GRANULARITY ADJUSTMENT FOR DYNAMIC MULTIPLE FACTOR MODELS : SYSTEMATIC VS UNSYSTEMATIC RISKS

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INTRODUCTION

Risk measures such as

- Value-at-Risk (VaR)
- Expected Shortfall (also called TailVaR)
- Distortion Risk Measures (DRM)

are the basis of the new risk management policies and regulations for Finance (Basel 2) and Insurance (Solvency 2)

Risk measures are used to

i) define the reserves (minimum required capital) needed to hedge risky investments

(Pillar 1 of Basel 2 regulation)

ii) monitor the risk by means of internal risk models

(Pillar 2 of Basel 2 regulation)

Risk measures have to be computed for large portfolios of individual contracts :

- portfolios of loans and mortgages
- portfolios of life insurance contracts
- portfolios of Credit Default Swaps (CDS)

and for derivative assets written on such large portfolios :

- Mortgage Backed Securities (MBS)
- Collateralized Debt Obligations (CDO)
- Derivatives on iTraxx
- Insurance Linked Securities (ILS) and longevity bonds

The value of portfolio risk measures may be difficult to compute even numerically, due to

i) the large size of the portfolio (between $\simeq 100$ and $\simeq 10,000 - 100,000$ contracts)

ii) the nonlinearity of risks such as default, loss given default, claim occurrence, prepayment, lapse

iii) the dependence between individual risks, which is induced by the systematic risk components

The granularity principle [Gordy (2003)] allows to :

- derive closed form expressions for portfolio risk measures at order $1/n$, where n denotes the portfolio size
- separate the effect of systematic and idiosyncratic risks

[Gouriéroux, Laurent, Scaillet (2000), Tasche (2000), Wilde (2001), Martin, Wilde (2002), Emmer, Tasche (2005), Gordy, Lutkebohmert (2007)]

The value of the portfolio risk measure RM_n is decomposed as

$$\begin{aligned} RM_n &= \text{Asymptotic risk measure (corresponding to } n = \infty) \\ &\quad + \frac{1}{n} \text{ Adjustment term} \end{aligned}$$

The asymptotic portfolio risk measure, called

Cross-Sectional Asymptotic (CSA) risk measure

captures the effect of systematic risk on the portfolio value

The adjustment term, called

Granularity Adjustment (GA)

captures the effect of idiosyncratic risks which are not fully diversified for a portfolio of finite size

WHAT IS THIS PAPER ABOUT ?

We derive the granularity adjustment of Value-at-Risk (VaR) for general risk factor models where the systematic factor can be

- multidimensional and
- dynamic

We apply the GA approach to compute the portfolio VaR in a dynamic model with stochastic default and loss given default

Outline

1. STATIC MULTIPLE RISK FACTOR MODEL

1.1 Homogenous Portfolio

The individual risks

$$y_i = c(F, u_i)$$

depend on the vector of systematic factors F and the idiosyncratic risks u_i

Distributional assumptions

A.1 : *F and $(u_1, \ldots, u_n)$ are independent*

A.2 : *$u_1, \ldots, u_n$ are independent, identically distributed*

The portfolio is homogenous since the individual risks are **exchangeable**

Example 1 : Value of the Firm model [Vasicek (1991)]

The risk variables y_i are default indicators

$$y_i = \begin{cases} 1, \text{ if } A_i < L_i \quad \text{(default)} \\ \\ 0, \text{ otherwise} \end{cases}$$

where A_i and L_i are asset value and liability [Merton (1974)]

The log asset/liability ratios are such that $\log(A_i/L_i) = F + u_i$

Thus we get the single-factor model

$$y_i = \mathbb{1}_{F + u_i < 0}$$

considered in Basel 2 regulation [BCBS (2001)]

