About equity models based on Additive processes

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Outline

Motivation

Some recent works consider non-homogeneous time Lévy model to describe the implied volatility curve.

What about the dynamic of the smile curve ? Non-homogeneity property is not convenience, sticky delta . . .

Nevertheless, for some contracts (European, barrier, or American-style exercise) pricing with a stochastic volatility model or pricing with the additive process which has the same characteristic function gives the same result.

Definition

Let $(\Omega, \mathcal{F}_t, \mathbb{P})$ be *a complete filtered probability space* .

Definition

A stochastic process $(X_t)_{t\geq 0}$ on $\mathbb{R}$ is an additive process if the following conditions are satisfied :

1. The increments $X_{t_0}, \; X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent random variables for any partition $0 \leq t_1 \leq \cdots \leq t_n, \; n > 0$.
2. $X_0 = 0$ a.s.
3. It is continuous in probability, that is, for every $t \geq 0$ and $\epsilon > 0$, it holds
$$\lim_{s\to t} \mathbb{P}\{|X_t - X_s| > \epsilon\} = 0.$$
4. $(X_t)_{t\geq 0}$ is an adapted cad-lag stochastic process.

No stationary increments : the law of $X_{t+h} - X_t$ can depend on t.

Exponential Additive models

The value of an option is defined as a discounted conditional expectation of its terminal payoff H under a risk-adjusted martingale measure $\mathbb{Q}$:

$$C_t = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} H(S_T) | \mathcal{F}_t \right].$$

In *exponential additive* models, the (risk-neutral) dynamics of S_t under $\mathbb{Q}$ is represented as the exponential of a additive process:

$$S_t = S_0 \, e^{(r-q)t + X_t^{0,0}}.$$

Here $(X_t)_{t>0}$ is an additive process. The interest rate r and the dividend rate are supposed to be 0.

Construction

- The simplest way is to consider some well-known processes like Lévy process listed in [CT04].
- An other approach is to use the additive process which has the same characteristic function as some well-known processes like time-change Lévy processes or affine processes.
- Consider the self-decomposable additive processes presented in [CGMY07] $X_t = t^\gamma X$, It follows that the characteristic function of X_t is of the form

$$\Phi(\xi, t) = \mathbb{E}\left[e^{\imath \xi X_t}\right] = e^{L(\xi) t^\gamma}, \quad L(\xi) = \int_{\mathbb{R}} e^{\imath \xi} - 1 - \imath \xi x 1_{|x|<1} \quad k(x) dx.$$

- Construct a function ψ which respects all good properties to define an additive process. For example we can work on the cumulant function by parameterization as which is done on local volatility model.

Analytical properties

Holder property of the characteristic function

Let $T > 0$ a fixed time and $(X_t)_{t\geq 0}$ be an additive process on $\mathbb{R}$ such that $\Phi(\xi, t)$ is a function of class $\mathcal{C}^1$ on $[0, T]$. Then the characteristic function of $(X_t)_{t\geq 0}$ is

$$\Phi(\xi, t) = \mathbb{E}\left[e^{\imath\xi X_t}\right] = \exp\left(\int_0^t \psi(\xi, s)ds\right),$$

for $\xi \in \mathbb{R}$ and

$$\psi(\xi, t) = -\frac{1}{2}\xi^2\sigma(t)^2 + \imath\xi\mu(t) + \int_{\mathbb{R}}\left(e^{\imath\xi x} - 1 - \imath\xi 1_{|z|<1}\right)\nu(t, dx).$$

We now call $t \to (\sigma(t), \mu(t), \nu(\cdot, t))$ the generating triplet of the additive process.

Analytical properties

Under the risk neutral probability $\mathbb{Q}$, the infinitesimal generator $\mathcal{L}$ is given by :

$$\mathcal{L}f(x) = \frac{\sigma(t)^2}{2}\left[\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\right] + \int_{\mathbb{R}} \nu(dz,t)\ \left[f(x+z) - f(x) - (e^z - 1)\frac{\partial f}{\partial x}(x)\right],$$

ad is type of Fourier multiplier.

From the Bochner's theorem, for all $t \in \mathbb{R}^+$, $\xi \to \psi(\xi,t)$ is a negative-definite and

$$\forall \xi \in \mathbb{R} \qquad |\psi(\xi,t)| \leq C_\psi(t)\left(1 + |\xi|^2\right),$$

where $C_\psi(t) = 2\sup_{|\xi|\leq 1}|\psi(\xi,t)|$.

