Optimal Securitization via Impulse Control

Rüdiger Frey (joint work with Roland C. Seydel)

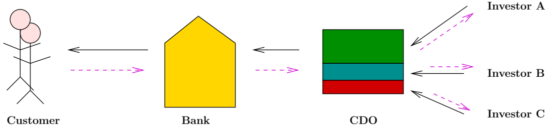
Mathematisches Institut Universität Leipzig and MPI MIS Leipzig

Bachelier Finance Society, June 2010

Overview

Securitization

Consider a commercial bank lending to customers. In a securitization transaction the bank sells part of its loan portfolio to investors in a bond-like form, passing on part of the risk and return.



Potential benefits of securitization

- On the macro level: possibly mitigation of concentration risk and easier refinancing for banks
- On the micro level securitization can be an important risk management tool for commercial banks (reduction of leverage)

Securitization is of course not problem-free but this is not our focus here.

Securitization ctd.

It is not costless to securitize (sell) loans:

- An ABS transaction typically entails fixed costs: Rating agency fees, legal costs, time spent . . .
- The more you sell, the lower the price investors want to pay (liquidity and agency problems)

Finding a good (optimal) securitization strategy is non-trivial

. . . it is not optimal to sell all at once, but rather distributed over time;

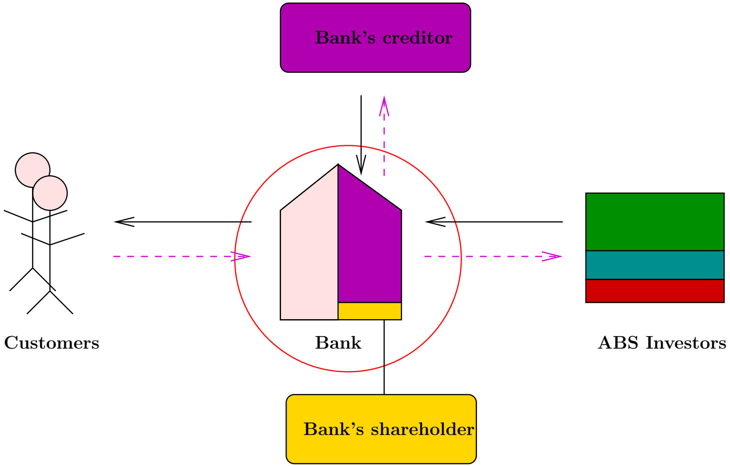
. . . it is not optimal to securitize all the time, but rather at discrete points in time.

Conclusion. Determination of an optimal securitization strategy (for the bank) leads to a a dynamic optimization problem under fixed and variable transaction costs. $\Rightarrow$ apply impulse control methods.

Overview

1. Introduction
2. Optimal Securitization Strategy
   - The model
   - Specific model inputs
   - Solution of the Optimization Problem
3. Numerical Results

The Agents Involved

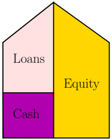


Modeling a commercial a bank ctd

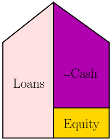
- Consider a commercial bank solely engaged in lending business. Loan portfolio is homogeneous and loans to customers have maturity ∞ (perpetuities) with nominal 1.
- Starting point is the fundamental balance sheet equation

$$\begin{aligned} \text{assets} &= \text{liabilities} \\ \text{cash} \cdot 1_{\text{cash}>0} + \text{loans} &= \text{equity} - \text{cash} \cdot 1_{\text{cash}<0} \end{aligned}$$

- Refinancing through negative cash (short-term refinancing): Assume there are no liquidity problems!



Cash > 0



Cash < 0

Model dynamics

State variables of the model:

- Nominal loan exposure $L_t =: X_t^1$.
- Cash balance $C_t =: X_t^2$ (positive or negative). Note that leverage equals $\pi_t = \frac{\text{loan exposure}}{\text{equity}} = \frac{L_t}{L_t + C_t}$.
- Economic state variable $M_t =: X_t^3$ that affects default rates

Dynamics of L: $dL_t = -dN_t + \beta_t dP_t$, where

- N_t is a point process with state dependent intensity $\lambda(M_{t-})L_{t-}$, representing the timing of defaults. (Bottom-up view: defaults are conditionally independent given $\mathcal{F}_\infty^M$ with intensity $\lambda(M_{t-})$.)
- The stochastic control $\beta_t \in \{0, 1\}$ allows to increase the loan exposure at the advent of a potential customer. Advents of customers are modelled by jumps of the standard Poisson process P.

