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A CDO option market model for standardized CDS index tranches

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Economic Motivation

Market Context

- CDO is a OTC Product ⇒ High Transaction Costs
- "Liquidity Gap" costs precious Basis Points

⇒ Initialization of a standardized synthetic CDO Market (CDX/iTraXX)

Modeling Constraints

- Credit Derivatives : "Static" Models ⇒ The investor does not pay for the Véga !
- Pricing of CDO tranches with option alike pay-offs (Deal Spread, Cumulative Loss as underlying)
- "Maturity Trap"

→ need for Spread Dynamics !

Standardization Assumptions

- Underlying CDS Portfolio restricted to components of CDX / iTraXX index series
- Pre-Set Attachment / Detachment Points

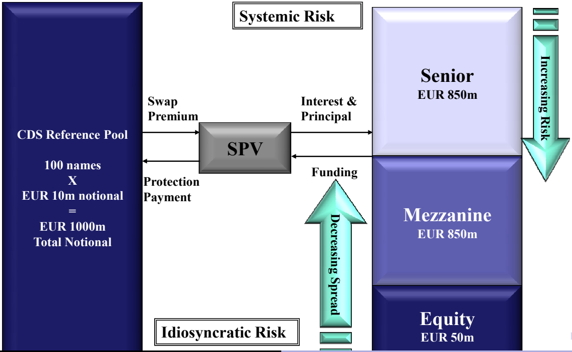
$\longrightarrow$ Success Story $\Rightarrow$ option trading possible

CDS Index Tranches

- CDS Index Tranches securitize CDS Index Series.
- Attachment / Detachment Points are standardized $[0\%, 3\%, 6\%, 9\%, 12\%, 22\%, 100\%]$
  $\Rightarrow$ improves liquidity, reduces ramp-up costs for structurers

Synthetic CDO Structure

- Only synthetic CDOs (CDOs on a CDS portfolio) allow for product standardization and hence for liquidity
- CDOs securitize credit spreads and issue tranches → Leverage



Related Literature

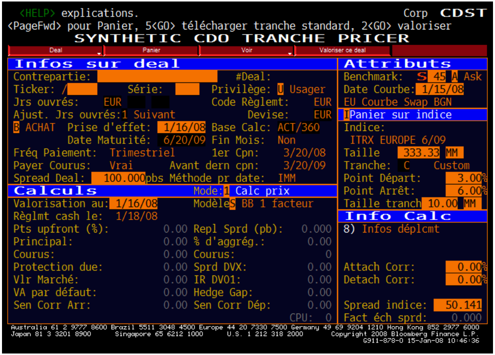
- BGM model : Arbitrage-Free model for other than instantaneous, continuously compounded forward rates
  - The idea is to chose a different numeraire other than the risk-free account
  - Leads to Black's formula $\longrightarrow$ we refer to as "market models"
  - First attempt to model a market-implied term structure of forward rates
- CDS option MM : Brigo & Mercurio transferred the idea of a market model into the credit derivatives environment
  - One-Period Spread modeling approach applied to the CDS market, with approximation constraints
- D. Filipovic, L. Overbeck and T. Schmidt. "Dynamic CDO Term Structure Modelling" (2008)

Scope

- provide a market model for tranche spread dynamics
- practitioner-oriented approximation method which avoids modeling the 125 underlying CDS
- incorporates CDS correl assumptions market-implied (wrong or false)
- underlier are traded tranche by tranche $\Rightarrow$ no need for cross tranche model
- every tranche has its own correl assumption $\Rightarrow$ individual asset
- Dynamics become even more relevant with upcoming CDS clearing chamber

Scope

We aim to provide a framework that justifies B&S market practice application.



Scope

- We consider a CDO tranche with AP $D\%$ and DP $E\%$ and tenor $[T_a; T_b]$.
- The aim consists in finding a recursive formula for market-implied Spread Dynamics !
- $\Rightarrow$ need for liquid market data.

Lemma

Let $\Pi_{CallCDO_{a,b}^{D,E}}(t,K)$ describe the $t-$time pay-off of a forward start call option written on standardized CDO tranche with boundaries $[D\%; E\%]$. The tenor is $[T_a; T_b]$. Within the Black & Scholes framework the Call option takes the value

$$\Pi_{CallCDO_{a,b}^{D,E}}(t,K) = \hat{C}_{a,b}^{D,E}(t) \times \left[S_{a,b}^{D,E}(t)N(d_1) - K \times N(d_2)\right] \tag{1}$$

with

$$\hat{C}_{a,b}^{D,E}(T_a) =: \sum_{i=a+1}^{b} \delta_i B(T_a, T_i) E_{Q^{T_i}}^{t}\left[X(T_i)\right]$$

and

$$d_{1,2} = \frac{ln\left(\frac{S_{a,b}^{D,E}(t)}{K}\right) \pm (T_a - t)\frac{1}{2}\int_t^{T_a} \sigma_{a,b}^2(s)ds}{\sigma_{a,b}(T_a - t)\sqrt{T_a - t}}$$

