Time dependent Heston model

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Agenda

1. Motivation: real-time pricing/calibration of financial products, while maintening accurate results

   $\rightsquigarrow$ Computational challenge.

2. Heston model: definition, Fourier transform . . .

3. Stochastic expansion using a proxy: general principles

4. Expansion results in time dependent Heston model

5. Numerical tests

A popular model generating volatility smile and skew: the Heston model

The dynamics of the asset S: $S_t = e^{\int_0^t (r_s - q_s)ds} e^{X_t}$, where X solves:

$$\begin{aligned} dX_t &= -\frac{1}{2} V_t dt + \sqrt{V_t} dW_t, \\ dV_t &= \kappa(\theta_t - V_t)dt + \xi_t \sqrt{V_t} dB_t, \\ d\langle W, B\rangle_t &= \rho_t dt. \end{aligned}$$

$(\rho_t, \theta_t, \xi_t)_t$ are time dependent coefficients $\rightsquigarrow$ **time dependent Heston model.**

In practice, coefficients are piecewise constant.

Ref: **[Heston '93]**, **[Lewis '00]**, **[Mikhailov and Nogel '03]**, **[Elices '08]**

LEMMA. Assume $\mathbf{V_0 > 0}$ and $\inf_{\mathbf{t \le T}} \frac{\mathbf{2\kappa\theta_t}}{\mathbf{\xi_t^2}} \ge \mathbf{1}$. Then, $\mathbb{P}(\forall \mathbf{t} \in [\mathbf{0}, \mathbf{T}] : \ \mathbf{V_t} > \mathbf{0}) = \mathbf{1}$.

In the following, we additionnally assume that $\sup_{\mathbf{t \le T}} |\rho_\mathbf{t}| < \mathbf{1}$ and $\xi_\mathbf{t} > \mathbf{0}$.

Lewis '00 closed formula via Fourier transform

$$Call_{Heston}(t, S_t, V_t; T, K) = S_t e^{-\int_t^T q_s ds} - \frac{K e^{-\int_t^T r_s ds}}{2\pi} \int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} e^{-izX} \phi_{t,T}(-z) \frac{dz}{z^2 - iz}$$

where $X = \log\left(\frac{S_t e^{-\int_t^T q_s ds}}{K e^{-\int_t^T r_s ds}}\right)$ and $\phi_{t,T}(z) = \mathbb{E}(e^{z(X_T - X_t)}|\mathcal{F}_t)$.

Since (X, V) is an affine model, $\phi_{t,T}(z)$ can be decomposed as

$$\phi_{t,T}(z) = \exp(A_{z,T}(t) + V_t B_{z,T}(t))$$

where $(A_{z,T}(.), B_{z,T}(.))$ solves a system of Ricatti equations.

- For **constant coefficients**, explicit solutions (using trigonometric functions), but a numerical integration (w.r.t. z) is still needed.
- For **time dependent coefficients**, numerical solutions for Ricatti equations and numerical integration.

Our approach: volatility of volatility expansion $\xi \to 0$

First used by Lewis '00 in the case of **constant coefficients**.

- His formula is of the form

$$\mathrm{Call}^{Heston}(T,K) = \mathrm{Call}^{BS}(T,K,\hat{\sigma}) + \sum_i \alpha_i \mathrm{Greek}_i^{BS}(T,K,\hat{\sigma}) + \text{expansion error},$$

  with explicit parameters $\hat{\sigma}$ and coefficients α_i.

  $\rightsquigarrow$ **very quick evaluation**.

- His approach relies on the Fourier transform and the explicit form of $\phi_{t,T}(z)$.
- No justification of the expansion is provided. Error estimation? order of convergence?

Our contribution:

- We obtain closed formula also in the time-dependent case.
- We provide error estimate.

Our methodology is based on Malliavin calculus and stochastic analysis.