Example 2 : Model with Stochastic Drift and Volatility

The risks are (opposite) asset returns

$$y_i = F_1 + (F_2)^{1/2} u_i$$

where factor $F = (F_1, F_2)'$ is bivariate and includes

- common stochastic drift F_1
- common stochastic volatility F_2

1.2 Portfolio Risk

The total portfolio risk is :

$$W_n = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} c(F, u_i)$$

and corresponds to either a Profit and Loss (P&L) or a Loss and Profit (L&P) variable

The distribution of the portfolio risk W_n is typically unknown in closed form due to risk dependence and aggregation

Numerical integration or Monte-Carlo simulation can be very time consuming

1.3 Asymptotic Portfolio Risk

Limit theorems such as the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT) cannot be applied to the sequence $y_1, \ldots, y_n$ due to the common factors

However, LLN and CLT can be applied conditionally on factor values !

This is the so-called condition of **infinitely fine grained portfolio** in Basel 2 terminology

By applying the CLT conditionally on factor F, for large n we have

$$W_n/n = m(F) + \sigma(F)\frac{X}{\sqrt{n}} + O(1/n)$$

where

- $m(F) = E[y_i|F]$ is the conditional individual expected risk
- $\sigma^2(F) = V[y_i|F]$ is the conditional individual volatility
- X is a standard Gaussian variable independent of F
- the term at order $O(1/n)$ is conditionally zero mean

1.4 Granularity Principle

i) Standardized risk measures

The VaR of the portfolio explodes when portfolio size $n \to \infty$

It is preferable to consider the VaR per individual asset included in the portfolio, that is the quantile of W_n/n

For a L&P variable the VaR at level α is defined by the condition

$$P[W_n/n < VaR_n(\alpha)] = \alpha$$

where $\alpha = 95\%, 99\%, 99.5\%$

ii) The CSA risk measure

A portfolio with infinite size $n = \infty$ is not riskfree since the systematic risks are undiversifiable !

In fact, for $n = \infty$ we have :

$$W_n / n = m(F)$$

which is stochastic

We deduce that the CSA risk measure $VaR_\infty(\alpha)$ is the quantile associated with the systematic component $m(F)$:

$$P[m(F) < VaR_\infty(\alpha)] = \alpha$$

[Vasicek (1991)]

iii) Granularity Adjustment for the risk measure

The main result in granularity theory applied to risk measures provides the next term in the asymptotic expansion of $VaR_n(\alpha)$ with respect to n in a neighbourhood of $n = \infty$

Theorem 1 : *We have*

$$VaR_n(\alpha) = VaR_\infty(\alpha) + \frac{1}{n} GA(\alpha) + o(1/n)$$

where

$$\begin{aligned} GA(\alpha) &= -\frac{1}{2} \left\{ \frac{d \log g_\infty(w)}{dw} E[\sigma^2(F)|m(F) = w] \right. \\ &\left. + \frac{d}{dw} E[\sigma^2(F)|m(F) = w] \right\}_{w = VaR_\infty(\alpha)} \end{aligned}$$

and g_∞ denotes the probability density function of $m(F)$

- The expansion in Theorem 1 is useful since $VaR_{\infty}(\alpha)$ and $GA(\alpha)$ do not involve large dimensional integrals !
- The second term in the expansion is of order $1/n$. Hence, the granularity approximation can be accurate, even for rather small values of n ($\simeq$ 100)
- For single-factor models Theorem 1 provides the granularity adjustment derived in Gordy (2003)
- Theorem 1 applies for general multi-factor models
- The expansion is easily extended to the other Distortion Risk Measures [Wang (1996, 2000)], which are weighted averages of VaR, in particular to the Expected Shortfall