Recall CDO Spread Determinants

Definition "CDO Premium Leg" :

- Sum of discounted Cash-Flows perceived by the Trancheholder

Definition "CDO Protection Leg" :

- Sum of the discounted reductions of a tranche's notional inherent to credit events which lead to a decrease in the Trancheholder's "spread revenue".

Definition "Fair Spread" :

- The t-time Fair Spread is the Spread the investor should have contracted instead of Deal Spread + Euribor/Libor at issuing date in order to allow the tranche quote at par at time t.

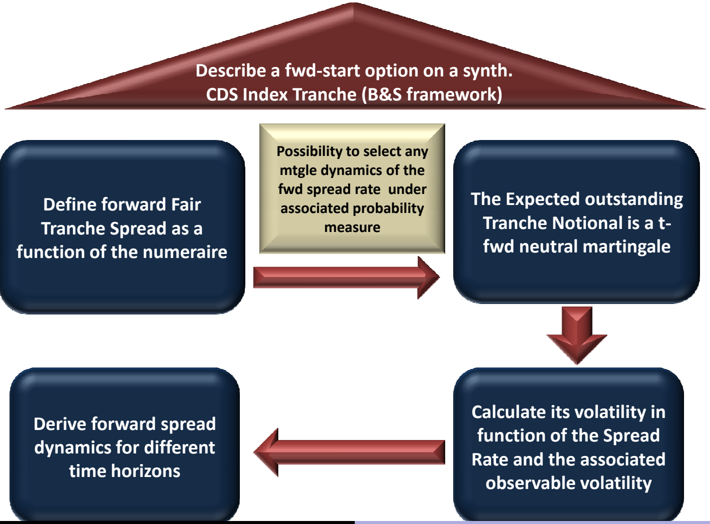
$$\text{Fair Spread} = \frac{\text{Protection Leg}}{\text{Premium Leg}}$$

Notional Erosion $\rightarrow$ Spread has to be calculated on the outstanding Tranche Notional

Definition Forward Fair Tranche Spread

$$\text{Fwd Fair Spread} = \frac{B(t, T_i)\text{Protection Leg}_i}{\sum_i B(t, T_i)\text{Premium Leg}_i}$$

The Model - Central Idea



Step 1- The Fwd Spread Dynamics

Definition

$$S_{a,b}^{D,E}(t) = \frac{Protleg(t)}{\text{Premleg(t)}}$$

Fwd-neutral Measure

$$\frac{dQ_{a,b}^{D,E}}{dQ} = \frac{\approx}{\text{Premleg(t)}}$$

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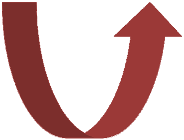

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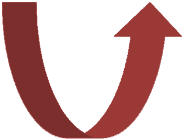


- Hence $S_{a,b}^{D,E}(t)$ is a $Q_{a,b}^{D,E}$ - martingale.
- $\frac{dS_{a,b}^{D,E}(t)}{S_{a,b}^{D,E}(t)} = \sigma_{a,b}(t)dW_t^{a,b}$
- $\frac{Y_{i-1}(t)}{Y_i(t)}$ introduces recursion

Step 2 - Shortfall Dynamics

The expected outstanding tranche notional $Y_i(t)$ is a Q^{T_i}-martingale. Its dynamics under the forward-neutral probability Q^t follows :

$$\frac{dY_i(t)}{Y_i(t)} = \gamma_i(t) dZ_t$$

Step 3 - Deriving the Volatility

Lemma

$\forall k \in [a+1; b]$ the volatility of the process Y_k related to tenor $[T_a, T_b]$ is given by

$$\gamma_k(t) = -\sum_{j=a+1}^{k} \left(\frac{\delta_j S_{j-1,j}^{D,E}(t)}{1 + \delta_j S_{j-1,j}^{D,E}(t)} \sigma_{j-1,j}(t) \right)$$