Analytical properties

Let us introduce β the smallest value in $(0,2)$, such that

$$\int_{|x|<1} |x|^{\beta} \nu(dx) < \infty.$$

In the case $\beta \leq 1$ (finite variation), we introduce the characteristic exponent,

$$\widetilde{\psi}(\xi, t) = \psi(\xi, t) - \imath\xi\gamma(t).$$

Proposition (Growth at infinity)

For all $0 \leq \beta \leq 2$,

$$\int_{\mathbb{R}} |x|^{\beta} \nu(dx) < \infty \Leftrightarrow \left|\widetilde{\psi}_r(\xi)\right| \underset{\infty}{\sim} \left|\widetilde{\psi}(\xi)\right| \underset{\infty}{\sim} |\xi|^{\beta},$$

and

$$\int_{\mathbb{R}^+} x^{\beta} |\nu(dx) - \nu(-dx)| < \infty \Leftrightarrow \left|\widetilde{\psi}_i(\xi)\right| \underset{\infty}{\lesssim} |\xi|^{\beta}, \quad \left|\widetilde{\psi}_i(\xi)\right| \underset{\infty}{\lesssim} \left|\widetilde{\psi}_r(\xi)\right|.$$

Overview of pricing methods

- Carr-Madan's method[CMS99] and Attari's method [Att04],
- Cosin expansion [FO09],
- Wiener Hopf factorization [KL09],
- PIDE methods.

Main advantages of PIDE methods:

- path-dependent options (barrier, Asian, loop back options, . . .).
- strongly non linear problem which appear in quantitative finance with discrete hedging or transaction cost problem.
- Dupire equation, see the last section, to solve one PIDE problem for all prices function of Strike and Maturity.

Galerkin methods

We consider thus the initial value problem

$$\frac{\partial u}{\partial t} - \mathcal{L}_t u = f \quad \text{in }]0, T] \times \mathbb{R}, \qquad u\,(t=0) = u_0 \quad \text{in } \mathbb{R}, \tag{1}$$

where u is typically the solution of the pricing equation associated to the additive process $(X_t)_{t\geq 0}$.$\mathcal{L}_t$ denotes the integro-differential operator. Using Fourier transform:

$$\forall\, \xi \in \mathbb{R} \qquad \frac{\partial \widehat{u}}{\partial t}\,(\xi) - \psi\,(\xi, t)\,\widehat{u}\,(\xi) = 0,$$

with $\widehat{u}\,(t = 0, \xi) = \widehat{u}_0\,(\xi)$. The variational form of the parabolic problem (1) is given by

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle - \langle \mathcal{L}u, v\rangle = (f, v)\,.$$

Using the Parseval identity,

$$\mathcal{E}_t\,(u, v) = -\int_{\mathbb{R}} \psi(\xi, t)\,\widehat{u}\,(\xi)\,\bar{\widehat{v}}\,(\xi)\,d\xi,$$

Galerkin methods

Proposition (Weak formulation case 1)

Suppose that β *is the smallest real value in* $(0,2)$ *such that :* $\int_{|x|<1} |x|^{\beta}\,\nu(dx) < \infty$ *then*

- *if* $\beta > 1$, *then the solution of the pricing equation* $u \in L^2\left(]0,T[;\mathcal{V}\right) \cap \mathcal{C}\left([0,T];\mathcal{H}_{\psi^\star}\right)$ *with* $\frac{\partial u}{\partial t} \in L^2\left(]0,T[;H^{-\beta/2}\right)$ *such that:* $\forall\ v \in \mathcal{H}_{\psi^\star}$, *for almost* $t \in [0,T]$,

$$ {}_{H^{-\beta/2}}\left\langle \frac{\partial u(t)}{\partial t}, v \right\rangle_{\mathcal{H}_{\psi^\star}} + \mathcal{E}\left(u(t), v\right) = {}_{H^{-\beta/2}}\left(f(t), v\right)_{\mathcal{H}_{\psi^\star}} $$

$$ u(0) = u_0, $$

has a unique solution. Moreover, there exist $C > 0$ *such that*

$$ \|u\|_{L^\infty([0,T],L^2(\mathbb{R}^n))} + \|u\|_{L^2(]0,T[;\mathcal{H}_{\psi^\star})} \leq C\left(\|u_0\|_{L^2} + \|f\|_{L^2(]0,T[;H^{-\beta/2})}\right). $$

Galerkin methods

Proposition (Weak formulation case 2)

- *if $\beta \leq 1$, then the solution of the pricing equation is obtained after the change of variable :*

$$\widetilde{u}(t,x) = u\left(t, x - \int_0^t \gamma(s)ds\right).$$

Then $\widetilde{u}$ is the unique solution of the pricing equation with the Fourier symbol $\widetilde{\psi}$. We have the weak formulation and an energy norm estimate for $\widetilde{u}$.

