Affine processes on positive semidefinite matrices

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1. Introduction
   - Definition of affine processes on S_d^+
2. Applications of S_d^+-valued affine processes in finance
   - Multivariate affine stochastic volatility models
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3. Characterization of affine processes on S_d^+
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   - Main theorem
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- state space S_d^+, the cone of symmetric $d \times d$-positive semidefinite matrices which is a subset of
- S_d, the vector space of symmetric $d \times d$-matrices equipped with scalar product $\langle x, y \rangle = \mathrm{Tr}(xy)$ (isomorphic to $\mathbb{R}^{d(d+1)/2}$).
- $(P_t)_{t \geq 0}$: semigroup associated to the Markov process which acts on bounded measurable functions $f : S_d^+ \to \mathbb{R}$,

$$P_t f(x) := \mathbb{E}_x[f(X_t)] = \int_{S_d^+} f(\xi) p_t(x, d\xi), \quad x \in S_d^+.$$

Definition

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An S_d^+-valued Markov process X is called affine if

1. it is stochastically continuous, that is, $\lim_{s\to t} p_s(x,\cdot) = p_t(x,\cdot)$ weakly on S_d^+ for every t and $x \in S_d^+$, and
2. its Laplace transform has exponential-affine dependence on the initial state,

$$\mathbb{E}_x\left[e^{-\langle u, X_t\rangle}\right] = e^{-\phi(t,u)-\langle\psi(t,u),x\rangle},$$

for all t and $u, x \in S_d^+$ and some functions $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \times S_d^+ \to S_d^+$.

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  - Multivariate stochastic volatility models.
  - Affine term structure models based on S_d^+-valued affine processes.
- Understanding of this class of processes:
  - Necessary admissibility conditions on the parameters of the infinitesimal generator.
  - Sufficient conditions for the existence of affine processes on S_d^+.

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- Risk neutral dynamics for the log-price process Y_t and the $\mathbb{R}_+$-valued variance process X_t:

$$dX_t = (b + \beta X_t)\, dt + \sigma \sqrt{X_t}\, dW_t, \qquad X_0 = x,$$
$$dY_t = \left(r - \frac{X_t}{2}\right) dt + \sqrt{X_t} dB_t, \qquad Y_0 = y.$$

  - B, W: correlated Brownian motions,
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- Efficient valuation of European options via Fourier methods since the moment generating function is explicitly known (up to the solution of an ODE) and of the following form

$$\mathbb{E}_{x,y}\left[e^{-uX_t+vY_t}\right] = e^{\Phi(t,u,v)+\Psi(t,u,v)x+vy}, \quad (u,v) \in \mathbb{C}^2.$$

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- ...obtain a consistent pricing framework for multi-asset options such as basket options,
- ...use them as a basis for financial decision-making in the area of portfolio optimization and hedging of correlation risk.

Model specification

- Multivariate stochastic volatility models consist of a *d*-dimensional logarithmic price process with risk-neutral dynamics

$$dY_t = \left(r\mathbf{1} - \frac{1}{2} X_t^{\text{diag}} \right) dt + \sqrt{X_t} dB_t, \quad Y_0 = y,$$

and stochastic covariation process $X = \langle Y, Y \rangle$.

  - B: d-dimensional Brownian motion,
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  - $\mathbf{1}$: the vector whose entries are all equal to one,
  - X^{diag}: the vector containing the diagonal entries of X.
- In order to qualify for a covariation process, X must be specified as a process in S_d^+. Affine dynamics for X guarantee tractability of the model.

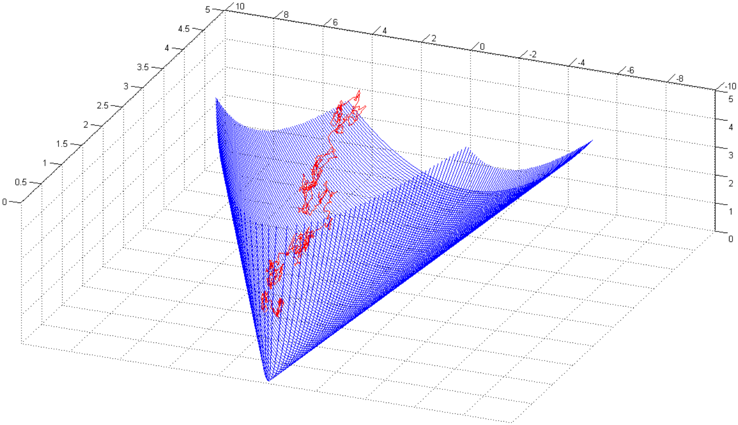
Prototype equation of an affine process in S_d^+