A generic approach for obtaining approximative closed formulas

We have already used this approach for Local Volatility Models, with or without lognormal jumps, with or without stochastic interest rates (see two works in *FS 2009*, *IJTAF 2010* plus one submitted).

1. Given the dynamics of the underlying asset X (or log-asset), **find a proxy** X^P for this model. For instance
   - **log-normal proxy** (Black-Scholes price)
   - **normal proxy** (Bachelier price)
   - **Merton** (log-normal diffusion + log-normal jumps)
   - ...

   Constraints: price and Greeks in this model should be given in closed forms.

2. **Expansion around the proxy:**
$$\mathbb{E}(\mathbf{h}(\mathbf{X_T})) = \mathbb{E}(\mathbf{h}(\mathbf{X_T^P})) + \mathbb{E}(\mathbf{h}'(\mathbf{X_T^P})(\mathbf{X_T} - \mathbf{X_T^P})) + \frac{\mathbf{1}}{\mathbf{2}}\mathbb{E}(\mathbf{h}''(\mathbf{X_T^P})(\mathbf{X_T} - \mathbf{X_T^P})^{\mathbf{2}}) + \cdots$$
Closed approximation forms for terms with derivatives (Malliavin calculus)
$\rightsquigarrow$ **Reversing the representation of Greeks using Malliavin calculus:**
$$\mathbb{E}(h'(X_T^P)(X_T - X_T^P)) = \sum_j \alpha_j \mathrm{Greek}_j^h(X^p) + \text{error}.$$
3. **Final formula:** $\mathbb{E}(\mathbf{h}(\mathbf{X_T})) = \mathbb{E}(\mathbf{h}(\mathbf{X_T^P})) + \sum_{\mathbf{i}} \alpha_{\mathbf{i}} \mathrm{Greek}_{\mathbf{i}}^{\mathbf{h}}(\mathbf{X_T^P}) + \mathrm{Error}.$
4. We **analyse the error**: depends on payoff regularity, on maturity and on model.

Specification of the proxy

We use a parametrization: for $\epsilon \in [0,1]$, define

$$dX_t^\epsilon = -\frac{V_t^\epsilon}{2}dt + \sqrt{V_t^\epsilon}dW_t, \qquad X_0^\epsilon = x_0,$$
$$dV_t^\epsilon = \kappa(\theta_t - V_t^\epsilon)dt + \epsilon\xi_t\sqrt{V_t^\epsilon}dB_t, \quad V_0^\epsilon = v_0,$$

The proxy is obtained by taking $\epsilon = 0$, that is **BS model with time-dependent volatility $(V_t^0)_t$.**

Analogies and differences

• There are some analogies with regular/singular pertubations in PDEs (see **[Fouque etal. '00]**). **BUT** within our approach, we are able to easily compute correction terms in explicit forms (even for time dependent coefficients, or including jumps...).

• There are also analogies with Watanabe's approach **[Watanabe '87]** about asymptotic expansion of Wiener functionals. **BUT** expansions and asymptotics are actually different.

Price expansion

Step 1. Integrate w.r.t. W:

$$e^{-\int_0^T r_t dt}\mathbb{E}[(K - e^{\int_0^T (r_t - q_t)dt + X_T^1})_+]$$

$$= \mathbb{E}[\text{Put}^{BS}(x_0 + \int_0^T \rho_t \sqrt{V_t^1} dB_t - \int_0^T \frac{\rho_t^2}{2} V_t^1 dt, \int_0^T (1-\rho_t^2) V_t^1 dt)],$$

Step 2. Expand w.r.t. ϵ around $\epsilon = 0$: for this, set $V_{i,t} \equiv \frac{\partial^i V_t^\epsilon}{\partial \epsilon^i}|_{\epsilon=0}$. It follows that

$$V_{0,t} = e^{-\kappa t}(v_0 + \int_0^t \kappa e^{\kappa s}\theta_s ds),$$

$$V_{1,t} = e^{-\kappa t}\int_0^t e^{\kappa s}\xi_s \sqrt{V_{0,s}} dB_s,$$

$$V_{2,t} = e^{-\kappa t}\int_0^t e^{\kappa s}\xi_s \frac{V_{1,s}}{(V_{0,s})^{\frac{1}{2}}} dB_s$$

Step 3. Expand Put^{BS} and reverse the Greeks...