2. EXAMPLES

Example 1 : Value of the firm model

$$y_i = \mathbb{1}_{-\Phi^{-1}(PD) + \sqrt{\rho}F^* + \sqrt{1-\rho}u_i^* < 0}$$

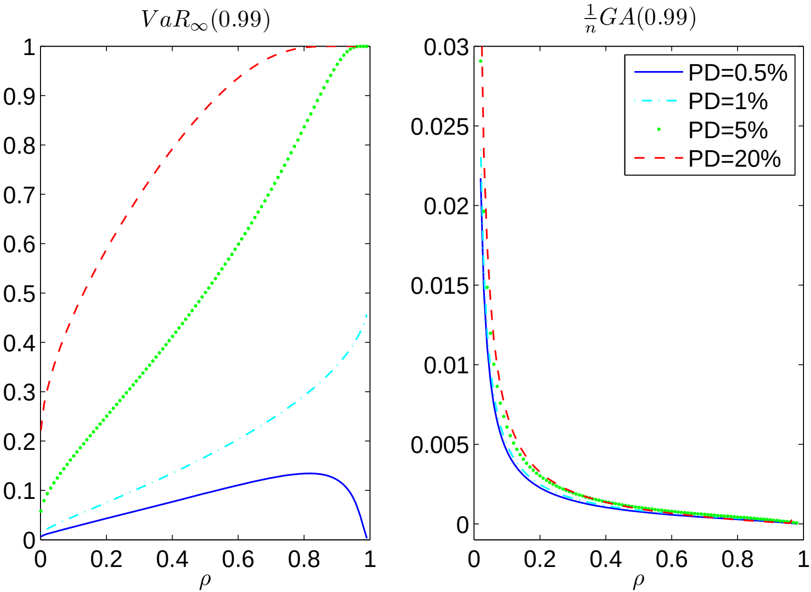
where $F^*, u_i^* \sim N(0,1)$ and PD is the unconditional probability of default and ρ is the asset correlation

$$VaR_\infty(\alpha) = \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)$$

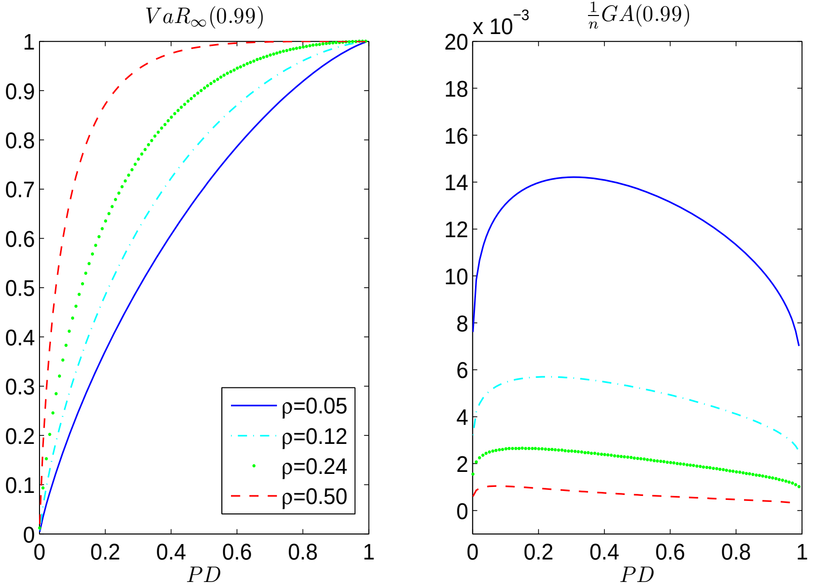
$$GA(\alpha) = \frac{1}{2}\left\{\frac{\sqrt{\frac{1-\rho}{\rho}}\Phi^{-1}(\alpha) - \Phi^{-1}\left[VaR_\infty(\alpha)\right]}{\phi\left(\Phi^{-1}\left[VaR_\infty(\alpha)\right]\right)} VaR_\infty(\alpha)[1 - VaR_\infty(\alpha)] \right.$$