Model dynamics (2)

Cash C. Cash position is affected by

- interest payed or earned on cash position given by $r_B C_t dt$ (r_B refinancing cost)
- interest $r_L L_t dt$ earned on loan position
- recovery payments in case of default $(1-\delta)dN_t$ (here LGD $\delta = 1$).
- cash-reduction because of issuing of new loans $-\beta_t dP_t$.

Hence we have the following cash-dynamics

$$dC_t = (r_B(X_t)C_t + r_L L_t)\,dt + dN_t - \beta_t dP_t$$

Economic state M. Markov switching process (continuous-time Markov chain) between two economic states, $M_t \in \{0, 1\}$.

The Securitization strategy (impulses)

Each securitization of $\zeta_i \geq 0$ loans at a stopping time τ_i has the following effects:

1. Reduce loan exposure: $L_{\tau_i} = \check{L}_{\tau_i -} - \zeta_i$.
2. Increase cash by market value $\eta(\cdot)$ of the amount sold minus fixed costs $c_f > 0$:

$$C_{\tau_i} = \check{C}_{\tau_i -} + \eta(\check{M}_{\tau_i -}, \zeta_i) - c_f$$

In summary, bring the process $X = (L, C, M)^T$ from an old state x to the new state $\Gamma(x, \zeta)$ with

$$\Gamma(x, \zeta) = (x_1 - \zeta, x_2 + \eta(x_3, \zeta) - c_f, x_3)^T$$

A sequence $\gamma = (\tau_i, \zeta_i)_{i \geq 1}$ is called an *impulse control strategy*.

The optimization problem

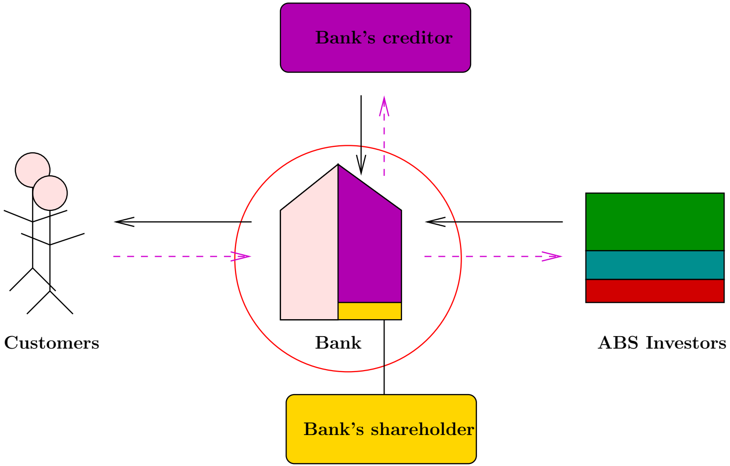
- We consider the model on the state space $S = \{x \in \mathbb{R}^3 \colon x_1 > -1,\ x_1 + x_2 > 0\}$ (as long as the bank does not default and $L_t \geq 0$.)
- Fix horizon date $T > t$ and some concave increasing utility function U. Let $\tau = \tau_S \wedge T$. We assume that the shareholders want the bank to maximize the expected utility of its liquidation value at τ_S,

$$J^{(\alpha)}(t,x) = \mathbb{E}^{(t,x)}\left[U\left(\max(\eta(M_\tau, L_\tau^\alpha) + C_\tau^\alpha, 0)\right)\right] \tag{1}$$

  by an choosing optimal stochastic and impulse control strategy $\alpha = (\beta, \gamma)$.
- Define the value function