Crucial technical results

LEMMA. For every $p > 0$, one has:

$$\sup_{\mathbf{0\leq\epsilon\leq 1}} \mathbb{E}\Big[\Big(\int_{\mathbf{0}}^{\mathbf{T}} \mathbf{V_t^\epsilon dt}\Big)^{\mathbf{-p}}\Big] \leq \frac{\mathbf{C}}{\mathbf{T^p}}.$$

(appeared in Bossy-Diop'07 under stronger conditions).

LEMMA. Define the volatility process $\sigma_t^\epsilon = \sqrt{V_t^\epsilon}$, which is governed by the SDE:

$$d\sigma_t^\epsilon = \Big(\Big(\frac{\kappa\theta_t}{2} - \frac{\epsilon^2\xi_t^2}{8}\Big)\frac{1}{\sigma_t^\epsilon} - \frac{\kappa}{2}\sigma_t^\epsilon\Big)dt + \frac{\epsilon\xi_t}{2}dB_t,\ \ \sigma_0^\epsilon = \sqrt{v_0},$$

Then $a.s.$, the application $\epsilon \mapsto \sigma_t^\epsilon$ is C^2 at $\epsilon = 0$. And for $k = 0, 1, 2$, we have

$$\Big\|\sup_{\mathbf{0\leq t\leq T}}\Big|\sigma_{\mathbf{t}}^\epsilon - \sum_{\mathbf{i=0}}^{\mathbf{k}}\frac{\epsilon^{\mathbf{i}}}{\mathbf{i!}}\partial_\epsilon^{\mathbf{i}}\sigma_{\mathbf{t}}^\epsilon|_{\epsilon=\mathbf{0}}\Big|\Big\|_{\mathbf{L_p}} \leq \mathbf{C_p}\Big[\epsilon\sup_{\mathbf{0\leq t\leq T}}\xi_{\mathbf{t}}\sqrt{\mathbf{T}}\Big]^{\mathbf{k+1}}.$$

Final formula

THEOREM. The put price is approximated by

$$e^{-\int_0^T r_t dt}\mathbb{E}[(K - e^{\int_0^T (r_t - q_t)dt + X_T^1})_+] = \mathrm{Put}^{BS}(x_0, var_T) + \sum_{i=1}^{2} a_{i,T} \frac{\partial^{i+1}\mathrm{Put}^{BS}}{\partial x^i y}(x_0, var_T)$$

$$+ \sum_{i=0}^{1} b_{2i,T} \frac{\partial^{2i+2}\mathrm{Put}^{BS}}{\partial x^{2i} y^2}(x_0, var_T) + \text{error},$$

where

$$var_T = \int_0^T V_{0,t}dt, \quad a_{1,T} = \omega_{0,T}^{(\kappa,\rho\xi V_{0,.}),(-\kappa,1)}, \quad a_{2,T} = \omega_{0,T}^{(\kappa,\rho\xi V_{0,.}),(0,\rho\xi),(-\kappa,1)},$$

$$b_{0,T} = \omega_{0,T}^{(2\kappa,\xi^2 V_{0,.}),(-\kappa,1),(-\kappa,1)}, \quad b_{2,T} = \frac{a_{1,T}^2}{2},$$

and $\omega_{t,T}^{(k,l)} = \int_t^T e^{ku} l_u du$ and $\omega_{t,T}^{(k_1,l_1),\cdots,(k_n,l_n)} = \omega_{t,T}^{(k_1,l_1\omega_{.,T}^{(k_2,l_2),\cdots,(k_n,l_n)})}$.