- The following affine dynamics for X have been proposed in the literature:

$$dX_t = (b + HX_t + X_t H^\top)dt + \sqrt{X_t}dW_t\Sigma + \Sigma^\top dW_t^\top \sqrt{X_t} + dJ_t,$$
$$X_0 = x \in S_d^+.$$

  - b: suitably chosen matrix in S_d^+,
  - H, Σ: invertible matrices,
  - W a standard $d \times d$-matrix of Brownian motions possibly correlated with B,
  - J a pure jump process whose compensator is an affine function of X.

Trajectory of a 2×2 positive semidefinite valued affine process



Term structure models based on affine processes with canonical state space

Here, X is $\mathbb{R}_+^m \times \mathbb{R}^n$-valued, $N = m + n$.

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- If the short rate r_t is specified as an affine function of an affine process, that is

$$r_t = l + \lambda^\top X_t, \quad l \in \mathbb{R}, \lambda \in \mathbb{R}^N,$$

then the zero coupon bond prices have exponential affine form

$$B_{t,T} = \mathbb{E}\left[e^{-\int_0^T r_s ds}\middle| X_t\right] = e^{G(t,T)+H(t,T)^\top X_t}.$$

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- The functions G and H solve a system of a generalized Riccati ODEs.

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  - For nonnegative short rates the state space has to be chosen to be $\mathbb{R}_+^m$. Due to admissibility conditions, this implies mutually independent positive factors.
  - The introduction of correlated factors induces a positive probability of negative yields.
- The use of S_d^+-valued affine processes allows for nonnegative affine term structure models with stochastically correlated risk factors while preserving tractability. By specifying the short rate like before as

$$r_t = l + \mathrm{Tr}(\lambda X_t), \quad l \in \mathbb{R}_+, \, \lambda \in S_d^+,$$

where X is now an affine process on S_d^+, the exponential affine form of the zero coupon prices is maintained.

Related Literature

- Theory of affine processes on $\mathbb{R}_+^m \times \mathbb{R}^n$:
  - Duffie, Filipović and Schachermayer [13]: Characterization of affine processes $\mathbb{R}_+^m \times \mathbb{R}^n$.
  - Keller-Ressel, Schachermayer and Teichmann [19]: Regularity.
  - etc.

Related Literature

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  - etc.
- Theory of affine processes on S_d^+:
  - Bru [5]: Existence and uniqueness (in law) of Wishart processes of type

$$dX_t = (\delta I_d)dt + \sqrt{X_t}dW_t + dW_t^\top \sqrt{X_t}, \quad X_0 \in S_d^+,$$

    for $\delta > d - 1$.
  - Barndorff-Nielsen and Stelzer [3]: Matrix-valued Lévy driven Ornstein-Uhlenbeck processes.

Related Literature

- Affine processes on S_d^+ - Applications in mathematical finance:
  - Buraschi et al. [6],
  - Da Fonseca et al. [9, 10, 11, 12],
  - Gourieroux and Sufana [15, 16],
  - Leippold and Trojani [20],
  - etc.

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  - Leippold and Trojani [20],
  - etc.
- Numerics and simulation of affine processes on S_d^+:
  - Ahdida and Alfonsi [1],
  - Gauthier and Possamai [14],
  - etc.

Feller property, regularity and related ODEs

Theorem

Let X be an affine process with state space S_d^+. Then, X is a Feller process and it is regular, that is the derivatives

$$F(u) = \partial_t \phi(t, u)|_{t=0+}, \qquad R(u) = \partial_t \psi(t, u)|_{t=0+}$$

exist and are continuous at $u = 0$.

