The Effect of Estimation in High–dimensional Portfolios

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Outline

1. Classical Portfolio Optimisation
2. Plug-In Strategies with Estimated Parameters
3. James-Stein-Shrinkage Applied to Strategies
4. L_1–Constrained Strategies - LASSO
5. Other Strategies
6. Application to Empirical Data

Optimal Portfolio Selection

- The asset prices: 1 bond $S_0(t) = e^{rt}$, d risky assets

$$dS_i(t) = S_i(t)[\mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t)], \quad S_i(0) > 0, \quad i = 1, \ldots, d,$$

  $r > 0$ interest rate, $\mu \in \mathbb{R}^d$ drift, $\sigma \in \mathbb{R}^{d \times d}$ volatility matrix of full rank (all constant), W d-variate Brownian motion.
- Investor has $T > 0$ fixed time horizon, $X_0 > 0$ constant initial wealth, chooses $\pi_i(t)$ fraction of the wealth invested in the ith asset at time t, resulting in time-t-wealth X_t with $dX_t = \sum_{i=0}^{d} \pi_i(t) X_t \frac{dS_i(t)}{S_i(t)}$.
- Investor seeks π to maximise $V(\pi) := \mathbb{E}[\log(X_T)]$.
- Optimal solution

$$\pi^* = \Sigma^{-1}(\mu - r1),$$

  where $\pi_0(t) = 1 - \sum_{i=1}^{d} \pi_i(t)$, $\pi = (\pi_1, \ldots, \pi_d)^T$, $\Sigma = \sigma\sigma^T$.

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The Problem

- What if we need to estimate μ?
- What if the number of risky assets $d \to \infty$?

Plug-in Merton Strategy with Estimated μ

General Unbiased Plug-in Estimator

- Estimate μ by $\hat{\mu}$ and define the plug-in strategy $\hat{\pi} = \Sigma^{-1}(\hat{\mu} - r1)$.
- Assume $\hat{\pi} \sim N(\Sigma^{-1}(\mu - r1), V_0^2)$, $V_0 \in \mathbb{R}^{d \times d}$, then

$$V(\hat{\pi}) = V(\pi^*) - \frac{T}{2} \operatorname{trace}(\Sigma V_0^2).$$

Specific Plug-in Estimator

- Observation period $[-t_{est}, 0]$ for $t_{est} > 0$.
- Set

$$\hat{\mu}_i = \frac{\log(S_i(0)) - \log(S_i(-t_{est}))}{t_{est}} + \frac{1}{2} \sum_{j=1}^{d} \sigma_{ij}^2.$$

- Then $\hat{\pi} \sim N(\Sigma^{-1}(\mu - r1), \Sigma^{-1}/t_{est})$.
- $V(\hat{\pi}) = V(\pi^*) - d\frac{T}{2t_{est}}$.
- There are realistic scenarios in which even $V(\hat{\pi}) \to -\infty$ as $d \to \infty$.

James-Stein-Type Shrinkage of the Strategy

The James-Stein-Strategy

Let $\hat{\pi} = \Sigma^{-1}(\hat{\mu} - r1)$, $\pi^0 \in \mathbb{R}^d$, $a > 0$ fixed constants. Consider

$$\hat{\pi}^{JS,\pi^0} = \left(1 - \frac{a}{(\hat{\pi} - \pi^0)^T \Sigma (\hat{\pi} - \pi^0)}\right)(\hat{\pi} - \pi^0) + \pi^0.$$

The Expected Utility for the JS-Strategy

Let $\hat{\mu} \sim N(\mu, \Sigma/t_{est})$, $K \sim \text{Poisson}(\lambda)$, $\lambda = (\pi^* - \pi^0)^T \Sigma (\pi^* - \pi^0)/2$:

$$V(\hat{\pi}^{JS,\pi^0}) = V(\hat{\pi}) + \frac{T}{2} a \left[2 \frac{d-2}{t_{est}} - a\right] \mathbb{E}\left[\frac{t_{est}}{d - 2 + 2K}\right].$$

$\hat{\pi}^{JS,\pi^0}$ dominates $\hat{\pi}$ for $0 < a < 2(d-2)/t_{est}$; optimal $a = (d-2)/t_{est}$.

Special Choices for π^0 and Optimal a

- $\pi^0 = \pi^*$: $V(\hat{\pi}^{JS,\pi^0}) = V(\pi^*) - \frac{T}{t_{est}}$.
- $\pi^0 = \frac{\beta}{d} 1$, $\beta \in \mathbb{R}$: In some situations $V(\hat{\pi}^{JS,\pi^0}) \to \infty$ as $d \to \infty$.

