The tracking error rate of the Delta-Gamma hedging strategy

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Delta Hedging Strategy (DHS)

- Risky assets S (with Black-Scholes model)
- Option to be hedged : $u(t, S) := \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}g(S_T)|S_t = S)$
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Portfolio value at time t : $V^{\Delta,N}(t,S_t) = \delta_t^0 S_t^0 + \delta_t S_t$

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The (discounted) Delta tracking error
$\bar{\mathcal{E}}_N^{\Delta} := e^{-rT}(V_T^{\Delta,N} - g(S_T))$

$$\bar{\mathcal{E}}_N^{\Delta} = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} (\delta_{t_i} - \delta_t)\mathrm{d}\bar{S}_t.$$

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  - For $g(x) = \mathbb{1}_{x \geq K}$,

$$(\mathbb{E}|\bar{\mathcal{E}}_N^{\Delta}|^2)^{\frac{1}{2}} \sim N^{-\frac{1}{4}}.$$

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- Moreover, one can reach the order $N^{-\frac{1}{2}}$ thanks to a convenient choice of a non regular time net.

▶ Both the payoff function regularity and the time net choice have an effect on the convergence order of the Delta hedging error.

▶ The Delta-Gamma Hedging Strategy (DGHS) $\equiv$ hold, between t_i and t_{i+1}, δ_{t_i} risky assets S and $\delta^C_{t_i}$ of another instrument whose price is $(C(t,S_t))_{0\le t\le T}$: (in dim 1)

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$$\delta^C_{t_i} := \frac{\partial^2_S u(t_i, S_{t_i})}{\partial^2_S C(t_i, S_{t_i})}, \qquad \delta_{t_i} := \partial_S u(t_i, S_{t_i}) - \frac{\partial^2_S u(t_i, S_{t_i})}{\partial^2_S C(t_i, S_{t_i})}\partial_S C(t_i, S_{t_i}).$$

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▶ **Our goal** : study, in dimension d,

- the link between the order of $\bar{\mathcal{E}}^{\Delta\Gamma}_N$ and the payoff regularity
- the effect of the rebalancing dates choice

- **Assets**

$$\begin{cases} S_0^j &= s_0^j, \\ \mathrm{d}S_t^j &= \mu_j S_t^j \mathrm{d}t + \sigma_j S_t^j \mathrm{d}\hat{W}_t^j, \end{cases}$$

- $\hat{W} = (\hat{W}^1, ..., \hat{W}^d)$ is a Brownian motion under the **historical probability** $\mathbb{P}$.
- $\langle \hat{W}^j, \hat{W}^k \rangle_t = \rho_{j,k} t$, and the matrix $(\rho_{j,k})_{1 \leq j,k \leq d}$ has full rank.
- **Risk-neutral probability** $\mathbb{Q}$:
  - $\lambda_j = \frac{\mu_j - r}{\sigma_j}$
  - $(W_t^j := \hat{W}_t^j + \lambda_j t)_{1 \leq j \leq d}$ is a $\mathbb{Q}$-Brownian motion

- **Hedging instruments** : for $0 \leq j < k \leq d$,
  $C^{j,k}(t, S^j, S^k) :=$
  $\mathbb{E}_{\mathbb{Q}}\left[e^{-r(T_2-t)}(S^k_{T_2} - K_{j,k}S^j_{T_2})_+ | S^j_t = S^j, S^k_t = S^k\right]$,
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  ($\longrightarrow$ closed **BS** and **Margrabe** formulas).
- **Hedging ratios** :
  - $\delta^{j,k}_{t_i}$ ($1 \leq j < k \leq d$, Exchange options)
  - $\delta^{0,l}_{t_i}$ ($1 \leq l \leq d$, Call options)
  - $\delta^{l}_{t_i}$ ($1 \leq l \leq d$, assets).

▶ with almost similar definitions to those in dim 1.

- **The option to hedge** :
  - $u(t,S) := \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}g(S_T)|S_t = S\right]$, with $S = (S^1, ..., S^d) \in \mathbb{R}_+^d$.

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  - Payoff : $\mathbb{E}_{\mathbb{P}}\left|g(S_T)\right|^{2p_0} < \infty$, for some $p_0 > 1$.

For $l, m, n = 1...d$, we define

$$\bar{u}(t) := e^{-rt} u(t, S_t) = \mathbb{E}_{\mathbb{Q}} \left[e^{-rT} g(S_T) | \mathcal{F}_t \right];$$

$$\bar{u}_l^{(1)}(t) := e^{-rt} \sigma_l S_t^l \partial_l u(t);$$

$$\bar{u}_{l,m}^{(2)}(t) := e^{-rt} \sigma_l \sigma_m S_t^l S_t^m \partial_{l,m}^2 u(t);$$

$$\bar{u}_{l,m,n}^{(3)}(t) := e^{-rt} \sigma_l \sigma_m \sigma_n S_t^l S_t^m S_t^n \partial_{l,m,n}^3 u(t).$$

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- $\mathbb{Q}$-**Martingales**.
- enable tricky calculus of the Itô decompositions.