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- By the regularity of X, it follows that the function ϕ and ψ are solutions of ODEs:

$$\frac{\partial \phi(t,u)}{\partial t} = F(\psi(t,u)), \quad \phi(0,u) = 0, \tag{1}$$

$$\frac{\partial \psi(t,u)}{\partial t} = R(\psi(t,u)), \quad \psi(0,u) = u \in S_d^+, \tag{2}$$

which we call generalized Riccati equations due to the particular form of F and R.

Infinitesimal generator

Theorem

If X is an affine process on S_d^+, then its infinitesimal generator is affine:

$$\begin{aligned}\mathcal{A}f(x) = 2\left\langle \left(\frac{\partial}{\partial x}\right) \alpha \left(\frac{\partial}{\partial x}\right), x\right\rangle f|_x + \langle b + B(x), \nabla f(x)\rangle \\ - (c + \langle \gamma, x\rangle) f(x) + \int_{S_d^+\setminus\{0\}} (f(x+\xi) - f(x))\, m(d\xi) \\ + \int_{S_d^+\setminus\{0\}} (f(x+\xi) - f(x) - \langle \chi(\xi), \nabla f(x)\rangle) \frac{\langle x, \mu(d\xi)\rangle}{\|\xi\|^2 \wedge 1},\end{aligned} \tag{3}$$

for some truncation function χ and admissible parameters

$$\left(\alpha, b, B(x) = \sum_{i,j} \beta^{ij} x_{ij}, c, \gamma, m(d\xi), M(x, d\xi) = \frac{\langle x, \mu\rangle}{\|\xi\|^2 \wedge 1}\right).$$

The functions F and R and existence of affine processes

Theorem

Moreover, $\phi(t,u)$ and $\psi(t,u)$ solve the differential equations (1) *and* (2)*, where F and R have the following form*

$$F(u) = \langle b, u\rangle + c - \int_{S_d^+\setminus\{0\}} (e^{-\langle u,\xi\rangle} - 1) m(d\xi),$$

$$R(u) = -2u\alpha u + B^\top(u) + \gamma - \int_{S_d^+\setminus\{0\}} \left(e^{-\langle u,\xi\rangle} - 1 + \langle \chi(\xi), u\rangle\right) \frac{\mu(d\xi)}{\|\xi\|^2 \wedge 1}.$$

Conversely, let $(\alpha, b, \beta^{ij}, c, \gamma, m, \mu)$ be an admissible parameter set. Then there exists a unique affine process on S_d^+ with infinitesimal generator (3)*.*

Relation to semimartingales

Corollary

Let X be a conservative affine process on S_d^+. Then X is a semimartingale. Furthermore, there exists, possibly on an enlargement of the probability space, a $d \times d$-matrix of standard Brownian motions W such that X admits the following representation

$$
\begin{aligned}
X_t = x &+ \int_0^t \left(b + \int_{S_d^+ \setminus \{0\}} \chi(\xi) m(d\xi) + B(X_s) \right) ds, \\
&+ \int_0^t \left(\sqrt{X_s} dW_s \Sigma + \Sigma^\top dW_s \sqrt{X_s} \right) \\
&+ \int_0^t \int_{S_d^+ \setminus \{0\}} \chi(\xi) \left(\mu^X(ds, d\xi) - (m(d\xi) + M(X_s, d\xi)) \, ds \right) \\
&+ \int_0^t \int_{S_d^+ \setminus \{0\}} (\xi - \chi(\xi)) \mu^X(ds, d\xi),
\end{aligned}
$$

where Σ is a $d \times d$ matrix satisfying $\Sigma^\top \Sigma = \alpha$ and μ^X denotes the random measure associated with the jumps of X.

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- constant jump term: Borel measure m on $S_d^+ \setminus \{0\}$,
- constant killing rate term: $c \in \mathbb{R}^+$.