Key of proof:

- Gårding inequality is obtained in Sobolev space.
- Continuity estimate come from the new Fourier symbol $\widetilde{\psi}$.

Viscosity solution

Let us suppose $f \in L^2\left(0, T; L^2(\mathbb{R}^n)\right)$. We investigate the smoothing problem associated to eq (1) on which we add to the operator $\mathcal{L}$ a diffusion term $-\varepsilon\Delta$:

$$\frac{\partial u_\varepsilon}{\partial t} - \mathcal{L}_\varepsilon u_\varepsilon = f, \quad \text{in }]0, T] \times \mathbb{R}^n, \qquad u_\varepsilon(t=0) = u_0 \text{ in } \mathbb{R}^n,$$

where

$$\mathcal{L}_\varepsilon u_\varepsilon = \varepsilon \Delta u_\varepsilon + \mathcal{L} u_\varepsilon.$$

Space discretization

Let $V_p \subset \mathcal{D}(a))$ be a subspace of dimension $p := dim V_p$ generated by a finite element basis $\Phi := \{\varphi_j : j = 1, ..., p\}$. We use the Galerkin approach,

$$u_p(t,x) = \sum_{j=1}^{p} u_j(t)\varphi_j(x) \ \in V_p.$$

For each time $t \in [0,T]$ the semi discrete problem of finding the coefficient vector $\bar{u}(t) = (u_1(t), \ldots, u_p(t))$ is an initial value problem for p ordinary differential equations

$$M\frac{\partial \bar{u}}{\partial t}(t) + A\bar{u}(t) = 0, \bar{u}(0) = \bar{u}_0,$$

where $\bar{u}_0$ denotes the coefficient vector of decomposition of the function u_0 on the basis Φ, and M, A denote the mass and stiffness matrices with respect to the basis of V_p ,*i.e.*,

$$M_{i,j} = (\varphi_j, \varphi_i), \qquad A_{i,j} = \mathcal{E}(\varphi_j, \varphi_i).$$

Two computational problems :

- how to compute the entries of the matrix ?
- how to solve the linear system for a full matrix $M - \Delta t\, A = K = \left(K_{i,j}\right)_{1 \le i,j \le p-1}$.

Methods	Solver for linear system	Computed entries
[CV05]	FB substitution	Special Function
[Ach08]	LU +FB substitution	Special Function
[MSW06]	Iterative method	Special Function
based Toeplitz	Iterative method	work for all ψ.

Table: Numerical methods for PIDE

Toeplitz based method

Definition

- An *p-by-p* matrix $T_p = \left(t_{i,j}\right)_{1 \leq i,j \leq p}$ is said to be Toeplitz if $t_{i,j} = \mathbf{t}_{i-j}$ *i.e.*if T_p is constant along its diagonals.
- The matrix is said to be circulant if each diagonal $\mathbf{t}_k$ further satisfies $\mathbf{t}_{p-k} = \mathbf{t}_{-k}$ for $0 \leq k \leq p-1$.

$$T_p = \begin{pmatrix} \mathbf{t}_0 & \mathbf{t}_{-1} & \cdots & \mathbf{t}_{-(p-1)} \\ \mathbf{t}_1 & \mathbf{t}_0 & \cdots & \mathbf{t}_{-(p-2)} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{t}_{p-1} & \mathbf{t}_{p-2} & \cdots & \mathbf{t}_0 \end{pmatrix}, \qquad C_p = \begin{pmatrix} \mathbf{c}_0 & \mathbf{c}_{p-1} & \cdots & \mathbf{c}_1 \\ \mathbf{c}_1 & \mathbf{c}_0 & \cdots & \mathbf{c}_2 \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{c}_{p-1} & \mathbf{c}_{p-2} & \cdots & \mathbf{c}_0 \end{pmatrix}.$$

Toeplitz based method

Fast matrix-vector multiplication

$$\sum_{j=1}^{p} \mathbf{t}_{i-j} X_j = B_i \qquad \sum_{k=-(p-1)}^{p-1} \mathbf{t}_k \widetilde{X}_{i-k} = B_i, \quad 1 \leq i \leq p.$$

The convolution product is performed, with only $O(p \log p)$ operations, using Fourier transform:

$$B_i = \left(\mathrm{IDFT} \left[\mathrm{DFT} \left[\widetilde{X} \right] \cdot \mathrm{DFT} \left[\mathbf{t} \right] \right] \right)_i, \quad 1 \leq i \leq p.$$

where $\cdot$ denote the point-wise multiplication product of two vectors.

