Multiple Defaults and Density Approach : Global and Default-free Information

Ying JIAO

LPMA Université Paris 7

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Introduction

- The progressive enlargement of filtration plays an essential role in the credit risk modelling.
- Consider multiple default times $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ on the market $(\Omega, \mathcal{A}, \mathbb{P})$.
- There are default-free market information $(\mathcal{F}_t)_{t\geq 0}$ and default information $\mathcal{D}_t^i = \sigma(\tau_i \wedge t)$, $i \in \Theta = \{1, \cdots, n\}$.
- The global market information $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t^1 \vee \cdots \vee \mathcal{D}_t^n$.
- The pricing and hedging problems are considered in $\mathcal{G}_t$, however it is technically difficult to work with $\mathcal{G}_t$ in general since it is not generated by continuous processes.

Single default: before-default and after-default

- For a single defaul τ, the before-default pricing on $\{\tau > t\}$ (e.g. Bielecki-Jeanblanc -Rutkowski) is to establish a relationship between $\mathcal{G}_t$ and $\mathcal{F}_t$ by the key lemma of Dellacherie and Jeulin-Yor: for any $\mathcal{A}$-measurable r.v. Y,

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}[Y|\mathcal{G}_t] = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}[Y\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]}{\mathbb{P}(\tau > t|\mathcal{F}_t)} \quad a.s.$$

- On the after-default set $\{\tau \leq t\}$, the default density approach (El Karoui-Jeanblanc-J.) is suitable: for any $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$-measurable $Y_T(\theta)$,

$$\mathbb{1}_{\{\tau\leq t\}} E[Y_T(\tau)|\mathcal{G}_t] = \mathbb{1}_{\{\tau\leq t\}} \frac{E\left[Y(T,\theta)\alpha_T(\theta)\middle|\mathcal{F}_t\right]}{\alpha_t(\theta)}\Big|_{\theta=\tau} a.s.$$

where $\alpha_t(\theta)$ is the conditional density of τ given $\mathcal{F}_t$ wrt the law of τ.

Generalization to multiple defaults

- The pricing problem in $\mathcal{G}_t$ is decomposed into a before-default problem and an after-default one.
- Each new problem is considered wrt the default-free information $\mathcal{F}_t$ and the impact of past default events can be examined.
- The $\mathcal{F}_t$-conditional law of default or its density will play an essential role.
- Applications to pricing with multiple defaults and to counterparty risks.

Decomposition on default scenarios

- Let $I \subset \Theta = \{1, \cdots, n\}$ and $\tau_I = (\tau_i)_{i \in I}$.
- Describe the default scenario by the event

$$A_t^I := \Big(\bigcap_{i\in I}\{\tau_i \leq t\}\Big) \cap \Big(\bigcap_{i\notin I}\{\tau_i > t\}\Big)$$

 at time t, I denotes the default set.

- All default scenarios: $\Omega = \cup_{I \subset \Theta} A_t^I$.
- Any $\mathcal{G}_t$-measurable random variable Y_t can be written in the decomposed form

$$Y_t = \sum_{I\subset\Theta} \mathbb{1}_{A_t^I} Y_t^I(\tau_I)$$

 where $Y_t^I(\cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^I)$-measurable on $\Omega \times \mathbb{R}_+^I$.

Random measure of defaults

- We describe the $\mathcal{F}_t$-conditional law of $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$.
- Random measure $\mu^{\boldsymbol{\tau}}$ on $(\Omega \times \mathbb{R}_+^n, \mathcal{F}_\infty \otimes \mathcal{B}(\mathbb{R}_+^n))$: for any positive and $\mathcal{F}_\infty \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function $h_\infty(\boldsymbol{s})$, $\boldsymbol{s} = (s_1, \cdots, s_n)$,

$$\mathbb{E}[h_\infty(\boldsymbol{\tau})] = \int h_\infty(\boldsymbol{s})\mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}).$$

- The restriction $\mu_t^{\boldsymbol{\tau}}$ of $\mu^{\boldsymbol{\tau}}$ on $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$ represents the conditional law of $\boldsymbol{\tau}$ on $\mathcal{F}_t$: for any positive and $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function $h_t(\boldsymbol{s})$,

$$\mathbb{E}[h_t(\boldsymbol{\tau})] = \int h_t(\boldsymbol{s})\mu^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}) = \int h_t(\boldsymbol{s})\mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}).$$

General pricing formula

Theorem
Let $Y_T(\boldsymbol{\tau})$ be the payoff function where $Y_T(\boldsymbol{s})$ is positive and $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^n)$- measurable. Then for any $t < T$,

$$\mathbb{E}[Y_T(\boldsymbol{\tau})|\mathcal{G}_t] = \sum_{I \subset \Theta} \mathbb{1}_{A_t^I} \frac{\int_{]t,\infty[^{I^c}} \mathbb{E}[Y_T(s_I)\mu_T^{\boldsymbol{\tau}}|\mathcal{F}_t](d\omega, d\boldsymbol{s})}{\int_{]t,\infty[^{I^c}} \mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s})}\Big|_{s_I = \tau_I} \qquad (1)$$

- On each default scenario, (1) is interpreted as a Randn-Nikodym derivative on $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^I)$ and depends on the past default events τ_I.
- We observe a jump of the price at each default time.
- Choose models of μ^τ for default correlations and explicit pricing results.