L_1–constrained Strategies - LASSO

General Idea

- Require that π satisfies $\|\pi\|_1 = \sum_{i=1}^{d} |\pi_i| \leq c$ for a constant $c \geq 0$.
- $V(\pi) \geq \log(X_0) + rT - T\mathbb{E}\left\{c \max_i |\mu_i - r| + \frac{c^2}{2} \max_{i,j} |\Sigma_{ij}|\right\}$.
- If $\max_i |\mu_i - r|$, $\max_{i,j} |\Sigma_{ij}|$ bounded, $V(\pi) \not\to -\infty$ as $d \to \infty$.

Specific Results

For $\Sigma = \eta^2(\rho 11^T + (1-\rho)I)$, $\eta > 0, 0 \leq \rho \leq 1$ analytic results for

- the optimal L_1-constrained strategies, if μ known.
- for the L_1-constrained plug-in strategy as $d \to \infty$:
  - the distribution of $\#\{i : \pi_i^* \neq 0\}$, if $\rho = 0$,
  - an upper bound on $\lim_{d\to\infty} \mathbb{P}(\#\{i : \pi_i^* \neq 0\} > k)$, if $\rho > 0$.

Other Strategies and Performance for $d \to \infty$

Other Norm Constraints

- L_0-restricted strategies: no degeneration of expected utility as $d \to \infty$.
- L_2-restricted strategies: degeneration possible.

Special L_1-Constraints

- $1/d$-strategy: Strategy that invests the same amount into all stocks, i.e. $\pi^{c/d} = \frac{c}{d}1$ for some $c > 0$.
- Equal Weighting of the most Extreme stocks (EWE):

$$\pi_{k_i}^{\text{EWE}} = \frac{c}{\beta d} \operatorname{sign}(\hat{a}_{k_i}) \mathbb{I}(i \leq \beta d), \quad i = 1, \ldots, d,$$

where $\hat{a}_i = \frac{\hat{\mu}_i - r}{\Sigma_{ii}}$, $c > 0$, $\beta \in (0,1)$ constants, k_i are such that $|\hat{a}_{k_1}| > |\hat{a}_{k_2}| > \cdots > |\hat{a}_{k_d}|$.

Example - Trading S&P500

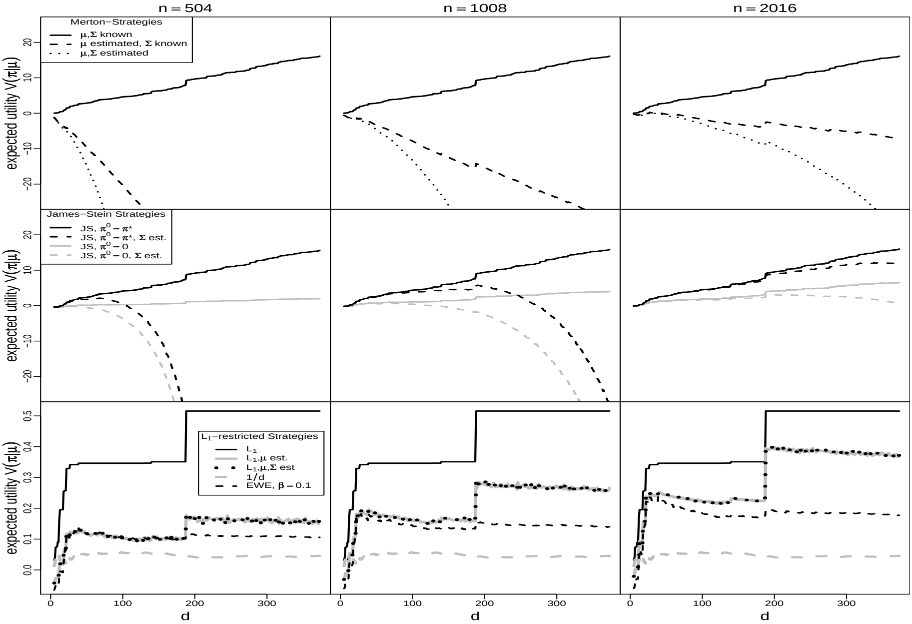
- Stocks in S&P 500 index on 01/01/2006 having daily returns for all trading days between 2001 and 2008 (373 stocks, n=2011 trading days, daily returns).
- Specific random ordering of stocks. Allow the strategies to invest in the first d stocks of this ordering.
- $X_0 = 1$, $r = 0.02$, roughly $T = 1$.
- Use of unbiased estimators based on observed stock prices at time points $0, \Delta, 2\Delta, \ldots, (n-1)\Delta$:

$$\hat{\mu}^{\text{data}} = \frac{1}{\Delta}\hat{\xi} + \frac{1}{2}\operatorname{diag}(\widehat{\Sigma}^{\text{data}}),$$

$$\widehat{\Sigma}^{\text{data}}_{\mu,\nu} = \frac{1}{\Delta(n-2)}\sum_{i=0}^{n-2}\left[R_\mu(i) - \hat{\xi}_\mu\right]\left[R_\nu(i) - \hat{\xi}_\nu\right]$$

