A Class of GIG Processes

An example of an example

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Chronology

1. The Brody-Hughston-Macrina (BHM) approach to information-based asset pricing is developed (2006-2008). See, e.g., Macrina (2006).
2. The gamma bridge information process is introduced for the modelling of cumulative gains/losses (BHM (2008)).
3. The BHM approach is extended to a class of Lévy-bridge information processes (H., Hughston and Macrina (2009)).
4. Lévy-bridge information is applied to non-life reserving (H., Hughston and Macrina (2010)).
5. The work presented here is based on an example from 4 which, in turn, is an example of 3.

GIG processes I

- We consider a class of increasing, stochastically-continuous processes, with stationary increments, defined over a finite time horizon $[0, T]$.
- In general, the increments of the processes are not independent.
- The *a priori* time-T distribution of the processes are generalized inverse-Gaussian (GIG).
- The processes are Markov.

GIG distribution

- The density of the GIG distribution is

$$f_{GIG}(x;\lambda,\delta,\gamma) = \mathbb{1}_{\{x>0\}} \left(\frac{\gamma}{\delta}\right)^{\lambda} \frac{x^{\lambda-1}\exp\left(-\frac{1}{2}\left(\delta^2 x^{-1} + \gamma^2 x^2\right)\right)}{K_\lambda[\gamma\delta]},$$

where $\delta, \gamma > 0$, $\lambda \in \mathbb{R}$, and $K_\nu[z]$ is the modified Bessel function.

- The kth moment of GIG random variable X is

$$\mathbb{E}[X^k] = \frac{K_{\lambda+k}[\gamma\delta]}{K_\lambda[\gamma\delta]} \left(\frac{\delta}{\gamma}\right)^k.$$

- The following identity is useful:

$$K_{n+1/2}[z] = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{j=0}^{n} (n+\tfrac{1}{2}, j)(2z)^{-j},$$

where (m,n) is Hankel's symbol

$$(m,n) = \frac{\Gamma[m+1/2+n]}{n!\Gamma[m+1/2-n]}.$$

GIG with $\lambda = n - 1/2$

- Fix $\gamma, c > 0$ and define

$$q_t^{(k)}(x) = f_{GIG}(x;,k - 1/2, ct, \gamma),$$

for $k \in \mathbb{N}_0$ and $t > 0$.

- $q_t^{(0)}(x)$ is an inverse-Gaussian density and has kth moment

$$m_t^{(k)} = \left[\frac{ct}{\gamma}\right]^k \sum_{j=0}^{k-1} (k - 1/2, j)(2ct\gamma)^{-j}.$$

- Fix $n \in \mathbb{N}_0$, then define the set of rational functions $\{w_{st}^{(k)}(x)\}_{k=0}^n$ by

$$w_{st}^{(k)}(x) = \frac{\binom{n}{k} m_{t-s}^{(n-k)} \sum_{j=0}^{k} \binom{k}{j} m_{T-t}^{(k-j)} x^j}{\sum_{j=0}^{n} \binom{n}{j} m_{T-t}^{(n-j)} x^j},$$

for $0 \le s < t < T$.

- It can be shown that $\sum_{k=0}^{n} w_{st}^{(k)}(x) = 1$.

Plot of $w_{st}^{(k)}$

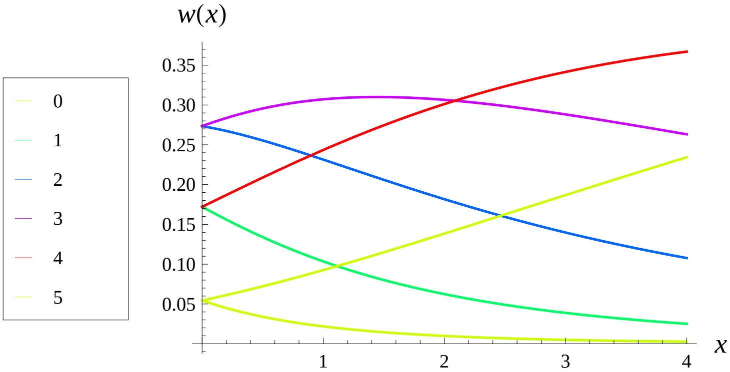


Figure: The rational functions $\{w_{st}^{(k)}\}$ for $n = 5$, $\gamma = 2$, $c = 2$, $s = 1$, $t = 3$, and $T = 5$.