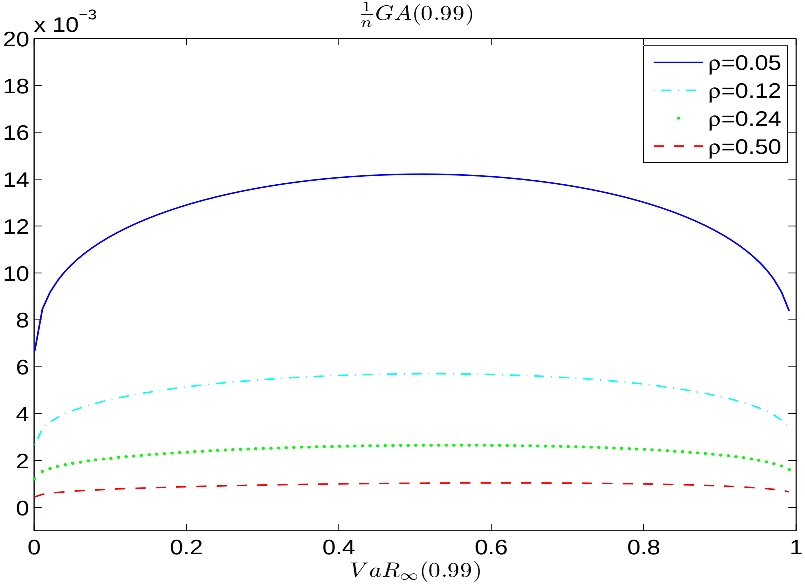
$$\left. + 2VaR_\infty(\alpha) - 1 \right\}$$

[cf. Emmer, Tasche (2005), formula (2.17)]



$VaR_\infty(0.99)$
$\frac{1}{n}GA(0.99)$
PD=0.5%
PD=1%
PD=5%
PD=20%
ρ
ρ


$VaR_\infty(0.99)$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 0.2 0.4 0.6 0.8 1
PD
ρ=0.05
ρ=0.12
ρ=0.24
ρ=0.50
$\frac{1}{n}GA(0.99)$
x 10^{-3}
20
18
16
14
12
10
8
6
4
2
0
0 0.2 0.4 0.6 0.8 1
PD



$\frac{1}{n}GA(0.99)$
x 10^-3
20
18
16
14
12
10
8
6
4
2
0
ρ=0.05
ρ=0.12
ρ=0.24
ρ=0.50
0
0.2
0.4
0.6
0.8
1
$VaR_\infty(0.99)$

Heterogeneity can be introduced into the model by including multiple idiosyncratic risks

Value of the firm model with heterogenous loadings

$$y_i = \mathbf{1}_{-\Phi^{-1}(PD) + \sqrt{\rho_i} F^* + \sqrt{1-\rho_i} v_i < 0} = c(F^*, u_i)$$

where $u_i = (v_i, \rho_i)'$ includes both firm specific shocks and factor loadings

Portfolio with heterogenous exposures

$$W_n = \sum_{i=1}^{n} A_i y_i = \sum_{i=1}^{n} A_i c(F^*, u_i) = \sum_{i=1}^{n} c(F^*, w_i)$$

where A_i are the individual exposures and $w_i = (u_i', A_i)'$

Example 2 : Stochastic Drift and Volatility

$$y_i \sim N(F_1, \exp F_2)$$

where $F = (F_1, F_2)' \sim N\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right]$

We have $m(F) = F_1$, $\sigma^2(F) = \exp F_2$ and

$$\frac{d}{dw} \log E[\sigma^2(F) | m(F) = w] = \frac{\rho\sigma_2}{\sigma_1} = \text{leverage effect !}$$

We deduce that :

$$\begin{aligned} VaR_\infty(\alpha) &= \mu_1 + \sigma_1 \Phi^{-1}(\alpha) \\ GA(\alpha) &= \frac{v_2^2}{2\sigma_1}[\Phi^{-1}(\alpha) - \rho\sigma_2] \exp\left(\rho\sigma_2\Phi^{-1}(\alpha) - \frac{\rho^2\sigma_2^2}{2}\right) \end{aligned}$$

where $v_2^2 = E[\exp F_2] = \exp\left(\mu_2 + \sigma_2^2/2\right)$

Example 3 : Stochastic Probability of Default and Loss Given Default

A zero-coupon corporate bond with loss at maturity :

$$y_i = Z_i LGD_i$$

where Z_i is the default indicator and LGD_i is the Loss Given Default

Conditional on factor $F = (F_1, F_2)'$, variables Z_i and LGD_i are independent such that

$$Z_i \sim \mathcal{B}(1, F_1), \quad LGD_i \sim Beta(a(F_2), b(F_2))$$

and

$$E\left[LGD_i | F\right] = F_2, \quad V\left[LGD_i | F\right] = \gamma F_2(1 - F_2)$$

with $\gamma \in (0, 1)$ constant

Example 3 : Stochastic Probability of Default and Loss Given Default (cont.)