Remark on the admissible parameters

- No constant diffusion part, linear part is of very specific form

$$\langle v, A(x)v\rangle = 4\langle x, v\alpha v\rangle \text{ for all } v \in S_d^+.$$

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- Jumps described by m are of finite variation, for the linear jump part we have finite variation for the directions orthogonal to the boundary while parallel to the boundary general jump behavior is allowed. Thus,
$$\int_{S_d^+\setminus\{0\}} \langle \chi(\xi), u\rangle M(x, d\xi) < \infty, \quad u \in S_d^+ \text{ with } \langle u, x\rangle = 0.$$

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$$\int_{S_d^+\setminus\{0\}} \langle \chi(\xi), u\rangle M(x, d\xi) < \infty, \quad u \in S_d^+ \text{ with } \langle u, x\rangle = 0.$$
- The linear drift part has to be inward pointing, that is
$$\langle B(x), u\rangle - \int_{S_d^+\setminus\{0\}} \langle \chi(\xi), u\rangle \, M\,(x, d\xi) \geq 0 \quad u \in S_d^+ \text{ with } \langle u, x\rangle = 0.$$

Remark on the admissible parameters

Very remarkable admissibility condition between the constant drift b and the linear diffusion coefficient α due to

$$\langle b, \nabla \det(x)\rangle + 2\left\langle \left(\frac{\partial}{\partial x}\right)\alpha\left(\frac{\partial}{\partial x}\right), x\right\rangle \det|_x \geq 0$$

for all $x \in \partial S_d^+$.

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For $d \geq 2$ the boundary of S_d^+ is curved and implies this relation between linear diffusion coefficient α and drift part b.

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- Extension of the model class.
  - General linear drift part $B(x) = \sum_{ij} \beta^{ij} x_{ij}$. This allows dependency of the volatility of one asset on the other ones which is not possible for $B(x) = Hx + xH^\top$. Example: $d = 2$ and

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  - Full generality of jumps (quadratic variation jumps parallel to the boundary).

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- Consider a multivariate stochastic volatility model:

$$dY_t = \left(r\mathbf{1} - \frac{1}{2} X_t^{\text{diag}} \right) dt + \sqrt{X_t} dB_t,$$

$$dX_t = (b + B(X_t))dt + \sqrt{X_t} dW_t \Sigma + \Sigma^\top dW_t^\top \sqrt{X_t} + dJ_t.$$

- Then, the moment generating function of the process (X, Y) has the following form:

$$\mathbb{E}_{x,y} \left[e^{-\operatorname{Tr}(uX_t) + v^\top Y_t} \right] = e^{\Phi(t,u,v) + \operatorname{Tr}(\Psi(t,u,v)x) + v^\top y}$$

for appropriate arguments $u \in S_d \times \mathrm{i} S_d$ and $v \in \mathbb{C}^d$. The functions Φ and Ψ solve a system of generalized Riccati ODEs.

Option pricing

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$$\pi_0 = e^{-rT}\mathbb{E}_{x,y}[f(Y_T)]$$

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- Efficient valuation of European options since
$$\begin{aligned}\pi_0 &= e^{-rT}\mathbb{E}_{x,y}\left[\left(\int_{\mathbb{R}^d} e^{(c+\mathrm{i}\lambda)^\top Y_T}\widetilde{f}(\lambda)d\lambda\right)\right] \\ &= \int_{\mathbb{R}} e^{\Phi(t,0,c+\mathrm{i}\lambda)+\mathrm{Tr}(\Psi(t,0,c+\mathrm{i}\lambda)x)+(c+\mathrm{i}\lambda)^\top y}\widetilde{f}(\lambda)d\lambda.\end{aligned}$$

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- Example Spread option: $c_2 < 0, c_1 + c_2 > 1$
$$(e^{y_1} - e^{y_2} - 1)^+ = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{(c+\mathrm{i}\lambda)^\top y}\frac{\Gamma(c_1 + c_2 - 1 + \mathrm{i}(\lambda_1 + \lambda_2))\Gamma(-c_2 - \mathrm{i}\lambda_2)}{\Gamma(c_1 + 1 + \mathrm{i}\lambda_1)}d\lambda_1 d\lambda_2.$$
This representation is due to Hurd and Zhou [18].

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where $\widetilde{\phi}$ and $\widetilde{\psi}$ satisfy the extended generalized Riccati equations

$$\partial_t \widetilde{\phi} = \widetilde{F}(\widetilde{\psi}) = F(\widetilde{\psi}) + l, \qquad \widetilde{\phi}(0,u) = 0,$$
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- Bond option prices (e.g.caps) are computed efficiently by Fourier pricing methods.

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- Outlook
  - Affine processes on other state spaces (symmetric cones).
  - Calibration of affine term structure models and multivariate stochastic volatility models.

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Thank you for your attention!