$$v(t,x) = \sup\{J^{(\alpha)}(t,x) \colon \alpha \text{ admissible}\}.$$

The players revisited



Modeling market value of loans $\eta(\cdot)$

- The loan portfolio of our bank consists of perpetuities (loans with maturity ∞, i.e., the nominal is never paid back).
- The risk-neutral value of one such perpetual loan is

$$p_m^\infty := \mathbb{E}\left[\int_0^\tau e^{-\rho s} r_L ds + e^{-\rho\tau}(1 - \delta(M_\tau)) \middle| M_0 = m \right],$$

  for τ the default time of the loan, and ρ risk-free interest rate. The vector p^∞ can be obtained by a simple inversion of the generator matrix of M.
- With constant default intensity λ: $p^\infty = (r_L + (1-\delta)\lambda)/(\rho + \lambda)$.
- To account for risk aversion, one possible choice for the market value of ζ loans is:

$$\eta(m, \zeta) := \zeta \cdot \min\left(1,\ p_m^\infty \cdot (1 - \delta\lambda(m))\right) < \zeta p_m^\infty \qquad (2)$$

Modeling refinancing cost $r_b(\cdot)$

Basic rule of thumb: On average, the bank's creditors want to earn the risk-free interest ρ, so they will demand a refinancing rate r_B according to

$$1 + \rho = (1 - PD) \cdot (1 + r_B), \tag{3}$$

where PD = probability of default of the bank over one year. Equation (3) leads to

$$r_B := \frac{\rho + PD}{1 - PD}. \tag{4}$$

Now, for a given loan amount ℓ and cash position c we define

$$PD := \mathbb{P}(\Delta L > \ell + c) = \mathbb{P}\left(\frac{\Delta L}{\ell} > \frac{\ell + c}{\ell}\right) \tag{5}$$

$\Rightarrow$ model the distribution of the $[0, 1]$-valued relative loss $\Delta L/\ell$.

Modeling refinancing cost $r_b(\cdot)$ ctd.

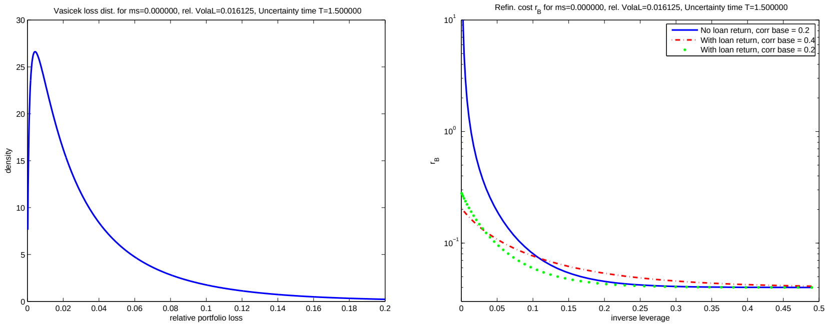
Goal: Model distribution of relative loss $\Delta L/\ell$ as seen by creditors

- Losses in our loan portfolio follow a Bernoulli mixture distribution (for fixed t) with path of M as common factor.
- Large portfolio approximation: typically a Bernoulli mixture distribution converges for granularity going to ∞ to a limiting distribution depending only on the common factor (see [McNeil et al., 2005])
- Choosing a probit-normal factor leads to the well-known continuous Vasicek loss distribution $V_{p,\varrho}$,

$$V_{p,\varrho}(x) = N\left[1/\sqrt{\varrho}(N^{-1}(x)\sqrt{1-\varrho} - N^{-1}(p))\right]$$

  ($p \in (0,1) \approx$ average default rate, $\varrho \in (0,1) \approx$ correlation)
- We take $p \approx \lambda(M_t)$; the parameter ρ can be used as additional risk-aversion parameter on behalf of creditors

Refinancing cost: examples



Loss distribution $V_{p,\rho}$ (left) and refinancing rate r_B (right) as function of the inverse leverage $(\ell + c)/\ell$.