In addition, the expansion error is a $\mathbf{O}\left([\sup_{\mathbf{0\le t\le T}} \xi_{\mathbf{t}}]^{\mathbf{3}}\mathbf{T}^{\mathbf{2}}\right)$.

Numerical tests

Table 1: Set of maturities and strikes used for the numerical tests.

T/K								
3M	70	80	90	100	110	120	125	130
6M	60	70	80	100	110	130	140	150
1Y	50	60	80	100	120	150	170	180
2Y	40	50	70	100	130	180	210	240
3Y	30	40	60	100	140	200	250	290
5Y	20	30	60	100	150	250	320	400
7Y	10	30	50	100	170	300	410	520
10Y	10	20	50	100	190	370	550	730

Implied BS volatilities of the closed formula, of the approximation formula and related errors (in bp), expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\theta = 6\%$, $\kappa = 3$, $\xi = 30\%$ and $\rho = -\mathbf{20}\%$.

3M	24.50%	23.07%	21.92%	21.16%	20.84%	20.91%	21.04%	21.21%
	24.04%	23.14%	21.93%	21.15%	20.82%	20.87%	21.06%	21.37%
	45.76	**-7.65**	**-1.25**	**0.38**	**2.35**	**3.68**	**-2.73**	**-16.51**
6M	25.68%	24.38%	23.31%	21.94%	21.65%	21.68%	21.88%	22.15%
	25.19%	24.45%	23.38%	21.93%	21.63%	21.64%	21.96%	22.47%
	49.49	**-7.75**	**-7.32**	**0.99**	**2.22**	**4.10**	**-8.10**	**-32.52**
1Y	26.20%	25.14%	23.65%	22.82%	22.47%	22.51%	22.72%	22.86%
	25.92%	25.23%	23.68%	22.81%	22.44%	22.49%	22.89%	23.17%
	28.04	**-8.22**	**-2.65**	**1.32**	**3.45**	**2.08**	**-16.41**	**-31.56**
2Y	26.03%	25.28%	24.29%	23.51%	23.18%	23.09%	23.17%	23.29%
	25.95%	25.35%	24.32%	23.50%	23.16%	23.08%	23.25%	23.56%
	7.83	**-6.41**	**-2.54**	**0.93**	**2.37**	**1.57**	**-8.04**	**-26.37**
3Y	26.06%	25.40%	24.57%	23.78%	23.47%	23.34%	23.36%	23.42%
	25.95%	25.44%	24.60%	23.78%	23.45%	23.32%	23.41%	23.58%
	11.21	**-3.39**	**-2.44**	**0.61**	**1.65**	**1.71**	**-5.11**	**-16.68**
5Y	25.83%	25.28%	24.47%	24.01%	23.75%	23.57%	23.55%	23.55%
	25.75%	25.30%	24.47%	24.01%	23.74%	23.56%	23.56%	23.65%
	8.29	**-1.76**	**-0.65**	**0.32**	**0.84**	**1.01**	**-1.92**	**-9.38**
7Y	26.02%	24.97%	24.56%	24.11%	23.86%	23.70%	23.65%	23.64%
	25.82%	24.99%	24.57%	24.11%	23.85%	23.69%	23.67%	23.70%
	20.23	**-1.59**	**-0.59**	**0.21**	**0.60**	**0.69**	**-1.50**	**-6.16**
10Y	25.43%	24.99%	24.49%	24.19%	23.97%	23.81%	23.75%	23.72%
	25.40%	25.00%	24.49%	24.18%	23.96%	23.80%	23.76%	23.76%
	3.46	**-0.94**	**-0.20**	**0.14**	**0.38**	**0.48**	**-0.95**	**-3.98**

Implied BS volatilities of the closed formula, of the approximation formula and related errors (in bp), expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\theta = 6\%$, $\kappa = 3$, $\xi = 30\%$ and $\rho = -\mathbf{50}\%$.