GIG processes II

- Fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.
- We define the Markov process $\{\xi_t\}_{0\leq t\leq T}$ by

$$\mathbb{Q}[\xi_t \in \mathrm{d}y \,|\, \xi_s] = \sum_{k=0}^{n} w_{st}^{(k)}(\xi_s) q_{t-s}^{(k)}(y-\xi_s)\,,$$

$$\mathbb{Q}[\xi_T \in \mathrm{d}y \,|\, \xi_s] = \frac{y^n q_{T-s}^{(0)}(y-\xi_s)}{\sum_{k=0}^{n} \xi_s^k m_{T-s}^{(n-k)}},$$

  for $0 \leq s < t < T$, and with initial condition $\xi_0 = 0$.
- Note that it is non-trivial to prove that $\{\xi_t\}$ is well defined.
- *A priori*, ξ_T has a GIG distribution with parameters $\delta = cT$, $\gamma > 0$, and $\lambda = n - 1/2$.
- The increment $\xi_t - \xi_s$ depends on the first n powers of ξ_s.

Moments of the terminal value

- The moments of ξ_T can be calculated as

$$\mathbb{E}\left[\xi_T^k \,\middle|\, \xi_t\right] = \frac{\sum_{j=0}^{n+k}\binom{n+k}{j} m_{T-t}^{(n+k-j)}\xi_t^j}{\sum_{j=0}^{n}\binom{n}{j} m_{T-t}^{(n-j)}\xi_t^j},$$

  for $k \in \mathbb{N}_+$.
- These moments form a class of martingales, and are rational functions of an increasing Markov process.
- The Laplace transform of ξ_T is

$$\mathbb{E}\left[e^{\frac{1}{2}\alpha^2\xi_T} \,\middle|\, \xi_t\right] = \frac{\sum_{k=0}^{n}\binom{n}{k}\bar{m}_{T-t}^{(n-k)}\xi_t^k}{\sum_{k=0}^{n}\binom{n}{k} m_{T-t}^{(n-k)}\xi_t^k} \exp\left(\tfrac{1}{2}\alpha^2\xi_t - (T-t)(\bar{\gamma}-\gamma)\right),$$

  for $0 < \alpha < \gamma$, where $\bar{\gamma} = \sqrt{\gamma^2 - \alpha^2}$, and $\bar{m}_t^{(k)}$ is the kth moment of the IG distribution with parameters $\delta = ct$ and $\gamma = \bar{\gamma}$.

The Non-Life Reserving Problem

- Consider a non-life insurance company that underwrites various risks for a particular year in return for premiums.
- The insurer *incurs* claims over the one year period. However:
  - there may be a delay between the incurred date and the reported date,
  - the total size of the claim may not be known when the claim is reported,
  - the claim may not be paid by a single cash flow on a single date.
- The insurer may be paying these claims for many years.
- The problem is: how much money should the insurer *reserve* at a given time to cover all future claim payments?
  - This has implications for the insurer's accounting, tax liability, solvency, capital adequacy, and investment strategy.

Preliminaries

We examine the problem of reserving for an insurance company.

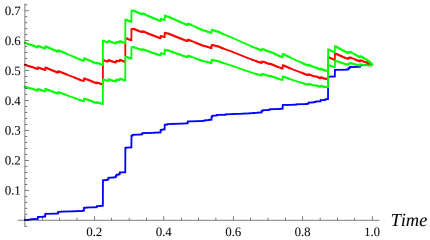
- We consider claims incurred from a single line of business during some *origin period* $[0, \bar{T}] \subset [0, T]$.
- The *ultimate loss* U_T is the total amount of claims paid.
- The insurer needs to hold reserves to cover future losses, and so wishes to estimate U_T, and to quantify the estimation error.
- The information used to estimate the reserves can be described by a *reserving filtration* $\{\mathcal{F}_t\}_{0 \leq t \leq T}$.
- At time $t < T$, the *best estimate (ultimate loss)* is $U_{tT} = \mathbb{E}[U_T \mid \mathcal{F}_t]$.

GIG-process model

We make the following assumptions:

1. All claims have been settled (paid) at time T.
2. U_T is a GIG random variable with parameters $\delta = cT$, γ, and $\lambda = n - 1/2$.
3. The (cumulative) paid-claims process $\{\xi_t\}$ is a GIG process with $\xi_T = U_T$.
4. The reserving filtration $\{\mathcal{F}_t\}$ is generated by $\{\xi_t\}$.