Numerical considerations

- We want to find an optimal impulse control for the bank
- It seems impossible to find an analytical solution
- Usual approach: solve numerically the HJBQVI[1] (partial integro-differential equation) by iterated optimal stopping and thus obtain the value function $v = \sup_\alpha \mathbb{E}[U(\text{wealth})]$
- From the value function, derive an (approximately) optimal strategy

[1] Hamilton-Jacobi-Bellman quasi-variational inequality

Value function v and HJBQVI

Our aim is to find v as solution of the HJBQVI (here partial difference equation)

$$\min(-\sup_{\beta\in\{0,1\}}\{u_t+\mathcal{L}^\beta u\}, u-\mathcal{M}u)=0 \qquad \text{in } [0,T)\times S \tag{6}$$

for $\mathcal{L}^\beta$ the infinitesimal generator of the state variable process $X=(L,C,M)$ where $\tilde{x}:=(x_1,x_2)=(\ell,c)$:

$$\begin{aligned}\mathcal{L}^\beta u(x) = &\left(u(\tilde{x}+\begin{pmatrix}-1\\0\end{pmatrix},x_3)-u(x)\right)\lambda(x_3)x_1\\ &+\left(u(\tilde{x}+\begin{pmatrix}\beta\\-\beta\end{pmatrix},x_3)-u(x)\right)\lambda_P\\ &+(u(\tilde{x},1-x_3)-u(x))\,\lambda_{x_3,(1-x_3)}+(r_B(x)x_2+r_Lx_1)u_{x_2}\end{aligned}$$

and $\mathcal{M}$ the intervention operator selecting the momentarily best impulse, $\mathcal{M}u(t,x)=\sup_\zeta\{u(t,\Gamma(x,\zeta))\}$.

HJBQVI: Existence and Uniqueness

We can prove using results from [Seydel, 2008]:

Theorem (Parabolic viscosity solution)

Assume that $c \mapsto r_B(\ell, c, m)$ *is continuous, and* U *continuous and bounded from below. Further assume that* $\liminf_{c \downarrow -\ell} r_B(\ell, c, \cdot) > r_L$ *for* $\ell > 0$*, and* $\eta(\cdot, \zeta) \leq \zeta$*. Then the value function* v *is the unique viscosity solution of (6), and it is continuous on* $[0, T] \times \mathbb{Z} \times \mathbb{R} \times \{0, 1\}$ *(i.e., continuous in time and in cash).*

Proof: See [Frey and Seydel, 2009].

Overview

1. Introduction
2. Optimal Securitization Strategy
   - The model
   - Specific model inputs
   - Solution of the Optimization Problem
3. Numerical Results

Basic data

- Power law utility $U(x) = \sqrt{x}$
- Symmetric Markov chain transition intensities of 0.3 for M
- Default intensities per loan: $\lambda(0) = 2.6\%$ in expansion and $\lambda(1) = 4.7\%$ in contraction, no loan default recovery ($\delta \equiv 1$)
- Risk-free rate $\rho = 0.04$, loan interest rate $r_L = 0.08$
- Market value η: A form slightly more procyclical than (2) $\Rightarrow$ proportional transaction costs of 0% ($\approx 6.5\%$) in expansion (contraction)
- Refinancing cost r_B is based on Vasicek loss distribution with with $p = 1.5\lambda$, and correlation $\varrho = 0.2$ (0.4) in expansion (contraction).
- Fixed transaction costs $c_f = 0.5$

State equations:

