# Two-Factor Capital Structure Models for Equity and Credit

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6th World Congress of the Bachelier Finance Society

# Outline

# Literature

1. Unlimited liabilities, reserve capital requirements and the taxpayer put option, E. Eberlein and D. Madan.
2. Modelling credit derivatives via time changed Brownian motion, T. R. Hurd and Z. Zhou
3. Do credit spreads reflect stationary leverage ratios, Pierre Collin-Dufresne and R. Goldstein
4. Liquidity and leverage, T. Adrian and Hyun Song Shin.
5. Time changed Markov processes in unified credit-equity modelling, R. Mendoza, P. Carr and V. Linetsky.
6. A unified framework for pricing credit and equity derivatives, E. Bayraktar and B. Yang.

# Empirical observations

Equity and credit markets are intrinsically connected:

- Cremers (2006) et. al. find option ATM implied volatility and option implied skews explain 32% of the variation in long-term credit spreads.
- Carr and Wu (2009) find cross-correlation between weekly changes of average CDS and option implied volatility as high as 0.79, between weekly changes of average CDS and option implied skew as high as 0.82 for eight companies.
- Deep out-of-money puts, equity default swaps can be used as hedge for credit risk, or exercised in carry trade with CDS.
- Hybrid securities, e.g. convertible bonds, callable bonds need to be priced by joint modelling.

# Empirical observations-continued


Figure: Historical daily VIX and CDX data with correlation 0.655.
Source: Bloomberg

# Empirical observations-continued


Figure: Historical daily VIX and SPX data with correlation -0.691.
Source: Bloomberg

# Structural model

Merton(1974), Black and Cox(1976), Leland and Toft(1996), Hull, Nelken and White(2004), Luciano and Schoutens(2006), CreditGrades$^{\mathrm{TM}}$(2002).

- Advantages: Having natural economic interpretations, straightforward in modelling seniority, recovery structure and hybrid securities, illuminating capital structure arbitrage (Yu 2005).
- Disadvantages: Stylized balance sheet(default trigger, agent incentives), computational expenses(e.g. need to calculate first passage time density functions or numerically invert Laplace tranform, price compound options), underlying usually unobservable.

# Reduced form model

Carr and Wu(2009), Bayraktar and Yang(2009), Carr and Linetsky(2006).

- Advantages: Flexible coupling between equity and credit underlying, more tractable in computation, some underlying observable.
- Disadvantages: No economic interpretation(for default).

General difficulties in both frameworks: multiname modelling (basket options), exotic derivatives (American options).

# Limitations in existing structural models

- One-factor: insufficient flexibility, equity and credit determined by one degree of freedom, leading to perfect dependence for all instruments; market being complete, making non-underlying redundant.
- Two-factor: Nielsen, Saa-Requejo, and Santa-Clara(1993), Briys and de Varenne(1997) and Schobel(1999) considered stochastic default boundaries. However, the secondary randomness is tied to the interest rate process, not the default trigger.
- Pure diffusion: produce implied volatility surface and CDS term structure of restrictive shapes.

# Default trigger as stochastical processes

- Collin-Dufresne and Goldstein (2001): mean-reverting leverage (debt/asset) dynamics are more consistent with empirical observations;
- Adrian and Shin (2009) find strong evidence for leverage (asset/equity) procyclicality in commercial banks and broker-dealer banks;
- Eberlein and Madan (2010) emphasize the role of risky liabilities in hedge funds coming from short positions.

# Adrian and Shin (2009)

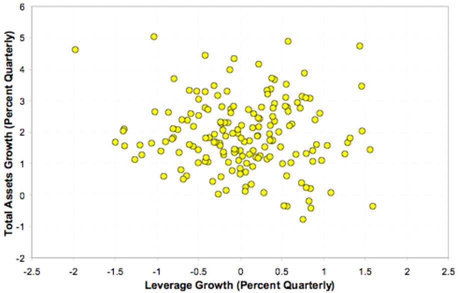



Figure: Total assets and leverage of non-financial, non-farm corporates.