Default density

Hypothesis

We say that $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ satisfies the density hypothesis if the measure $\mu^{\boldsymbol{\tau}}$ is absolutely continuous wrt $\mathbb{P} \otimes \nu^{\boldsymbol{\tau}}$ where $\nu^{\boldsymbol{\tau}}$ is the law of $\boldsymbol{\tau}$.

- $\nu^{\boldsymbol{\tau}}$ is the marginal measure $\nu^{\boldsymbol{\tau}}$ of $\mu^{\boldsymbol{\tau}}$ on $\mathcal{B}(\mathbb{R}_+^n)$, i.e., $\nu^{\boldsymbol{\tau}}(U) = \mu^{\boldsymbol{\tau}}(\Omega \times U)$, $\forall U \in \mathcal{B}(\mathbb{R}_+^n)$.
- Denote by $\alpha_t(\cdot)$ the density of $\mu^{\boldsymbol{\tau}}$ wrt $\mathbb{P} \otimes \nu^{\boldsymbol{\tau}}$ on $(\Omega \times \mathbb{R}_+^n, \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n))$,

$$\mu_t^{\boldsymbol{\tau}}(d\omega, d\boldsymbol{s}) = \alpha_t(\boldsymbol{s})\,\mathbb{P}(d\omega) \otimes \nu^{\boldsymbol{\tau}}(d\boldsymbol{s}).$$

- For any positive Borel function on $\mathbb{R}_+^n$,

$$\mathbb{E}[f(\boldsymbol{\tau})|\mathcal{F}_t] = \int_{\mathbb{R}_+^n} f(\boldsymbol{s})\alpha_t(\boldsymbol{s})\nu^{\tau}(ds).$$

Pricing with density

- For the k^{th}-to-default swap, consider the ordered set of defaults $\tau_{(1)} < \cdots < \tau_{(n)}$.
- The key term for the pricing is the indicator default process $\mathbb{1}_{\{\tau_{(k)}>T\}}$ with respect to the market filtration $\mathcal{G}_t$:

$$\mathbb{E}[\mathbb{1}_{\{\tau_{(k)}>T\}}|\mathcal{G}_t] = \sum_{|I|<k} \mathbb{1}_{A_t^I} \sum_{J\supset I,\, |J|<k} \frac{\int_{]T,\infty[^{J^c}} \int_{]t,T]^{J\setminus I}} \alpha_t(\boldsymbol{s})\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}{\int_{]t,\infty[^{I^c}} \alpha_t(\boldsymbol{s})\nu^{\boldsymbol{\tau}}(d\boldsymbol{s})}\Big|_{s_I=\tau_I}$$

- For CDO, the cumulative loss $L_t = \sum_{i=1}^{n} \mathbb{1}_{\tau_i \leq t}$, pricing via the ktD

$$(L_T - a)_+ = \sum_{k\geq a} \min(k-a, 1)\mathbb{1}_{\{\tau_{(k)}\leq T\}}.$$

- Both ktD and CDO depend on the ordered successive defaults.

Several remarks

- The density approach can also be applied to ordered defaults, making a link with the top-down models for the loss process L.
- The joint density characterize the correlation structure of defaults in a dynamic manner.
- Part of the joint density can be deduced from the individual default intensity processes. To obtain the whole term structure, we need more information.

Application: a contagion risk model

- The density approach and the decomposition methodology can be applied to multiple credit problems in a general way.
- Consider a portfolio of assets whose value process S is subjected to contagion default risks: an N-dimensional $\mathbb{G}$-adapted process $S_t = \sum_{I\subset\Theta} \mathbb{1}_{A_t^I} S_t^I(\tau_I)$.
- The asset value is affected by the counterparty defaults and has a regime switching at each default

$$dS_t^I(s_I) = S_t^I(s_I) * (\mu_t^I(s_I)dt + \Sigma_t^I(s_I)dW_t), \quad t > s_{\vee I}$$

and there is jump given default

$$S_{s_{\vee I}}^I(s_I) = S_{s_{\vee I}-}^J(s_J) * (\mathbf{1} - \gamma_{s_{\vee I}}^{J,k}(s_J)),$$

where $k = \min\{i \in I | s_i = s_{\vee I}\}$, $J = I \setminus \{k\}$ and $\gamma^{J,k}(\cdot)$ is $\mathcal{P}_{\mathbb{F}} \otimes \mathcal{B}(\mathbb{R}_+^J)$-measurable