We get a two-factor model

$F_1 = P[Z_i|F]$ is the conditional Probability of Default

$F_2 = E[LGD_i|F]$ is the conditional Expected Loss Given Default

The two factors F_1 and F_2 can be correlated

We derive the CSA risk measure and the GA with

$$\begin{aligned} m(F) &= F_1 F_2 \\ \sigma^2(F) &= \gamma F_2(1-F_2)F_1 + F_1(1-F_1)F_2^2 \end{aligned}$$

3. DYNAMIC RISK FACTOR MODEL

3.1 The model

Past observations are informative about future risk !

Consider a dynamic framework where the factor values include all relevant information

Static relationship between individual risks and systematic factors

$$y_{i,t} = c(F_t, u_{i,t})$$

A.3 : *The* $(u_{i,t})$ *are iid and independent of* (F_t)

(F_t) *is a Markov stochastic process*

The dynamics of individual risks are entirely due to the underlying dynamic of the systematic risk factor

3.2 Standardized portfolio risk measure

Future portfolio risk per individual asset

$$W_{n,t+1}/n = \frac{1}{n}\sum_{i=1}^{n} y_{i,t+1}$$

The dynamic VaR is defined by the equation :

$$P[W_{n,t+1}/n < VaR_{n,t}(\alpha)|I_{n,t}] = \alpha$$

where information $I_{n,t}$ includes all current and past individual risks $y_{i,t}, y_{i,t-1}, \ldots$, for $i = 1, \ldots, n$, but not the factor values

The quantile $VaR_{n,t}(\alpha)$ depends on the date t through the information $I_{n,t}$

3.3 Granularity adjustment

i) A first granularity adjustment

The general theory can be applied conditionally on the current factor value F_t. The conditional VaR is defined by :

$$P[W_{n,t+1}/n < VaR_n(\alpha, F_t)|F_t] = \alpha$$

and we have :

$$VaR_n(\alpha, F_t) = VaR_\infty(\alpha, F_t) + \frac{1}{n} GA(\alpha, F_t) + o(1/n)$$

where $VaR_\infty(\alpha, F_t)$ and $GA(\alpha, F_t)$ are computed as in the static case, but with an additional conditioning with respect to F_t

ii) A second granularity adjustment

However, the expansion above cannot be used directly since the current factor value is not observable !

Let the conditional pdf of y_{it} given F_t be denoted $h(y_{it}|f_t)$

The **cross-sectional maximum likelihood estimator** of f_t

$$\hat{f}_{nt} = \arg\max_{f_t} \sum_{i=1}^{n} \log h(y_{it}|f_t)$$

provides a consistent approximation of f_t as $n \to \infty$

Replacing f_t by $\hat{f}_{n,t}$ introduces an approximation error of order $1/n$ that requires an additional granularity adjustment

Use the approximate filtering distribution of F_t given $I_{n,t}$ at order $1/n$ derived in Gagliardini, Gouriéroux (2009) to get

$$VaR_n(\alpha) = VaR_\infty(\alpha, \hat{f}_{n,t}) + \frac{1}{n} GA(\alpha, \hat{f}_{n,t}) + \frac{1}{n} GA_{filt}(\alpha) + o(1/n)$$

Term $\frac{1}{n} GA_{filt}(\alpha)$ is an additional granularity adjustment of the risk measure due to non observability of the systematic factor