# Adrian and Shin (2009)

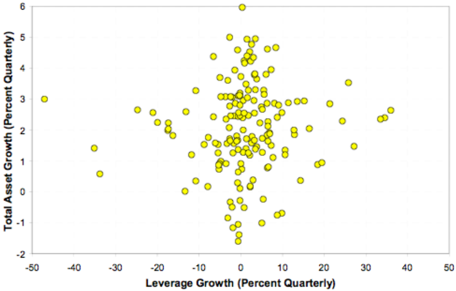



Figure: Total assets and leverage of commercial banks.

# Adrian and Shin (2009)

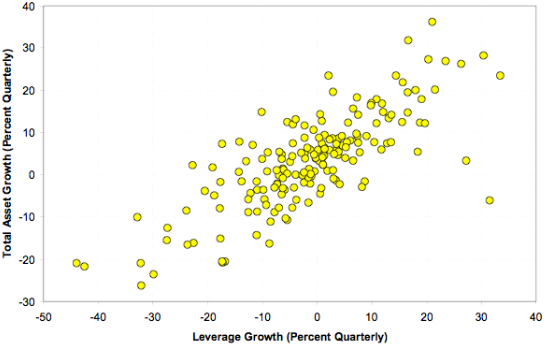



Figure: Total assets and leverage of security brokers and dealers.

# Model features

- Stochasticity in correlated asset and default trigger processes;
- Time change the calendar time to incorporate jumps in the underlying processes;
- We do not model strategic default as the default trigger is exogenous. The default is triggered as the first passage time (FPT) of the second kind. The survival probability has a spectral expansion form;
- The stock price is a vulnerable option of the asset value, with local volatility and state-dependent jumps;
- European call option prices can be written as a two-dimensional fast Fourier transform (FFT), promising efficient computation.

# Geometric BM model

## Assumption

- *Risk neutral probability space* $(\Omega, \mathcal{F}, \mathbb{Q})$*;*
- *Constant interest rate* $r$*;*
- *Two-factor capital structure: firm asset* $\tilde{V}_t = e^{v_0 + (r - \frac{1}{2}\sigma_v^2)t + \sigma_v W_t^v}$, *firm debt* $\tilde{D}_t = e^{d_0 + (r - \frac{1}{2}\sigma_d^2)t + \sigma_d W_t^d}$, *corr*$(W_t^v, W_t^d) = \rho$*;*
- *The dynamics of the firm equity and log-leverage is determined by the equations*

$$\tilde{S}_t = \max(\tilde{V}_t - \tilde{D}_t, 0), \quad \tilde{X}_t = \log \tilde{V}_t / \tilde{D}_t;$$

- *The time of default* $t^{(1)}$ *is the first passage time of the first kind* $t^{(1)} = \inf\{t | \tilde{X}_t \leq 0\}$, *after which the above processes are all stopped.*

# Lévy subordinated BM model

## Assumption

- *The "market clock" is an independent Lévy subordinator* $G_t$*, in particular, a gamma process with drift, having characteristics* $(1-a, 0, \nu)$ *where* $a \in (0,1)$ *and* $\nu(x) = ae^{-x/b}/(bx)$*, the Lévy measure, has support on* $\mathbb{R}^+$*;*
- *The natural filtration* $\mathcal{F}_t$ *contains* $\sigma\{G_u, W_v : u \le t, v \le G_t\}$*, and is assumed to satisfy the "usual conditions";*
- *The Lévy subordinated processes are* $V_t := \tilde{V}_{G_t}, \quad D_t := \tilde{D}_{G_t}, \quad X_t := \tilde{X}_{G_t}$*;*
- *The time of default* $t^{(2)}$ *is the first passage time of the second kind, after which the time change is stopped;*
- *The market observables are stock prices, implied volatility surface and CDS term structure.*

# First passage time of the second kind

## Definition

For any LSBM $X_t$ with $X_0 = x \geq 0$ the first passage time of the second kind is the $\mathcal{F}$-stopping time

$$t^{(2)} = \inf\{t | G_t \geq \tau\}$$

where $\tau = \inf\{t | \tilde{X}_t \leq 0\}$.