$$dL_t = -dN_t \qquad dC_t = (r_B(X_t)C_t + r_L L_t)\,dt$$

$$\Gamma(t, x, \zeta) = (x_1 - \zeta, x_2 + \eta(x_3, \zeta) - c_f, x_3)^T$$

Optimal impulses

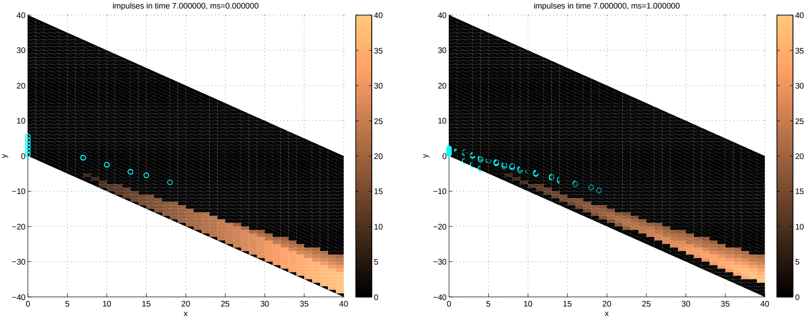


Figure: Impulses in expansion (left) and contraction (right) for $T = 7$. The light areas mark the impulse departure points (with the lightness indicating how far to the left the impulses goes, i.e., how many loans are sold), the cyan circles represent the corresponding impulse arrival points

Further results of numerical analysis

- Even with substantial transaction costs, securitization is a useful risk management tool for the bank: utility indifference argument shows that value of bank is increased substantially by the possibility of securitization
- Risk-dependent refinancing cost creates a major incentive to securitize
- Optimal securitization strategy is largely influenced by form of transaction cost
  - Low fixed cost $c_f \Rightarrow$ more transactions
  - Strongly procyclical market value of loans (high transaction costs in contraction), $\Rightarrow$ only little securitization in contraction, but more loans are securitized in expansion
    Weakly procyclical market value $\Rightarrow$ High securitization activity in contraction
- Additional control of loan exposure (β) had only small effect

Cash value of securitization

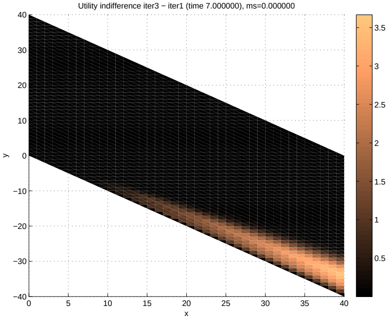


Figure: Cash value of securitization in expansion ($M = 0$) and in contraction (right), for $T = 7$. For every x we display the cash amount a such that $v_3(x_1, x_2 - a) = v_1(x_1, x_2)$ (v_3 being the value function with impulses, v_1 without

Impulse and stochastic control

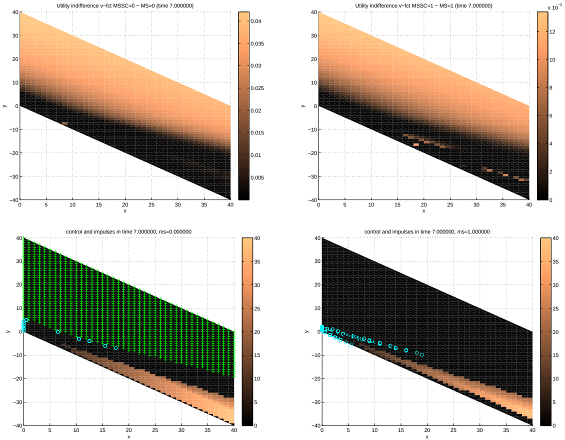


Figure: Impulse and stochastic control: Cash value of additional stochastic control (top row), and optimal strategy (bottom row) in expansion (left) and contraction (right), for $T = 7$. Business arrival intensity $\lambda_P = 2$

Literature

Frey, R. and Seydel, R. C. (2009).
Optimal securitization of credit portfolios via impulse control.
Working paper.

McNeil, A. J., Frey, R., and Embrechts, P. (2005).
Quantitative risk management.
Princeton Series in Finance. Princeton University Press, Princeton, NJ.
Concepts, techniques and tools.

Øksendal, B. and Sulem, A. (2005).
Applied stochastic control of jump diffusions.
Universitext. Springer-Verlag, Berlin.

Seydel, R. C. (2008).
General existence and uniqueness of viscosity solutions for impulse control of jump-diffusions.
MPI MIS Preprint 37/2008.