3M	26.13%	24.29%	22.60%	21.11%	19.95%	19.22%	19.03%	18.92%
	25.57%	24.43%	22.63%	21.11%	19.90%	18.99%	18.91%	19.57%
	56.55	**-14.06**	**-2.51**	**0.19**	**4.35**	**23.24**	**11.67**	**-64.22**
6M	27.47%	25.81%	24.31%	21.85%	20.92%	19.80%	19.55%	19.47%
	26.89%	25.97%	24.44%	21.84%	20.89%	19.50%	19.61%	21.11%
	58.13	**-16.68**	**-12.19**	**0.82**	**3.38**	**29.46**	**-5.28**	**-164.16**
1Y	27.96%	26.57%	24.34%	22.68%	21.51%	20.49%	20.19%	20.11%
	27.67%	26.75%	24.39%	22.66%	21.43%	20.24%	20.77%	21.73%
	29.08	**-18.08**	**-5.01**	**1.53**	**7.49**	**24.84**	**-58.18**	**-162.76**
2Y	27.56%	26.51%	24.93%	23.34%	22.31%	21.30%	20.95%	20.73%
	27.52%	26.65%	24.98%	23.33%	22.25%	21.15%	21.19%	22.20%
	4.11	**-14.03**	**-4.75**	**1.43**	**5.50**	**14.43**	**-23.17**	**-146.81**
3Y	27.53%	26.56%	25.22%	23.61%	22.66%	21.81%	21.39%	21.16%
	27.42%	26.66%	25.26%	23.60%	22.62%	21.71%	21.53%	22.04%
	11.28	**-9.11**	**-4.59**	**1.06**	**3.97**	**9.79**	**-14.43**	**-88.86**
5Y	27.11%	26.25%	24.83%	23.83%	23.10%	22.28%	21.94%	21.66%
	27.01%	26.31%	24.84%	23.82%	23.08%	22.23%	21.98%	22.14%
	9.64	**-5.22**	**-1.23**	**0.62**	**1.98**	**5.14**	**-4.04**	**-47.56**
7Y	27.35%	25.67%	24.92%	23.93%	23.23%	22.55%	22.22%	21.98%
	27.03%	25.71%	24.93%	23.93%	23.21%	22.52%	22.25%	22.28%
	31.65	**-3.57**	**-1.09**	**0.43**	**1.46**	**3.26**	**-3.91**	**-30.07**
10Y	26.40%	25.66%	24.70%	24.01%	23.40%	22.82%	22.50%	22.29%
	26.36%	25.68%	24.70%	24.00%	23.39%	22.80%	22.53%	22.48%
	4.15	**-2.43**	**-0.35**	**0.29**	**0.93**	**2.02**	**-2.65**	**-18.89**

Implied BS volatilities of the closed formula, of the approximation formula and related errors (in bp), expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\theta = 6\%$, $\kappa = 10$, $\xi = 1$ and $\rho = -50\%$.