$GA_{filt}(\alpha)$ is given in closed form in the paper

4. DYNAMIC MODEL FOR DEFAULT AND LOSS GIVEN DEFAULT

A Value of the Firm model [Merton (1974), Vasicek (1991)] with non-zero recovery rate and dynamic systematic factor

Risk variable is percentage loss of debt holder at time t :

$$y_{i,t} = \mathbb{1}_{A_{i,t} < L_{i,t}} \left(1 - \frac{A_{i,t}}{L_{i,t}}\right) = \left(1 - \frac{A_{i,t}}{L_{i,t}}\right)^{+}$$

where $A_{i,t}$ and $L_{i,t}$ are asset value and liability at t

The loss $L_{i,t} y_{i,t}$ is the payoff of a put option written on the asset value and with strike equal to liability

The log asset/liability ratios follow a linear single factor model :

$$\log\left(\frac{A_{i,t}}{L_{i,t}}\right) = F_t + \sigma u_{i,t}$$

where $(u_{i,t}) \sim IIN(0,1)$ and (F_t) are independent

The systematic risk factor F_t follows a stationary Gaussian AR(1) process :

$$F_t = \mu + \gamma(F_{t-1} - \mu) + \eta\sqrt{1-\gamma^2}\varepsilon_t,$$

where $(\varepsilon_t) \sim IIN(0,1)$ and $|\gamma| < 1$

The model is parameterized by 4 structural parameters μ, η, σ and γ, or equivalently by means of :

- $PD = P\left[\log\left(\frac{A_{i,t}}{L_{i,t}}\right) < 0\right]$ probability of default
- $ELGD = E\left[1 - \frac{A_{i,t}}{L_{i,t}} | \frac{A_{i,t}}{L_{i,t}} < 1\right]$ expected loss given default
- $\rho = corr\left[\log\left(\frac{A_{i,t}}{L_{i,t}}\right), \log\left(\frac{A_{j,t}}{L_{j,t}}\right)\right]$ asset correlation ($i \neq j$)
- γ first-order autocorrelation of the factor

The parameterization by *PD*, *ELGD*, ρ, γ is convenient for calibration !

The cross-sectional maximum likelihood approximation of the factor value at date t is given by :

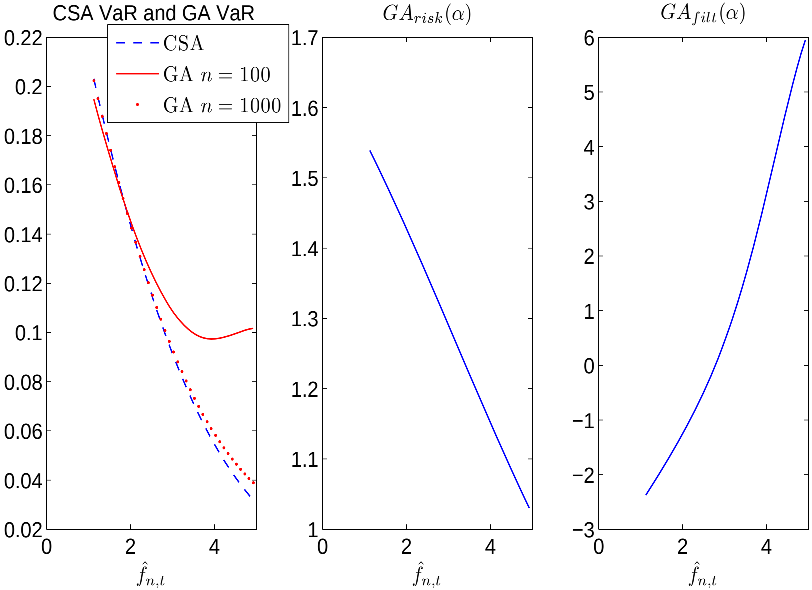
$$\hat{f}_{n,t} = \arg\max_{f_t} \left\{ -\frac{1}{2\sigma^2} \sum_{i:y_{i,t}>0} \left[\log(1-y_{i,t}) - f_t\right]^2 + (n-n_t)\log\Phi(f_t/\sigma) \right\}$$

where $n_t = \sum_{i=1}^{n} \mathbb{1}_{y_{i,t}>0}$ denotes the number of defaults at date t