Viscosity solutions (1)

Definition (Viscosity solution)

A function $u \in \mathcal{PB}([0,T] \times \mathbb{R}^d)$ is a (viscosity) subsolution of (6) if for all $(t_0, x_0) \in [0,T] \times \mathbb{R}^d$ and $\varphi \in \mathcal{PB} \cap C^{1,2}([0,T) \times \mathbb{R}^d)$ with $\varphi(t_0, x_0) = u^*(t_0, x_0)$, $\varphi \geq u^*$ on $[0,T) \times \mathbb{R}^d$,

$$\min \left(- \sup_{\beta \in B} \left\{ \frac{\partial \varphi}{\partial t} + \mathcal{L}^\beta \varphi + f^\beta \right\}, u^* - \mathcal{M} u^* \right) \leq 0$$

in $(t_0, x_0) \in S_T$, and

$$\min \left(u^* - g, u^* - \mathcal{M} u^* \right) \leq 0$$

in $(t_0, x_0) \in \partial^+ S_T$ (the parabolic boundary). [...]

Viscosity solutions (2)

Definition (Viscosity solution (cont'd))

A function $u \in \mathcal{PB}([0,T] \times \mathbb{R}^d)$ is a (viscosity) supersolution of (6) if for all $(t_0, x_0) \in [0,T] \times \mathbb{R}^d$ and $\varphi \in \mathcal{PB} \cap C^{1,2}([0,T) \times \mathbb{R}^d)$ with $\varphi(t_0, x_0) = u_*(t_0, x_0)$, $\varphi \le u_*$ on $[0,T) \times \mathbb{R}^d$,

$$\min\left(-\sup_{\beta \in B}\left\{ \frac{\partial \varphi}{\partial t} + \mathcal{L}^\beta \varphi + f^\beta \right\}, u_* - \mathcal{M}u_* \right) \;\; \geq \;\; 0$$

in $(t_0, x_0) \in S_T$, and

$$\min\left(u_* - g, u_* - \mathcal{M}u_* \right) \;\; \geq \;\; 0$$

in $(t_0, x_0) \in \partial^+ S_T$.
A function u is a viscosity solution if it is sub- and supersolution.

Impulses in time

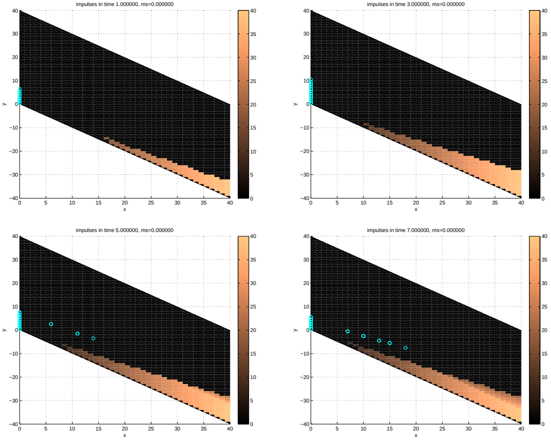


Figure: Impulses in expansion for different T: top left $T = 1$, top right $T = 3$, bottom left $T = 5$ and bottom right $T = 7$

Optimal impulses, no Markov-switching

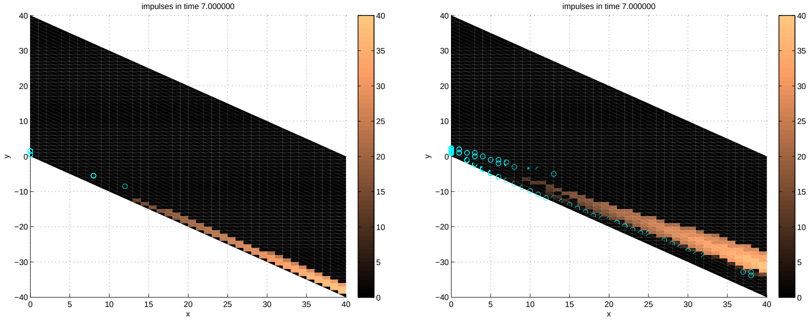


Figure: Impulses without Markov switching, for only expansion (left) and only contraction (right) for $T = 7$. Market value according to procyclical form (a), corresponding to 0% (about 17%) proportional transaction costs in expansion (contraction). For the colour code, see the explanations in Figure 1. Otherwise, same data as in Figure **??**