Step 4 - The Fwd one-period Spread Dynamics

Corollary

Consider a deal with tenor $[T_a, T_b]$ *and tranche* $[D, E]$*. The dynamics of the forward one-period Fair Tranche Spread on tenor* $[T_{i-1}, T_i]$ *is given by :*

$$\frac{dS^{D,E}_{i-1,i}(t)}{S^{D,E}_{i-1,i}(t)} = \sigma_{i-1,i}(t)\rho \sum_{j=a+1}^{i} \left(\frac{\delta_j S^{D,E}_{j-1,j}(t)}{1+\delta_j S^{D,E}_{j-1,j}(t)} \left(\sigma_{j-1,j}(t)\right)' \right) dt + \sigma_{i-1,i}(t) dZ_t$$

More precisely, for a deal with tenor $[T_{i-1}, T_i]$*, the forward one-period Fair Tranche Spread dynamics for the same tenor amounts to :*

$$\frac{dS^{D,E}_{i-1,i}(t)}{S^{D,E}_{i-1,i}(t)} = \frac{\delta_i S^{D,E}_{i-1,i}(t)}{1+\delta_i S^{D,E}_{i-1,i}(t)} |\sigma_{i-1,i}(t)|^2 dt + \sigma_{i-1,i}(t) dW_t$$

Step 5 - The Multi-Period Extension

Lemma

Again consider a deal with tenor $[T_a, T_b]$ and tranche $[D, E]$. The forward multi-period spread dynamics with the same tenor, note $S_{a,b}^{D,E}$, can be written as

$$\frac{dS_{a,b}^{D,E}(t)}{S_{a,b}^{D,E}(t)} = (\Lambda(t) + \varsigma(t))\,\rho\,(\Lambda(t))'\,dt - (\Lambda(t) + \varsigma(t))\,dZ_t$$

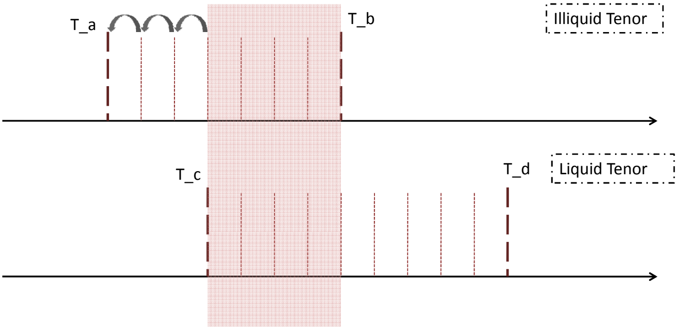
with

$$\Lambda(t) = \sum_{i=a+1}^{b} \frac{\delta_i A(t, T_i) Y_i(t)}{\hat{C}_{a,b}^{D,E}(t)} \gamma_i(t)$$

$$\varsigma(t) = \frac{A(t, T_b) Y_b(t)}{A(t, T_a) Y_a(t) - A(t, T_b) Y_b(t)} \gamma_b(t)$$

Conclusion

- Market Model allows for calibration of options with bespoke exercise periods to options with more liquid tenors thanks to multi-period fwd Tranche Spread Dynamics ⇒ More realistic prices.



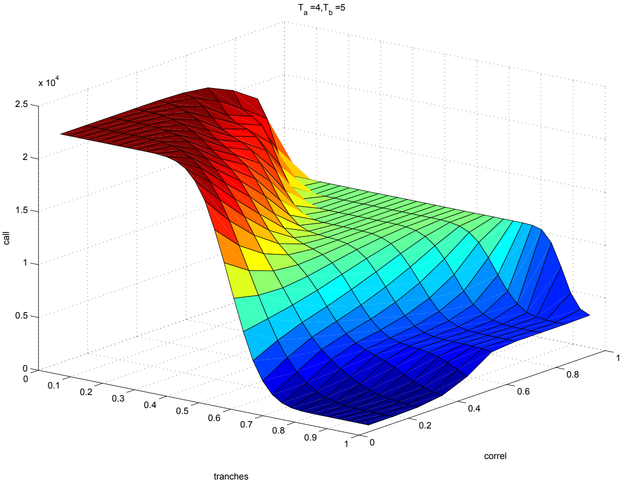
Conclusion

- Possibility of pricing options on tranches with future ramp-up dates $\Rightarrow$ Fwd spread is no longer a martingale $\Rightarrow$Calculate expectations of the fwd spread dynamics !
- Fwd spread dynamics allow for modeling of deals with complicated pay-offs !
- The investor finally pays for the Véga !