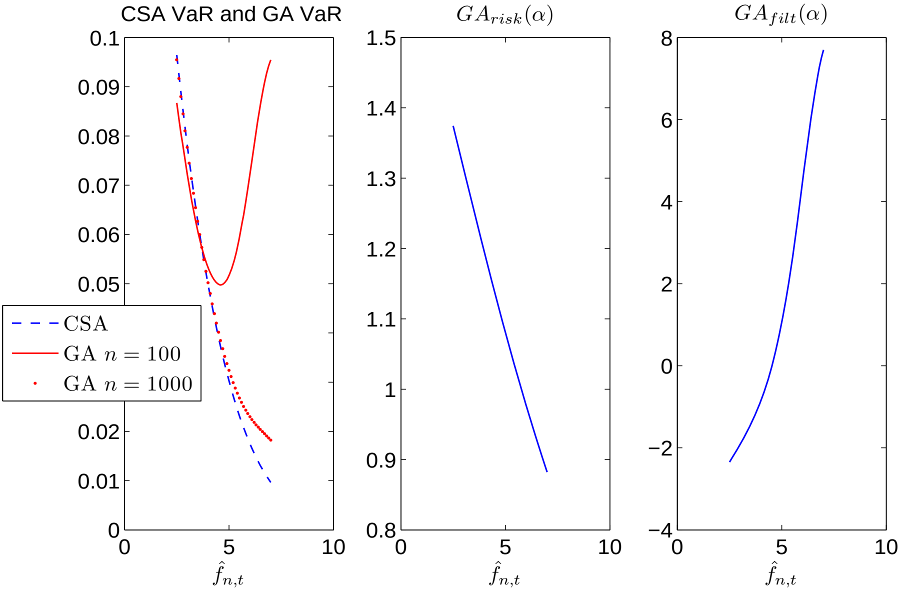
i.e. a Tobit Gaussian cross-sectional regression !

The filtering distribution depends on the available information through the current and lagged factor approximations $\hat{f}_{n,t}$ and $\hat{f}_{n,t-1}$ and the default frequency n_t/n

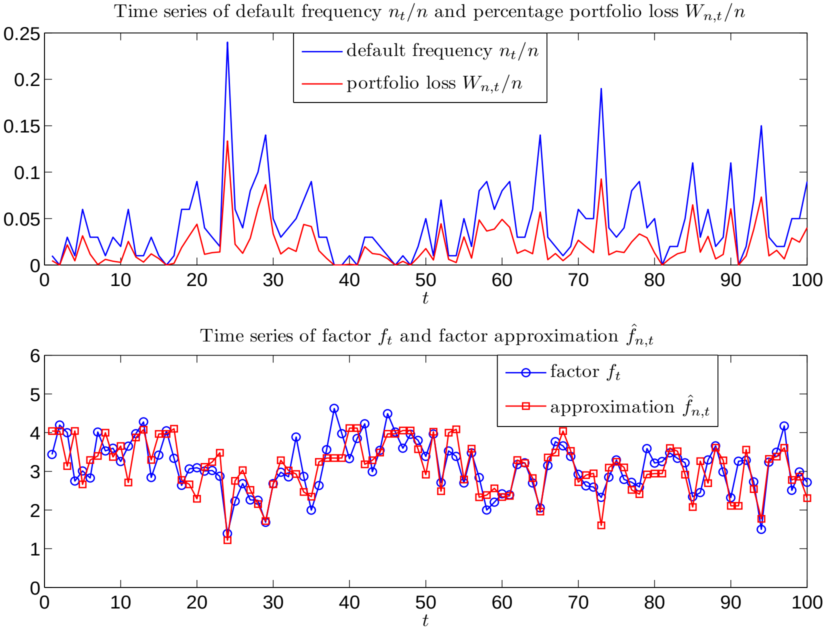
Parameters: $PD = 5\%$, $ELGD = 0.45$, $\rho = 0.12$, $\gamma = 0.5$. VaR: $\alpha = 99.5\%$