Toeplitz based method

Entries of the matrix operator

Using Euler implicit time discretization, the matrix operator is a Toeplitz matrix with :

$$T(k) = \int_{\mathbb{R}} \mathcal{G}(\xi)\, e^{\imath k\xi}\ d\xi, \quad \mathcal{G}(\xi) = \left(1 - \int_{t_{n-1}}^{t_n} \psi(\xi/h, s) ds\right) h \left|\varphi(\xi)\right|^2,$$

$$\widehat{\mathbf{t}}_q = \int_{\mathbb{R}} \mathcal{G}\left(\frac{\pi q}{p - \frac{1}{2}} - \xi\right) D_{p-1}(\xi)\, d\xi.$$

Introducing $\mathcal{S}(\xi) = \sum_{m=-\infty}^{\infty} \mathcal{G}(\xi - 2\pi m)$.

$$\widehat{\mathbf{t}}_q = (\mathcal{S} \star D_{p-1})\left(\frac{\pi q}{p - \frac{1}{2}}\right) \approx \mathcal{S}\left(\frac{\pi q}{p - \frac{1}{2}}\right).$$

Numerical results

European price under CGMY process

S	P	P_{ref}	Error %	S	P	P_{ref}	Error %
80.25	22.44	22.44	0.34	80.57	22.20	22.20	0.08
85.21	18.80	18.80	0.50	85.21	18.80	18.80	0.17
90.48	15.36	15.36	0.66	90.12	15.58	15.58	0.27
94.17	13.22	13.23	0.75	95.31	12.62	12.62	0.34
100.	10.33	10.34	0.82	100.	10.34	10.34	0.37
104.08	8.63	8.64	0.81	104.91	8.32	8.33	0.36
110.51	6.45	6.46	0.73	10.07	6.59	6.59	0.31
115.02	5.24	5.25	0.64	115.48	5.14	5.14	0.23
119.72	4.22	4.22	0.53	120.20	4.13	4.13	0.16

Table: Price of European contract

$\psi(\xi,t) = -\imath\mu\xi + C\Gamma(-Y)\left[G^Y - (G+\imath\xi)^Y + M^Y - (M-\imath\xi)^Y\right]$, Algorithm parameters: $p = 200$ - left, (resp. $p = 500$ right) space step, $N = 500$ number of time steps, S spot price. We solve the linear system using iterative solver (GMRES) with circulant preconditionner $\widehat{\mathbf{c}}_q = 2\,(\mathrm{Re}\,\mathcal{S})\left(\frac{\pi q}{p-\frac{1}{2}}\right)$ (at most 20 iterations).

Numerical Results

Down and out put price under CGMY process

s	p	p_{ref}	error %
90.95	4.43	4.45	1.96
95.86	3.87	3.89	1.38
101.04	3.42	3.43	1.65
105.95	3.04	3.05	1.33
111.09	2.69	2.71	2.07
115.86	2.40	2.40	0.35
120.85	2.13	2.13	0.21
126.04	1.88	1.89	1.05
91.17	13.22	0.252	0.75

Table: price of down and out put option

barrier at $S = 90$ and rebate of 50%. Reference price computed by Wiener Hopf factorization method.

Calibration by PIDE

The vector of unknown parameters θ is found by minimizing numerically the squared norm of the difference between market and model prices

$$\theta^\star = \arg\inf \sum_{i=1}^{N} \omega_i \left(P^i_{obs} - P^\theta(T_i, K_i) \right)^2 ,$$

where $(T, K) \to P^\theta(T, K)$ solve the Dupire PIDE.

Proposition

If X_t follows an exponential additive model, then the pseudo-differential operator ψ_d of P^θ is given by:

$$\psi_d(\xi, t) = \psi_b(t, -(u + \imath)) = \overline{\psi_b(t, \xi + \imath)}.$$

Proof, the price is homogeneous of order 1 in (S, K),

$$P(t, \lambda S, T, \lambda K) = \lambda P(t, S, T, K).$$

Calibration by PIDE, still undergoing work . . .

Following the method proposed in [Ach08],

- solve Dupire equation,
- solve adjoint problem to compute distribution of the fitting error,
- compute gradient direction.

We only need to solve 2 PIDE at each step of each step of the optimization problem. Can also be used for calibration on American options.

Discussion

- Extension of Galerkin method for finite variation process, using method of characteristic for transport dominated problem .
- New approach based on Toeplitz system to solve the PIDE by Galerkin method.
- Extension to more general process: stochastic volatility models ....

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