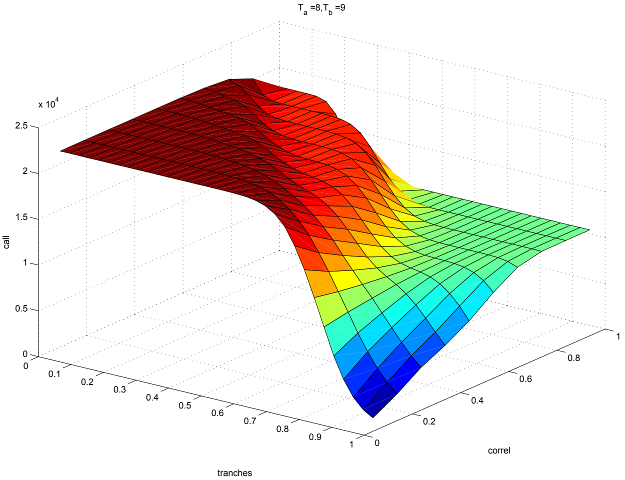
Implementation (1)

Interest rate assumptions	Deterministic
Recovery Rate	40% (market standard)
Implied volatilities of fwd-start options (in the money) for 3 maturities : 1 months, 3 months and 7 months	Semi-parametric approach (cf Gatarek [?])
Strike Spread K	0,02
Method used for implying the default intensity	student-t copula, degrees of freedom chosen in analogy to Hull [?]
Dataset used	iTraXX Series 8, daily data throughout may 2008 Bloomberg ID : ITRXEB58 (most recent and liquid series in May 2008)
Forward rates, yield curve	Provided by SGSS, Euro-VL

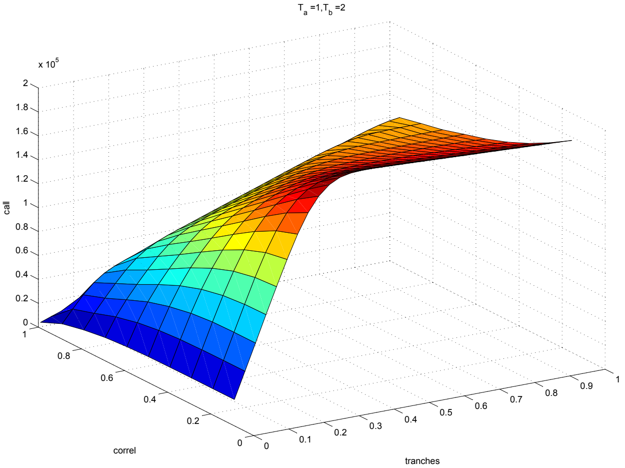
Implementation (1)



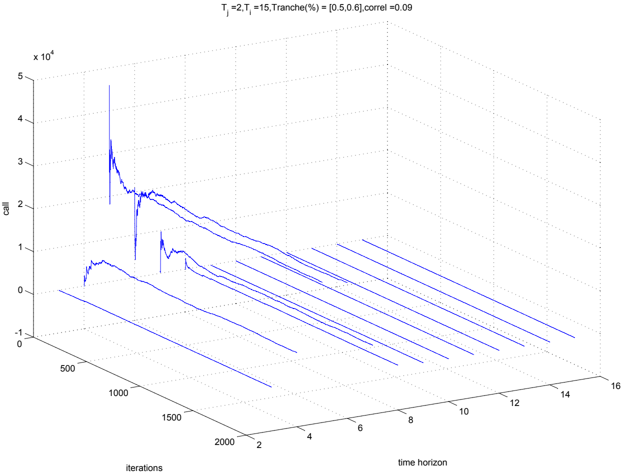
Implementation (2)



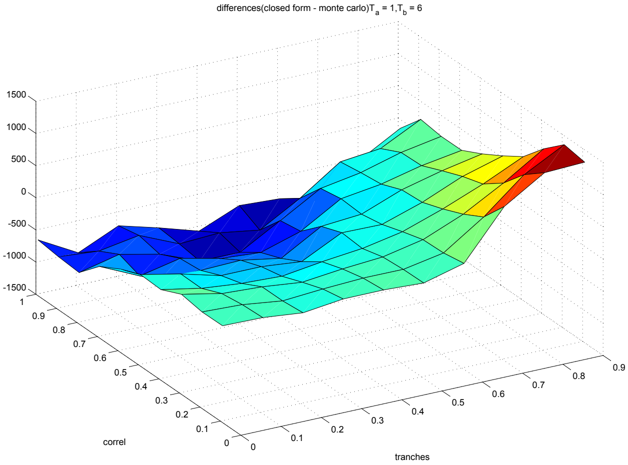
Implementation (3)



Implementation (4)



Implementation (5)



Outlook

- Approach might serve to model bespoke CDOs.
- The spread on a CDO tranche can be replicated by a Call Spread on the CDO's cumulative Loss Given Default (LGD) with strikes being the respective Attachment/Detachment Points.
- Hence by modeling the LGD dynamics there should be a way to price bespoke CDO tranches.

THANK YOU FOR YOUR ATTENTION !