Utility maximization of wealth

- Investment strategy is characterized by a $\mathbb{G}$-predictable process π which satisfies $\pi_t = \sum_I \mathbb{1}_{A^I_{t-}} \pi^I_t(\tau_I)$
- The wealth process X has the decomposed form $X_t = \sum \mathbb{1}_{A^I_t} X^I_t(\tau_I)$ such that

$$dX^I_t(s_I) = X^I_t(s_I)\pi^I_t(s_I) \cdot \left(\mu^I_t(s_I)dt + \Sigma^I_t(s_I)dW_t\right), \quad t > s_{\vee I}$$

  and

$$X^I_{s_{\vee I}}(s_I) = X^J_{s_{\vee I}-}(s_J)(1 - \pi^J_{s_{\vee I}}(s_J) \cdot \gamma^{J,k}_{s_{\vee I}})$$

- Optimal investment $\mathbb{E}[U(X_T)]$ for admissible trading strategies π or equivalently $(\pi^I(\cdot))_{I \subset \Theta}$. The admissible set $\mathcal{A}^I(s_I)$ such that $\int_0^T |\pi^I_t(s_I)\sigma^I_t(s_I)|^2 dt < \infty$ and $\pi^I_t(s_I) \cdot \gamma^{I,i}_t < 1$ for any $i \notin I$ and any $t \in]s_{\vee I}, T]$.

Optimization decomposition

- The global problem can be treated by the general theory of optimization.
- Motivations for adopting the decomposition methodology :
  - financially, to analyze the impact of past defaults on the optimal trading strategy
  - mathematically, to avoid the difficulty related to jump processes
- By decomposition of the terminal wealth

$$\mathbb{E}[U(X_T)] = \sum_{I\subset\Theta} \mathbb{E}[\mathbb{1}_{A_T^I} U(X_T^I(\tau_I))] = \sum_{I\subset\Theta} \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{A_T^I} U(X_T^I(\tau_I))|\mathcal{F}_T]\right]$$

$$= \mathbb{E}\Big[\sum_{I\subset\Theta} \int_{[0,T]^I\times]T,\infty[^{I^c}} U(X_T^I(s_I))\alpha_T(\mathbf{s})d\mathbf{s}\Big].$$

- The idea is to consider a family of optimization problems at each default scenario.

We shall treat the optimization problem in a backward and recursive way. Let

$$J_\Theta(x, \mathbf{s}, \pi^\Theta) := \mathbb{E}[U(X_T^\Theta(\mathbf{s}))\alpha_T(\mathbf{s}) \mid \mathcal{F}_{s_{\vee\Theta}}]_{X^\Theta_{s_{\vee\Theta}}(\mathbf{s})=x}$$

and

$$V_\Theta(x, \mathbf{s}) = \underset{\pi^\Theta \in \mathcal{A}^\Theta(\mathbf{s})}{esssup}\ J_\Theta(x, \mathbf{s}, \pi^\Theta).$$

We define recursively for $I \subset \Theta$,

$$J_I(x, s_I, \pi^I) := \mathbb{E}\Big[U(X_T^I(s_I)) \int_{]T,+\infty[^{I^c}} \alpha_T(\mathbf{s}) ds_{I^c} + \sum_{i \in I^c} \int_{]s_{\vee I},T]} V_{I\cup\{i\}}\big(X_{s_i}^{I\cup\{i\}}(s_{I\cup\{i\}}), s_{I\cup\{i\}}\big) ds_i \,\Big|\, \mathcal{F}_{s_{\vee I}}\Big]_{X^I_{s_{\vee I}}(s_I)=x}$$

and correspondingly

$$V_I(x, s_I) := \underset{\pi^I \in \mathcal{A}^I(s_I)}{esssup}\, J_I(x, s_I, \pi^I).$$

- We obtain a family of optimization problems $(V_I(x, s_I))_{I \subset \Theta}$.
- The whole system need to be dealt with in a recursive manner backwardly, each problem concerning the filtration $(\mathcal{F}_t)_{t \geq 0}$ and the time interval $[s_{\vee I}, T]$.
- At each step, the problem V_I involves the resolution of other ones $V_{I \cup \{i\}}$.
- By resolving recursively the problems, we obtain a family of optimal strategies $(\widehat{\pi}^I(\cdot))_{I \subset \Theta}$, which shall give the optimal strategy of the initial optimization problem.

Theorem

$$\sup_{\pi \in \mathcal{A}} \mathbb{E}[U(X_T)]_{X_0 = x} = V_{\emptyset}(x). \tag{2}$$

Conclusions

- The default density approach provides a suitable framework for financial problems concerning multiple defaults.
- By the two applications to pricing and to optimal investment, we show that the initial problem in global information filtration can be decomposed to problems in default-free information filtration (often a Brownian filtration), with which we are more familiar.
- The decomposition method also allows to analyze the contagion default impact in an explicit manner.

Related works

- El Karoui, N., Jeanblanc, M. and Jiao, Y. (2009), “What happens after a default: the conditional density approach”, *Stochastic Processes and their Applications*, 120(7), 1011-1032.
- El Karoui, N., Jeanblanc, M. and Jiao, Y. (2010), “Modelling successive default events”, working paper.
- Jiao, Y. (2010), “Multiple defaults and contagion risks with global and default-free information”, working paper.
- Jiao, Y. and Pham, H. (2009), “Optimal investment with counterparty risk: a default-density model approach”, to appear in *Finance and Stochastics*.

Thanks for your attention !