_t^\infty \rho(x)\Psi\left(\pm\xi^*\alpha(x), \mp\frac{\sigma\phi(x)}{\alpha(x)}, \frac{t}{\nu}\right)\mathrm{d}x. \end{aligned} \tag{19}$$

Here

$$\alpha(x) = \sqrt{\frac{1-\nu\mu\phi(x)-\frac{1}{2}\nu\sigma^2\phi(x)^2}{\nu}}, \tag{20}$$

and

$$\Psi(a,b,c) = \int_0^\infty \mathcal{N}\left(\frac{a}{\sqrt{u}}+b\sqrt{u}\right)\frac{u^{c-1}\mathrm{e}^{-u}}{\Gamma(c)}\mathrm{d}u. \tag{21}$$

Geometric variance-gamma process family: bond price

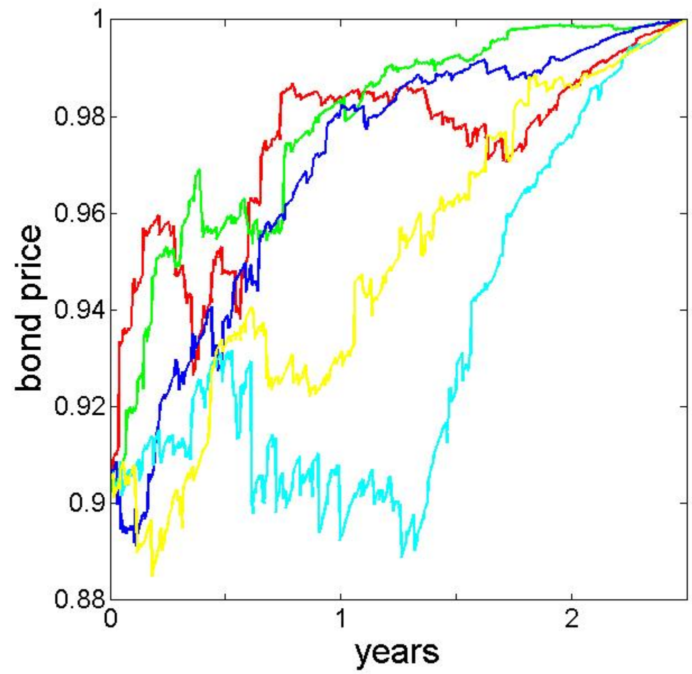

Figure 5: Simulation of the bond price based on a geometric variance-gamma martingale. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$, $\mu = 0.2$, $\sigma = 0.2$ $\nu = 0.1$.

Geometric variance-gamma process family: short rate

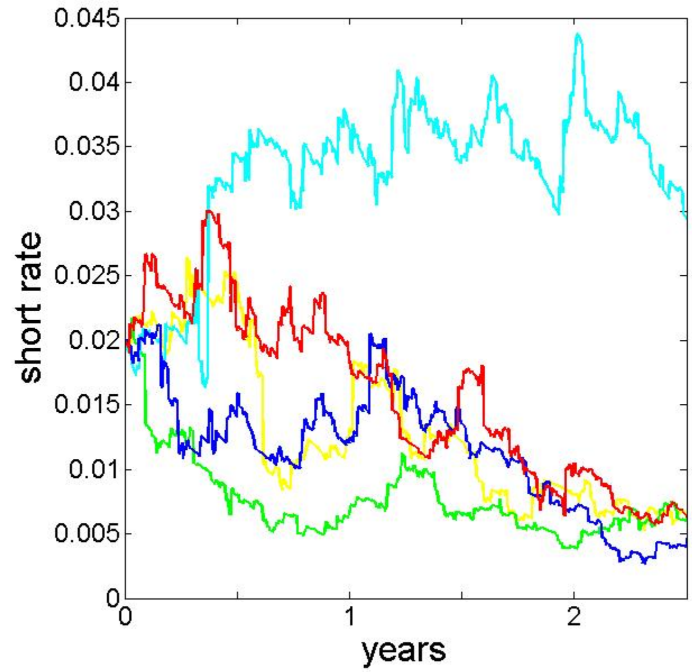

Figure 6: Simulation of the short rate based on a geometric variance-gamma martingale. The parameters are $\rho(x) = \mathrm{e}^{-0.02x}$, $\phi(x) = \mathrm{e}^{-0.025x}$, $\mu = 0.2$, $\sigma = 0.2$ $\nu = 0.1$.

Sensitivity analysis—option delta

In general we deduce that

$$C_{0t} = \int_T^\infty \rho(x) m_t(x) \mathrm{d}x - K \int_t^\infty \rho(x) m_t(x) \mathrm{d}x \tag{22}$$

where

$$m_t(x) = \mathbb{E}\left[\Theta\left(\int_T^\infty \rho(x) M_{tx} \mathrm{d}x - K \int_t^\infty \rho(x) M_{tx} \mathrm{d}x\right) M_{tx}\right]. \tag{23}$$

How does one determine the sensitivity of the option price C_{0t} on the initial price P_{0T} of the underlying?

Recall that the initial bond price is given by

$$P_{0T} = \int_T^\infty \rho(x) \mathrm{d}x. \tag{24}$$

In the case of interest-rate term structure, the initial price P_{0T} of the bond is a functional of the term structure density $\rho(x)$.

Likewise, the option price C_{0t} is a functional of $\rho(x)$.

Therefore, to determine the option sensitivity, we are required to employ the method of *functional derivative*

$$\frac{\delta C_{0t}}{\delta \rho} = \lim_{\varepsilon \to \infty} \frac{1}{\varepsilon} \left(C_{0t}[\rho + \varepsilon \eta] - C_{0t}[\rho] \right). \tag{25}$$

We consider a single perturbation, we take the perturbation to be a constant shift

$$R_{0T} \to R_{0T} + \varepsilon \tag{26}$$

of the initial yield curve R_{0T}.

Under this perturbation, we have

$$\rho(x) \to \rho(x) + \varepsilon(P_{0x} - x\rho(x)). \tag{27}$$

Note that the functional derivative of the initial bond price under this perturbation is given by

$$\frac{\delta P_{0T}}{\delta \rho} = -TP_{0T}. \tag{28}$$

Hence, by use of the chain rule we have

$$\frac{\delta C_{0t}}{\delta P_{0T}} = \frac{\delta \rho}{\delta P_{0T}} \frac{\delta C_{0t}}{\delta \rho} = -\frac{1}{TP_{0T}} \frac{\delta C_{0t}}{\delta \rho}. \tag{29}$$

After a short calculation we find that

$$\Delta = \int_T^\infty \frac{(x\rho(x) - P_{0x})}{TP_{0T}}\, m_t(x)\mathrm{d}x - K\int_t^\infty \frac{(x\rho(x) - P_{0x})}{TP_{0T}}\, m_t(x)\mathrm{d}x, \qquad (30)$$

In this way we are able to compute the option delta for a range of Lévy martingales $\{M_{tx}\}$.