# First passage time of the second kind-continued

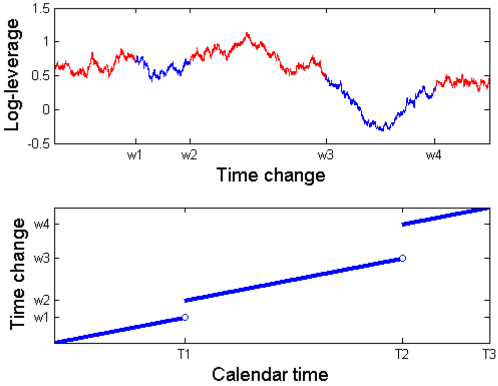



Figure: Realized sample paths of a log-leverage process and a subordinator(bottom). LSBM is the red path.

# Stock dynamics

$S_t := \tilde{V}_{G_t \wedge t^{(2)}} - \tilde{D}_{G_t \wedge t^{(2)}}$
Due to Itô's formula, conditional on no default

$$\frac{dS_t}{S_{t_-}} = rdt + \frac{\sqrt{1-a}}{e^{v_{t_-}} - e^{d_{t_-}}}(\sigma_v e^{v_{t-}} dW_t^v - \sigma_d e^{d_{t-}} dW_t^d)$$

$$+\frac{1}{e^{v_{t_-}} - e^{d_{t_-}}} \left[ e^{v_{t_-}} \cdot \int_{\mathbb{R}\setminus 0} (e^x - 1) N^v(dt, dx) - e^{d_{t-}} \cdot \int_{\mathbb{R}\setminus 0} (e^x - 1) N^d(dt, dx) \right]$$

where $N^v$ and $N^d$ are (dependent) VG random measure.
This model has local volatility $\sigma(v_t, d_t)$ and state-dependent jumps.

# CDS pricing

Now the firm's log-leverage ratio $X_t$ is a $\mathbb{Q}$-LSBM with "drift" $\beta_Q = -\frac{\sigma_v^2 - \sigma_d^2}{2(\sigma_v^2 + \sigma_d^2 - 2\sigma_v\sigma_d\rho)}$, "variance" $\sigma^2 = \sigma_v^2 + \sigma_d^2 - 2\sigma_v\sigma_d\rho$.

1. For any $t > 0, x \geq 0$ the risk-neutral survival probability $P^{(2)}(t,x) := \mathbb{E}_x^Q[\mathbf{1}_{\{t^{(2)} > t\}}]$ is given by

$$\frac{e^{-\beta_Q x}}{\pi} \int_{-\infty}^{\infty} \frac{z \sin(zx)}{z^2 + \beta_Q^2} e^{-\psi(\sigma^2(z^2+\beta_Q^2)/2,t)} dz,$$

   where the Laplace exponent of $G_T$ is

$$\psi^{VG}(u,t) := -\log E[e^{-uG_t}] = t[(1-a)u + \frac{a}{b}\log(1+bu)].$$

2. We calculate the fair swap rate for a CDS contract with maturity $T = N\Delta t$, with premiums paid in arrears on dates $t_k = k\Delta t, k = 1, \ldots, N$, and the default payment of $(1-R)$ paid at the end of the period when default occurs.

# European call pricing

A European call option with strike $K$ and maturity $T$ has time $t$ predefault value

$$\begin{aligned}\mathrm{Call}_t^{KT} &= \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(V_T - D_T - K)^{+}\mathbf{1}_{\{t^{(2)}>T\}}|\mathcal{F}_t] \\ &= \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(V_T - D_T - K)^{+}|\mathcal{F}_t] \\ &-\mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}(V_T - D_T - K)^{+}\mathbf{1}_{\{t^{(2)}<T\}}|\mathcal{F}_t]\end{aligned}$$