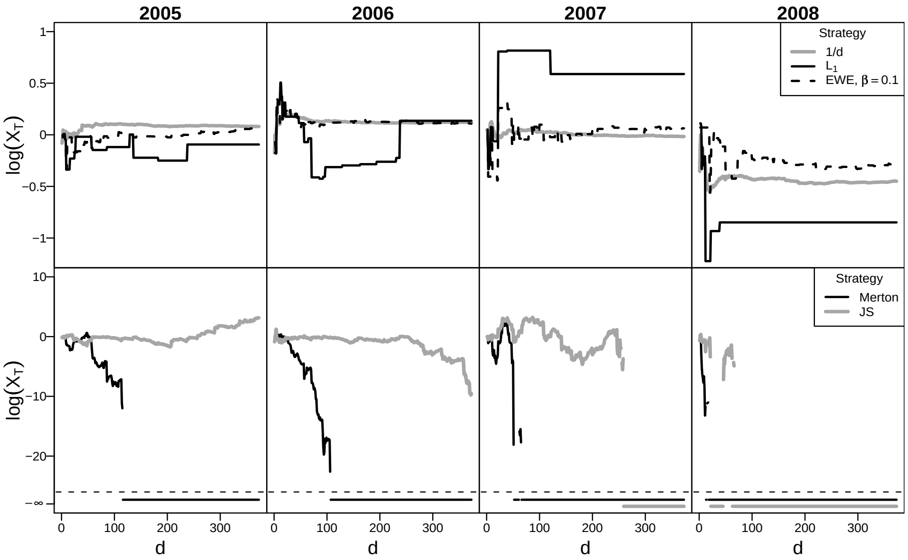
for $\mu, \nu = 1, \ldots, d$, where $R_\mu(i) = \log\left(\frac{S_\mu((i+1)\Delta)}{S_\mu(i\Delta)}\right)$,
$\hat{\xi}_\mu = \frac{1}{n-1}\sum_{i=0}^{n-2} R_\mu(i)$.

Analytic and Simulation Results



Expected utility plotted against the number d of available stocks.

Out-of-Sample Performance



$\log(X_T)$ with $T = 1$ year plotted against the number d of available stocks.

Summary

Main Contributions

- Quantification of the effect of estimation in vast portfolios (unknown μ and large d).
- Analysis of strategies which are less affected by estimation.
- Analytic formulae for James-Stein and optimal L_1-constrained strategies.

Specific Conclusions

- Estimation effects must not be ignored in vast portfolios!
- Simple plug in strategies have a loss through estimation linear in d.
- James-Stein shrinkage performs better than simple plug in strategies.
- L_1-constrained strategies cannot degenerate.
- L_1-constrained strategies and particularly the EWE-strategy and $1/d$ strategy perform well also in out-of-sample tests.

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Analytic Results for LASSO with $\Sigma = \eta^2 I$

Optimal L_1-constrained strategy with known μ

Suppose $|\mu_1 - r| > |\mu_2 - r| > \ldots > |\mu_d - r|$. Then $\pi^\dagger = \frac{1}{\eta^2}(\mu - r1)$ is a solution to the L_1-constrained optimisation problem if $\|\pi^\dagger\|_1 \leq c$. Otherwise, the unique solution is

$$\pi^* := \frac{1}{\eta^2}(\text{sign}(\mu_1 - r)(|\mu_1 - r| - a), \ldots, \text{sign}(\mu_k - r)(|\mu_k - r| - a), 0, \ldots, 0)^T,$$

$k = \min\left\{l \in \{1, \ldots, d\} : c \leq \frac{1}{\eta^2} \sum_{i=1}^{l} (|\mu_i - r| - |\mu_{l+1} - r|)\right\}$,
$a = \frac{1}{k}\left[\eta^2 c - \sum_{i=1}^{k} |\mu_i - r|\right]$ and $\mu_{d+1} = r$.

Plug-in strategy with iid normally distributed estimators $\hat{\mu}_1, \ldots, \hat{\mu}_d$

Let $c = \alpha c_d$, $c_d > 0$ norming constants from extreme value theory for folded normal distribution $FN(\mathbb{E}(\hat{\mu}_1 - r), \mathbb{V}ar(\hat{\mu}_1))$. Then

$$\#\{i : \pi_i^* \neq 0\} \xrightarrow{\mathcal{L}} K + 1 \quad (d \to \infty),$$

where K is a Poisson distribution with expected value $\alpha\eta^2$.