3M	31.51%	28.04%	24.74%	21.83%	19.94%	19.45%	19.58%	19.85%
	30.68%	28.99%	24.95%	21.71%	19.38%	18.05%	19.76%	22.93%
	82.46	**-94.66**	**-21.22**	**12.10**	**56.44**	**140.23**	**-18.10**	**-308.17**
6M	31.45%	28.86%	26.52%	22.69%	21.36%	20.11%	20.05%	20.20%
	30.83%	29.59%	26.98%	22.58%	21.09%	19.14%	20.64%	24.03%
	62.40	**-73.58**	**-46.52**	**11.30**	**26.99**	**97.22**	**-59.12**	**-383.12**
1Y	30.09%	28.30%	25.44%	23.34%	21.89%	20.76%	20.49%	20.45%
	29.87%	28.72%	25.54%	23.28%	21.70%	20.30%	21.65%	23.17%
	21.52	**-42.32**	**-10.69**	**6.02**	**19.45**	**46.13**	**-115.72**	**-271.22**
2Y	28.45%	27.27%	25.51%	23.73%	22.58%	21.48%	21.12%	20.90%
	28.46%	27.47%	25.57%	23.71%	22.50%	21.28%	21.42%	22.75%
	-0.53	**-20.08**	**-6.39**	**2.42**	**8.11**	**19.97**	**-30.34**	**-184.76**
3Y	28.08%	27.05%	25.61%	23.88%	22.86%	21.96%	21.51%	21.27%
	27.98%	27.16%	25.66%	23.86%	22.81%	21.83%	21.67%	22.30%
	9.78	**-11.59**	**-5.41**	**1.39**	**4.91**	**12.13**	**-16.04**	**-102.46**
5Y	27.40%	26.52%	25.04%	24.00%	23.23%	22.38%	22.03%	21.75%
	27.31%	26.58%	25.05%	23.99%	23.21%	22.33%	22.07%	22.26%
	9.15	**-5.98**	**-1.31**	**0.71**	**2.20**	**5.85**	**-3.93**	**-51.20**
7Y	27.56%	25.84%	25.06%	24.05%	23.33%	22.63%	22.29%	22.05%
	27.24%	25.88%	25.08%	24.05%	23.31%	22.59%	22.33%	22.36%
	32.00	**-3.83**	**-1.14**	**0.47**	**1.57**	**3.57**	**-3.88**	**-31.56**
10Y	26.53%	25.77%	24.80%	24.09%	23.47%	22.88%	22.55%	22.34%
	26.49%	25.80%	24.80%	24.09%	23.46%	22.86%	22.58%	22.53%
	4.02	**-2.57**	**-0.36**	**0.31**	**0.97**	**2.15**	**-2.64**	**-19.49**

Put prices of the closed formula, of the approximation formula and related errors (in bp), expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\theta = 6\%$, $\kappa = 10$, $\xi = 1$ and $\rho = -50\%$.