# GBM model

In Hurd and Zhou(2010), a vanilla spread option can be priced by FFT

$$\mathrm{Spr}(S_0;T) = e^{-rT}E[(S_{1T} - S_{2T} - 1)^+]$$
$$= (2\pi)^{-2}e^{-rT}\iint_{\mathbb{R}^2+i\epsilon} e^{ius_0'}\exp[iu(rTe - \sigma^2T/2)' - u\Sigma u'T/2]\hat{P}(u)d^2u$$

where $\hat{P}(u) = \frac{\Gamma(i(u_1+u_2)-1)\Gamma(-iu_2)}{\Gamma(iu_1+1)}$ . and $s_0 = [\log S_{10}, \log S_{20}]$.
We can also price a down-and-in spread option

$$\mathrm{ExSpr}(S_0;T) = e^{-rT}E[(S_{1T} - S_{2T} - 1)^+ 1_{\min_{0<t<T}(S_{1t}-S_{2t})<0}]$$
$$= C\cdot(2\pi)^{-2}e^{-rT}\iint_{\mathbb{R}^2+i\epsilon} e^{iuMs_0'}\exp[iu(rTe - \sigma^2T/2)' - u\Sigma u'T/2]\hat{P}(u)d^2u$$

where $M$ is a $2\times 2$ constant matrix and $C$ is a constant.

# LSBM models

Simply replace the term

$$\exp[iu(-\sigma^2 T/2)' - u\Sigma u' T/2]$$

by

$$E[\exp[iu(-\sigma^2 G_T/2)' - u\Sigma u' G_T/2]] = e^{\psi^{VG}(iu\sigma^2/2 + u\Sigma u'/2, t)}$$

# Producing empiricals

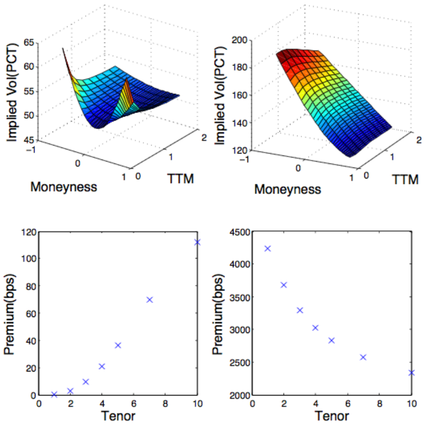



Figure: Left: Implied vol. and CDS term structure for a high quality (large $X = 3$) firm; Right: Implied vol. and CDS term structure for a low quality (small $X = 0.4$) firm. Hypothetic model parameters $S = 1, \sigma_v = 0.5, \sigma_d = 0.2, \rho = 0.1, a = 0.5, b = 0.5, R = 0.4.$

# Producing empiricals-continued

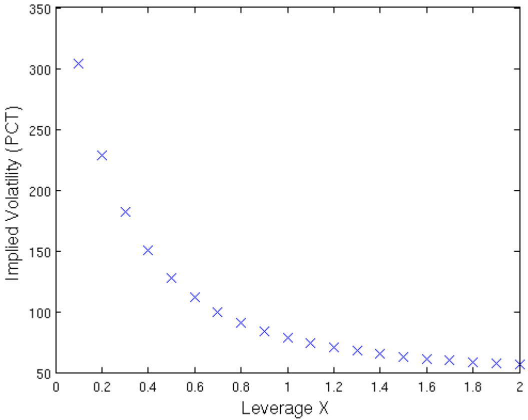



Figure: Implied volatility decreases with $X$, i.e. increases with PD, producing the empirical correlation. Hypothetic model parameters $S = 1, \text{strike} = 1, \text{TTM} = 1, \sigma_v = 0.5, \sigma_d = 0.2, \rho = 0.1, a = 0.5, b = 0.5$.