Tracking error decompostion

Theorem

$$\overline{\mathcal{E}}_N^{\Delta\Gamma}(g,\pi) = - \sum_{i=0}^{N-1} \sum_{l,m,n=1}^{d} \int_{t_i}^{t_{i+1}} \int_{t_i}^{t} \int_{t_i}^{s} \left(\bar{u}_{l,m,n}^{(3)}(r) + R_{l,m,n}^{i,(3)}(r) \right) \mathrm{d}W_r^n \mathrm{d}W_s^m \mathrm{d}W_t^l .$$

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- NB. For **DHS** : $\overline{\mathcal{E}}_N^{\Delta}(g,\pi) = -\sum_{i=0}^{N-1}\sum_{l,m=1}^{d}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t}\left(\bar{u}_{l,m}^{(2)}(s) + R_{l,m}^{i,(2)}(s)\right)\mathrm{d}W_s^m\mathrm{d}W_t^l.$

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- One has to estimate $\mathbb{E}_{\mathbb{P}}\left|\bar{u}^{(3)}_{l,m,n}(r)\right|^2$ and $\mathbb{E}_{\mathbb{P}}\left|R^{i,(3)}_{l,m,n}(r)\right|^2$: **the regularity of g plays a key role**.

When $\mathbb{E}|g(X_T)|^2 < +\infty$, we define

$$V_{t,T}(g) := \mathbb{E}_{\mathbb{P}} \left| g(S_T) - \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_t}(g(S_T)) \right|^2 .$$

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Definition

For some $\alpha \in (0,1]$,

$$\mathbf{L}_{2,\alpha} = \left\{ g \text{ t.q. } \mathbb{E}(g(S_T)^2) + \sup_{0 \le t < T} \frac{V_{t,T}(g)}{(T-t)^\alpha} < +\infty \right\} .$$

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- If $g(x) = (x - K)_+^a$ with $a \in (\frac{1}{2}, 1]$, then $g \in \mathbf{L}_{2,1}$!
- If $g(x) = \mathbb{1}_D(x)$, then $g \in \mathbf{L}_{2,\frac{1}{2}}$!

Key estimates

- For $1 \le l, m, n \le d$ and $0 \le t < T$, and using the usual Malliavin representation of Greeks,

$$\mathbb{E}_{\mathbb{P}} \left| \bar{u}_l^{(1)}(t) \right|^2 \le C \frac{V_{t,T}(g)}{(T-t)},$$

$$\mathbb{E}_{\mathbb{P}} \left| \bar{u}_{l,m}^{(2)}(t) \right|^2 \le C \frac{V_{t,T}(g)}{(T-t)^2},$$

$$\mathbb{E}_{\mathbb{P}} \left| \bar{u}_{l,m,n}^{(3)}(t) \right|^2 \le C \frac{V_{t,T}(g)}{(T-t)^3}.$$

Integrands estimates

- bound for $\mathbb{E}_{\mathbb{P}} \left| \bar{u}^{(3)}_{l,m,n}(t) \right|^2 : \frac{C}{(T-t)^{3-\alpha}}$ if $g \in \mathbf{L}_{2,\alpha}$.

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- For $R^{i,(3)}_{l,m,n}(t)$, it is more intricate !

$$R^{i,(3)}_{l,m,n}(t) = \cdots - \sum_{0 \le j < k \le d} \delta^{j,k}_{t_i} \bar{C}^{j,k,(3)}_{l,m,n}(t) - \ldots$$

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  - terms $\frac{\bar{C}^{j,k,(2)}_{l,m}(t)}{\bar{C}^{j,k,(2)}_{l,m}(t_i)}$ et $\frac{\bar{C}^{j,k,(3)}_{l,m,n}(t)}{\bar{C}^{j,k,(2)}_{l,m}(t_i)}$ (with $t_i \le t \le t_{i+1}$)

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▶ using the closed formulas, we obtain that these terms belong to $\mathbf{L}_p$ ($p \ge 2$) if and only if $|\pi| \le \pi^{\text{threshold}}$.