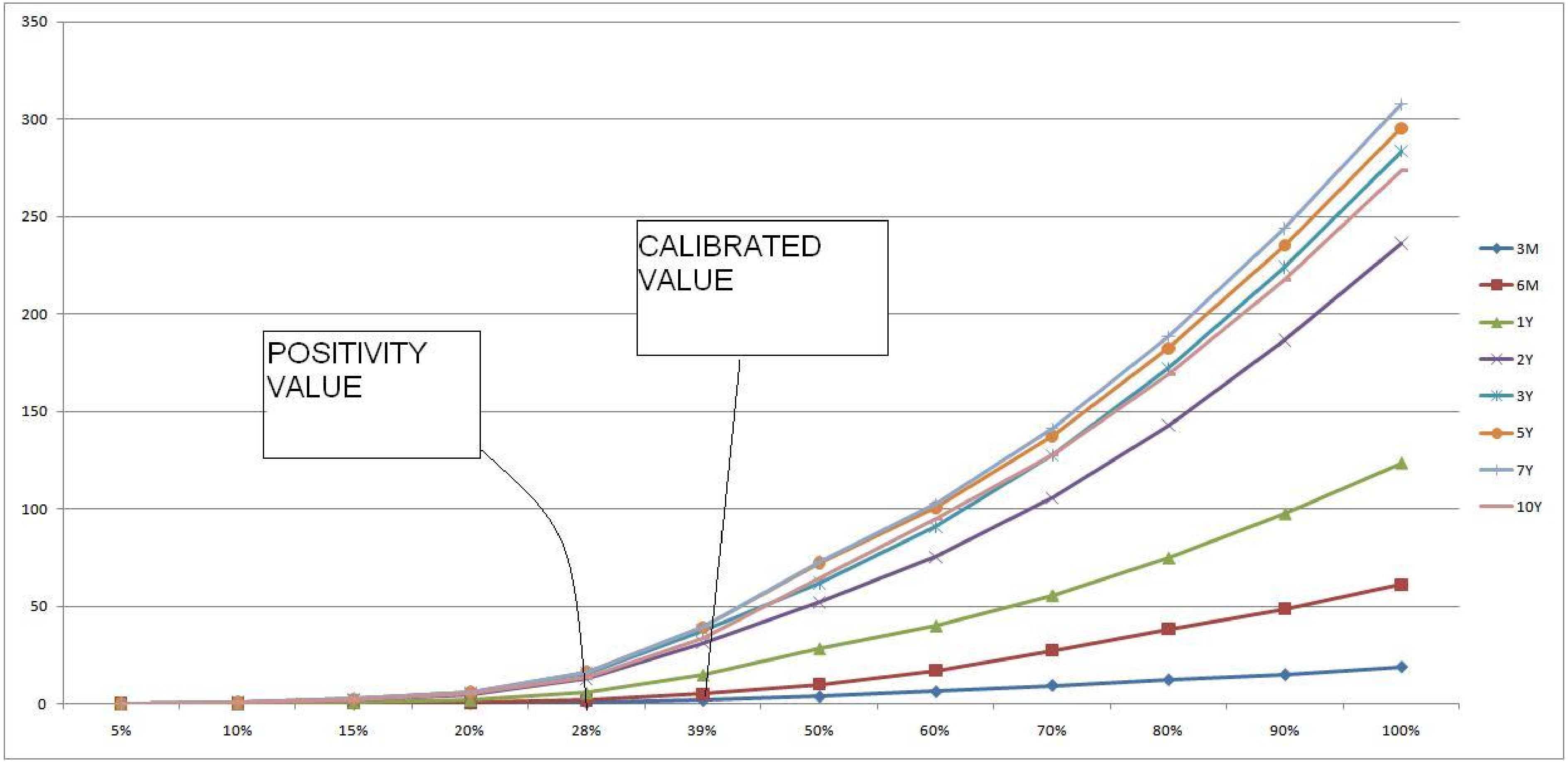
3M	30.05	20.30	11.28	4.35	0.95	0.13	0.04	0.01
	30.04	20.35	11.31	4.33	0.87	0.08	0.05	0.05
	0.99	**-4.95**	**-2.80**	**2.41**	**7.37**	**4.62**	**-0.31**	**-3.51**
6M	40.06	30.28	20.96	6.40	2.54	0.21	0.05	0.01
	40.05	30.32	21.02	6.36	2.47	0.15	0.06	0.06
	0.92	**-3.90**	**-5.83**	**3.18**	**6.51**	**5.23**	**-1.28**	**-4.72**
1Y	50.08	40.31	22.37	9.29	2.71	0.24	0.04	0.02
	50.07	40.33	22.40	9.26	2.65	0.21	0.07	0.06
	0.41	**-2.60**	**-2.58**	**2.39**	**5.95**	**3.19**	**-2.52**	**-3.89**
2Y	60.10	50.39	32.54	13.33	4.18	0.41	0.09	0.02
	60.10	50.40	32.56	13.31	4.14	0.38	0.10	0.05
	-0.01	**-1.58**	**-1.82**	**1.35**	**3.67**	**2.26**	**-1.17**	**-2.89**
3Y	70.06	60.28	42.09	16.38	5.09	0.71	0.13	0.03
	70.05	60.28	42.11	16.37	5.06	0.69	0.14	0.06
	0.17	**-0.73**	**-1.46**	**0.94**	**2.74**	**2.19**	**-0.86**	**-2.22**
5Y	80.04	70.25	44.15	21.15	7.76	1.02	0.26	0.06
	80.03	70.25	44.16	21.15	7.74	1.01	0.27	0.08
	0.11	**-0.36**	**-0.57**	**0.61**	**1.72**	**1.49**	**-0.38**	**-1.68**
7Y	90.00	70.56	53.46	24.96	8.45	1.31	0.32	0.09
	90.00	70.57	53.46	24.96	8.44	1.30	0.32	0.10
	0.06	**-0.44**	**-0.47**	**0.47**	**1.42**	**1.16**	**-0.46**	**-1.42**
10Y	90.02	80.31	55.47	29.67	10.51	1.85	0.44	0.13
	90.02	80.31	55.48	29.67	10.50	1.84	0.45	0.14
	0.03	**-0.19**	**-0.20**	**0.36**	**1.09**	**0.95**	**-0.42**	**-1.23**