# Error bounds

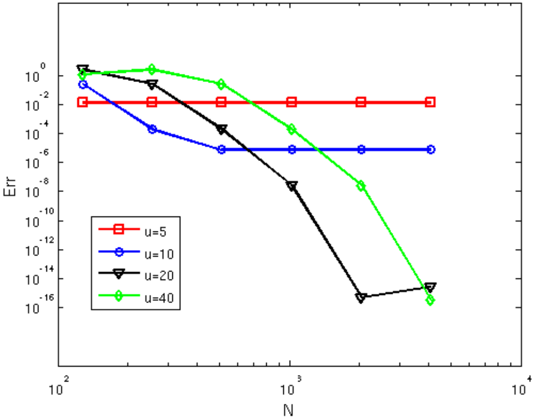



Figure: Truncation and discretization error bounds for the VG model. The benchmark uses $N = 2^{13}, u = 80$.Hypothetic model parameters strike $= 1$, TTM $= 1, \sigma_v = 0.5, \sigma_d = 0.2, \rho = 0.1, a = 0.5, b = 0.5$.

# Objective function

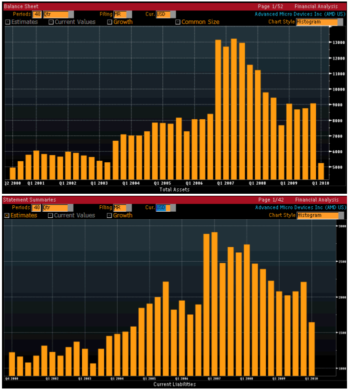
Let $Y$ denote market observed prices and $\hat{Y}$ denote our model prices, the objective function is defined as

$$\mathbb{J}(\theta) = \sum_{j=1}^{n} \frac{|\hat{Y}_j(\theta) - Y_j|^2}{Y_j^2}$$

The summation contains available market data of a certain day. Then the calibration is the minimizer of the objective function.

$$\hat{\theta} = \text{argmin}_{\theta} \mathbb{J}(\theta)$$

# AMD balance sheet




Top: AMD quarterly asset; Bottom: AMD quarterly current liability.

# AMD balance sheet-continued

From Q2 2000 to Q1 2010 (40 Quarters):
Total assets = Liabilities+Share holders' equity

- Standard deviation of asset return: 0.277
- Standard deviation of current liabilities return: 0.249
- Correlation: 0.735
- Log-(Assets/Cur.Lia.): 1.373(Q3 2008), 1.488(Q3 2006)

# AMD calibrations

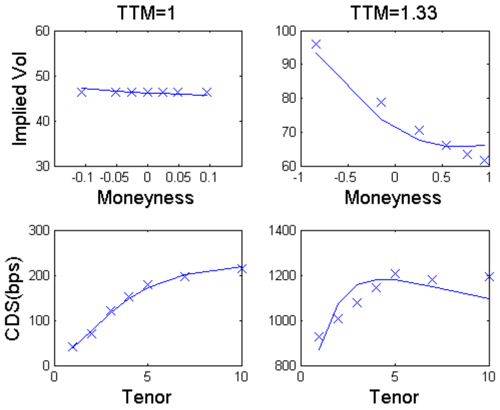



Figure: Market data ("X") VS model data("-"). Left: 07/12/2006; Right: 09/10/2008

# AMD calibrations-continued

Table: Two-day calibration for VG model.

| date | $\sigma_v$ | $\sigma_d$ | $\rho$ | a | b | X | R |
|---|---|---|---|---|---|---|---|
| 07/12/06 | 0.32 | 0.13 | 0.64 | 0.60 | 1.20 | 0.90 | 0.58 |
| 09/10/08 | 0.39 | 0.044 | -0.19 | 0.88 | 1.22 | 0.73 | 0.22 |

# Conclusions

1. We have presented an integrated structural model for pricing equity and credit;
2. We have derived efficient pricing formulas based on FFT technique;
3. The VG model is implemented with market calibration, yet other LSBM models should work under the same principle;
4. More efficient and consistent calibration technique is under study;
5. We have also calibrated for Ford Motor Company. Fixed model parameters+floating state variables give very good fit for CDS term structure over 3-year span. Co-calibration with implied volatility data is under study.