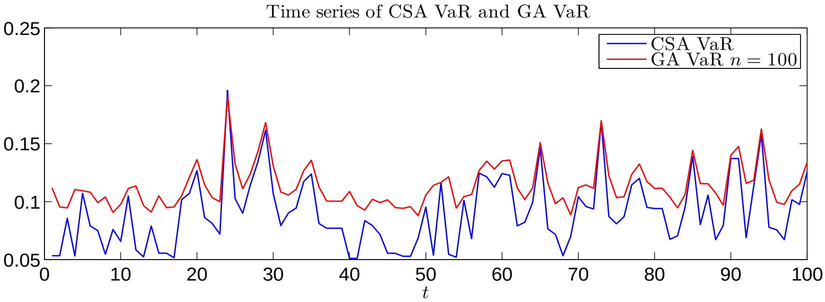
Parameters: $PD = 1.5\%$, $ELGD = 0.45$, $\rho = 0.12$, $\gamma = 0.5$. VaR: $\alpha = 99.5\%$

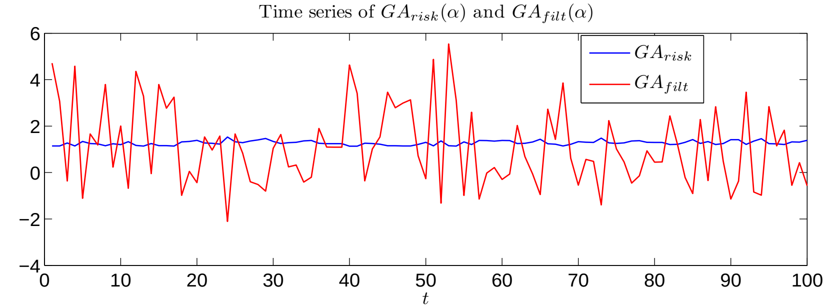


Parameters: $PD = 5\%$, $ELGD = 0.45$, $\rho = 0.12$, $\gamma = 0.5$. Portfolio: $n = 100$



Parameters: $PD = 5\%$, $ELGD = 0.45$, $\rho = 0.12$, $\gamma = 0.5$. Portfolio: $n = 100$





Backtesting of CSA VaR and GA VaR

$$H_t = \mathbf{1}_{W_{n,t}/n \geq VaR_{n,t-1}(\alpha)} - \alpha$$

	CSA	GA
$E[H_t]$	0.008	-0.001
$Corr(H_t, H_{t-1})$	-0.007	-0.004
$Corr(H_t, H_{t-2})$	0.002	-0.000
λ_2	-0.007	0.002
$Corr\left(H_t, \hat{f}_{n,t-1}\right)$	0.054	-0.022
$Corr\left(H_t, \hat{f}_{n,t-2}\right)$	0.005	0.002
$Corr\left(H_t, w_{n,t-1}\right)$	-0.034	0.019
$Corr\left(H_t, w_{n,t-2}\right)$	-0.002	0.002

5. CONCLUDING REMARKS

For large homogenous portfolios, closed form expressions of the VaR and other distortion risk measures can be derived at order $1/n$

Results apply for a rather general class of risk models with multiple factors in a dynamic framework

Two granularity adjustments are required :

The first GA concerns the conditional VaR with current factor value assumed to be observed

The second GA accounts for the unobservability of the factor

The GA principle appeared in Pillar 1 of the Basel Accord in 2001, concerning minimum required capital

It has been moved to Pillar 2 in the most recent version of the Basel Accord in 2003, concerning internal risk models

The recent financial crisis has shown the importance of distinguishing between idiosyncratic and systematic risks when computing reserves !

The GA technology can be useful for this purpose, e.g. by allowing to

- fix different risk levels for CSA and GA VaR
- smooth differently these components over the cycle when computing the reserves