Piecewise constant parameters. Implied Black-Scholes volatilities of the **closed formula**, of the **approximation formula** and of the **averaging formula**. $v_0 = 4\%$, $\kappa = 3$. The piecewise constant functions θ, ξ and ρ are equal respectively at each interval of the form $]\frac{i}{4}, \frac{i+1}{4}[$ to $4\% + i \times 0.05\%$, $30\% + i \times 0.5\%$ and $-20\% + i \times 0.35\%$.

6M	24.09%	22.59%	21.30%	19.63%	19.33%	19.58%	19.92%	20.31%
	23.09%	22.60%	21.43%	19.61%	19.30%	19.58%	20.19%	20.93%
	24.09%	22.59%	21.30%	19.63%	19.33%	19.58%	19.92%	20.31%
1Y	23.95%	22.66%	20.76%	19.70%	19.37%	19.69%	20.12%	20.36%
	23.12%	22.66%	20.81%	19.68%	19.32%	19.78%	20.62%	21.05%
	23.95%	22.66%	20.76%	19.70%	19.37%	19.69%	20.12%	20.35%
2Y	23.26%	22.30%	21.01%	19.99%	19.66%	19.83%	20.09%	20.37%
	22.84%	22.33%	21.04%	19.96%	19.62%	19.90%	20.43%	21.02%
	23.26%	22.30%	21.01%	19.98%	19.66%	19.83%	20.09%	20.37%
3Y	23.28%	22.40%	21.27%	20.26%	19.96%	20.02%	20.23%	20.43%
	22.81%	22.38%	21.33%	20.24%	19.93%	20.04%	20.47%	20.90%
	23.28%	22.40%	21.27%	20.26%	19.96%	20.02%	20.23%	20.42%
5Y	23.22%	22.46%	21.34%	20.77%	20.54%	20.54%	20.65%	20.80%
	22.88%	22.44%	21.35%	20.77%	20.52%	20.55%	20.76%	21.09%
	23.22%	22.46%	21.34%	20.77%	20.54%	20.54%	20.64%	20.79%
7Y	23.86%	22.36%	21.81%	21.26%	21.06%	21.06%	21.16%	21.27%
	23.25%	22.39%	21.82%	21.26%	21.05%	21.07%	21.23%	21.45%
	23.86%	22.37%	21.81%	21.26%	21.06%	21.06%	21.15%	21.26%
10Y	23.59%	22.96%	22.30%	21.97%	21.82%	21.83%	21.92%	22.02%
	23.46%	22.98%	22.30%	21.97%	21.81%	21.84%	21.96%	22.12%
	23.59%	22.96%	22.30%	21.97%	21.82%	21.83%	21.92%	22.01%

The mean absolute error for the prices (in bps), expressed as a function of the vol. of vol. and computed for each maturity.

Constant calibrated parameters taken from **[Baski, Cao and Chen '97]**.



Comparison of computational times

Tests performed on 2, 6 GHz Pentium PC, to compute 64 numerical values (8 maturities $\times$ 8 strikes).

- For the previous examples on constant coefficients:
  - 4.71 ms using the approximation formula
  - 301 ms using the closed formula
- For the previous example on piecewise constant coefficients:
  - 40.2 ms using the approximation
  - 2574 ms using the closed formula

Speed up by a factor 100 to 600.

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