Best-estimate process simulations



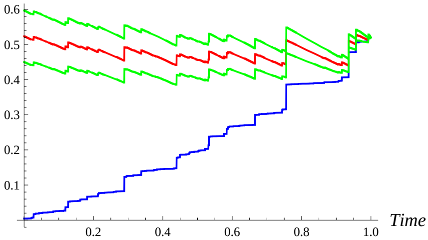


Figure: Paid-claims process (blue) and best-estimate process (red) with $n = 2$, $T = 1$, $c = 5$, $\gamma = 5$. The green lines give the best estimate $\pm$ one standard deviation.

VaR and CVaR

- The $\mathcal{F}_t$-conditional distribution function of the ultimate loss $U_T = \xi_T$ is

$$F_t(u) = \frac{\int_{\xi_t}^{u} y^n q_{T-t}^{(0)}(y - \xi_t)\, \mathrm{d}u}{\sum_{k=0}^{n} \binom{n}{k} m_{T-t}^{(n-k)} \xi_t^k}.$$

- The value-at-risk at level α is defined as

$$\mathrm{VaR}_\alpha = F_t^{-1}(\alpha), \qquad \alpha \in (0,1),$$

  and can be found by numerical inversion.
- At time t, the conditional value-at-risk at level α is defined as

$$\mathrm{CVaR}_\alpha = \mathbb{E}[U_T \,|\, U_T > \mathrm{VaR}_\alpha, \xi_t].$$

- A short calculation yields

$$\mathrm{CVaR}_\alpha = \frac{\sum_{k=0}^{n+1} \binom{n+1}{k} m_{T-t}^{(n-k+1)} \xi_t^k - \int_{\xi_t}^{\mathrm{VaR}_\alpha} u^{n+1} q_{T-t}^{(0)}(u - \xi_t)\, \mathrm{d}u}{(1-\alpha) \sum_{k=0}^{n} \binom{n}{k} m_{T-t}^{(n-k)} \xi_t^k}.$$

Tail-risk plots

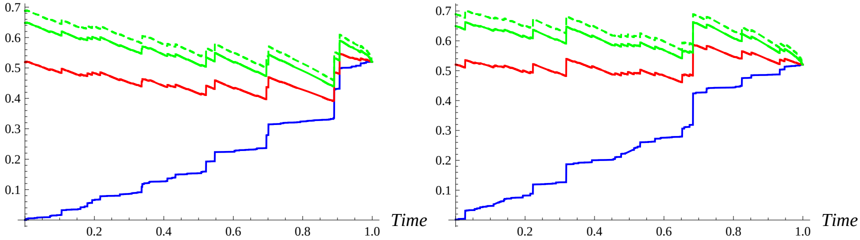


Figure: Paid-claims process (blue) and best-estimate process (red) with $n = 2$, $T = 1$, $c = 5$, $\gamma = 10$. The solid green line is the 95% VaR, and the dotted green line is the 95% CVaR.

Extreme Events

- For $0 < t < T$ we have

$$\lim_{x\to\infty} \frac{\mathbb{Q}\left[U_T > x \,|\, \xi_t\right]}{\mathbb{Q}\left[U_T > x\right]} = \frac{m_T^{(n)} \exp\left\{\frac{1}{2}\gamma^2\xi_t - ct\gamma\right\}}{\sum_{k=0}^{n} \binom{n}{k} m_{T-t}^{(n-k)} \xi_t^k} > 0.$$

- This shows that the tail of the conditional distribution of U_T is as heavy as the tail of the *a priori* distribution.
- This is a desirable property if the insurer is exposed to catastrophic losses.
- "The size of a catastrophe does not diminish with time."
- Note, on the other hand, that if $\{X_t\}$ is a Brownian motion, geometric Brownian motion, gamma process, or VG process then

$$\lim_{x\to\infty} \frac{\mathbb{Q}[X_T > x \,|\, X_t]}{\mathbb{Q}[X_T > x]} = 0.$$

Derivation of the GIG process

- Let $\{S_t\}$ be a stable-1/2 subordinator. That is, $\{S_t\}$ is an increasing Lévy process with Laplace transform

$$\log \mathbb{E}[e^{-\alpha S_t}] = -ct\sqrt{\frac{\alpha}{2}}, \quad \text{for } c > 0.$$

- Let X be a GIG random variable with parameters $\delta = cT$, $\gamma > 0$, and $\lambda = n - 1/2$.
- Then the conditioned process

$$\{S_t\}\big|_{S_T = X} \qquad (0 \leq t \leq T) \tag{1}$$

is a Lévy random bridge (LRB).
- LRBs are Markov processes, and analysis of the transition law of (1) show that it is identical in law to the GIG process $\{\xi_t\}$.

References

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