▶ If $|\pi| \le \pi^{\text{threshold}}$, then, for $0 \le t_i \le t < t_{i+1} \le T$,

$$\mathbb{E}_{\mathbb{P}}\left|R^{i,(3)}_{l,m,n}(t)\right|^2 \le \frac{C}{(T-t)^2}.$$

Corollary

Assume $g \in \mathbf{L}_{2,\alpha}$ *(for some* $\alpha \in (0,1]$*) and* $\mathbb{E}_{\mathbb{P}} \left|g(S_T)\right|^{2p_0} < \infty$ *for some* $p_0 > 1$*. Then, if* $|\pi| \leq \pi^{\text{threshold}}$*, and for* $0 \leq t < T$*,*

$$\mathbb{E}_{\mathbb{P}} \left| \bar{u}^{(3)}_{l,m,n}(t) + R^{i,(3)}_{l,m,n}(t) \right|^2 \leq \frac{C}{(T-t)^{3-\alpha}}.$$

Time nets

For some $\beta \in (0, 1]$,

$$\pi^{(\beta)} := \left\{ t_k^{(N,\beta)} := T - T\left(1 - \frac{k}{N}\right)^{\frac{1}{\beta}}, 0 \leq k \leq N \right\}.$$

NB.

- $\pi^{(1)}$ = uniform grid.
- For $\beta < 1$, the points in $\pi^{(\beta)}$ are more concentrated near T.

Theorem (with uniform grid)

Assume $g \in \mathbf{L}_{2,\alpha}$ *and* $\mathbb{E}_{\mathbb{P}} |g(S_T)|^{2p_0} < \infty$ *for some* $p_0 > 1$.

- **Regular grid** $\pi^{(1)}$. *For* N *sufficiently large to ensure* $|\pi^{(1)}| = \frac{T}{N} \le \pi^{\text{threshold}}$, *one has*

$$\left(\mathbb{E}_{\mathbb{P}} \left|\overline{\mathcal{E}}_N^{\Delta\Gamma}(g, \pi^{(1)})\right|^2\right)^{1/2} = \mathcal{O}\left(\frac{1}{N^{\alpha/2}}\right).$$

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▶ tight estimate for $\alpha < 1$ (if $\alpha = 1$, the rate may go from $N^{\frac{1}{2}}$ to N).

▶ DGHS with a regular grid does **not** improve the rate of convergence.

Main result

Theorem (with non regular grid)

- **Non regular grid** $\pi^{(\beta)}$, $\beta \in (0,1)$. *For N sufficiently large to ensure* $|\pi^{(\beta)}| \leq \pi^{\text{threshold}}$, *one has*

$$\left(\mathbb{E}_{\mathbb{P}}\left|\overline{\mathcal{E}}_N^{\Delta\Gamma}(g,\pi^{(\beta)})\right|^2\right)^{1/2} = \begin{cases} \mathcal{O}(\frac{1}{N^{\frac{\alpha}{2\beta}}}) \text{ if } \beta \in (\frac{\alpha}{2},1), \\ \mathcal{O}(\frac{\sqrt{\log N}}{N}) \text{ if } \beta = \frac{\alpha}{2}, \\ \mathcal{O}(\frac{1}{N}) \text{ if } \beta \in (0,\frac{\alpha}{2}). \end{cases}$$

- **NB.** These estimates are equal to those we observe numerically.

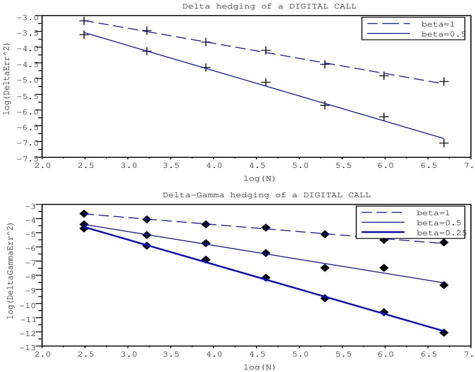

Figure: For a Digital Call : at the top (DHS), $\log(\mathbb{E}_{\mathbb{P}}|\overline{\mathcal{E}}_N^{\Delta}(g, \pi^{(\beta)})|^2)$ vs $\log(N)$. At the bottom (DGHS), $\log(\mathbb{E}_{\mathbb{P}}|\overline{\mathcal{E}}_N^{\Delta\Gamma}(g, \pi^{(\beta)})|^2)$ vs $\log(N)$.

Remark on the convergence in distribution

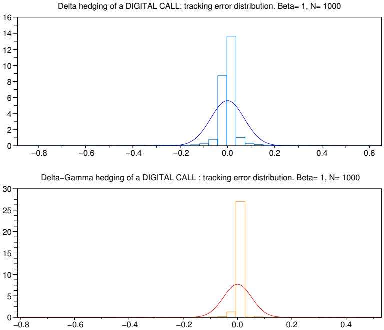

Figure: Distributions of the DHS (at the top) and DGHS (at the bottom) tracking errors for a Digital Call

$\rightarrow$ Convergences in $\mathbf{L}_2$ and in distribution are different.

Further research

- Extension to more general model for S

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- Rate of convergence in distribution of the DGHS tracking error ?