Dynamic Consumption and Portfolio Choice with Ambiguity about Stochastic Volatility

Gonçalo Faria
FEP-UP, CEF-UP, RGEA-U.Vigo

João Correia-da-Silva
FEP-UP, CEF-UP

Cláudia Ribeiro
FEP-UP, CEF-UP

6th World Congress of the Bachelier Finance Society
Toronto, Canada

June 24th, 2010

Outline

- Outline:
  - Concept: Ambiguity vs Risk
  - Motivation
  - Model
  - Analytical Solution
  - Simulation Results
  - Conclusions

1. Concept: Ambiguity vs Risk.

- Knight (1921): conceptual distinction between ambiguity and risk.
  - Risk: uncertainty that can be described by a single probability distribution. *"known unknown".*
  - Ambiguity: uncertainty than cannot be described by a single probability distribution. *"unknown unknown".*

- Ellsberg (1961): experimental evidence supporting Knightian distinction between ambiguity and risk – Ellsberg Paradox.

1. Concept: Ambiguity vs Risk.

Variation on Ellsberg (1961, QJE) 2-colour, 2-urn experiment :

Ambiguous Urn – 100 Balls	**Risky Urn – 100 Balls**
?? Red	**50** Red
?? Blue	**50** Blue

- **Question:** Placed in a choice situation, which urn does the typical agent choose to drawn a ball?

- **Answer: Strict preference for betting on the Risky urn.** Why?
  - The chance of winning (50% in this case) is "safe" and well understood.

- **Implication from** $\boldsymbol{RR \succ AR}$ **and** $\boldsymbol{RB \succ AB}$:

  - As $Pr(RR)=Pr(RB)=0.5$, then implied "subjective" probabilities are $Pr(AR) < 0.5$ and $Pr(AB) < 0.5$. Paradox!

  - Standard Additive Probability can not represent Ellsberg evidence about agent's behavior in such uncertain context.

1. Concept: Ambiguity vs Risk.

- Mainstream Theory of Choice in Economics for the last 60 years:
  - (EU) Expected Utility Theory (von Neumann and Morgenstern, 1944):
    - Probabilities of the possible states of nature are known.
  - (SEU) Subjective Expected Utility Theory (Savage, 1954):
    - Probabilities are not necessarily known, but agents still behave as if they were maximizing an expected utility function, using their subjective probability beliefs.
  - Both EU and SEU ignore ambiguity, reducing all uncertainty to risk.
- Gradually, ambiguity is being incorporated in decision theory since 90's: (i) further empirical evidence; (ii) theoretical developments (Multiple-Priors Approach and Robust Control).

2. Motivation: Research Question.

- ❑ What is the impact on the dynamic consumption and portfolio choices from the ambiguity about the stochastic precision*?
  - ❑ Is stochastic precision relevant to portfolio choice?

** Note: precision is the reciprocal of variance (volatility) of the risky asset's return.*

2.1 Motivation: Literature Review.

- Large literature on the portfolio choice problem without ambiguity considerations.
    - Few of those works explore the problem with stochastic precision.
- Few and recent literature focuses on portfolio choice with ambiguity aversion, but:
    - Ambiguity is about the expected (excess) return of the risky asset.
    - No explicit stochastic process for precision.

2.2 Motivation: Ambiguity about Expected Precision?

- This paper introduces Ambiguity aversion:
  - within a setting with an explicit process for the stochastic precision.
  - about the expected value of precision of the risky asset's return.
- Why?
  - Precision: not observed by investors - intuitive reason to assume they may feel ambiguous on it.
  - Precision's expected value: the most intuitive parameter to which investors pay attention.
  - Analytical tractability.

3 Model: Major Guidelines – Investment Opportunity Set.

- Chacko and Viceira (2005) – base model:
  - For dynamic consumption and portfolio choice.
  - Instantaneous return of the risky asset given by:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}} dW_S \tag{1}$$

  - Precision y_t follows a mean-reverting square-root process described by :

$$dy_t = \kappa\left(\theta - y_t\right) dt + \sigma\sqrt{y_t} dW_y \; , \tag{2}$$

where:

- μ - expected return of the risky asset
- $E(y_t) = \theta$;
- W_s and W_y are standard Brownian Motions; assumed $dW_y dW_S = \rho dt$, $\rho > 0$.

3 Model: Major Guidelines - Preferences

- Preferences are represented by the Stochastic Differential Utility function introduced by Duffie and Epstein (1992), with the utility process:

$$J = E_t \left[\int_t^\infty f\left(C_s, J_s\right) ds \right], \qquad (3)$$

where:

- C_s - current consumption
- J_s - continuation utility on C at time t=s
- $f\left(C_s, J_s\right)$ - normalized aggregator that generates J . It is a function ot, among others:
  - $\gamma > 0$ - coefficient of relative risk aversion
  - $\psi > 0$ - elasticity of intertemporal substitution of consumption
  - $\beta > 0$ - the rate of time preference.

3.1 Model: Our contribute.

Our contribution:

- Assume ambiguity about $E(y_t) = \theta$.

- Following Gilboa and Schmeidler (1989) Max-Min framework and applying the Saddle Point Theorem [Fan(1953), Sion(1958)] :

  - Investors have a set of priors, the interval $[\underline{\theta}, \overline{\theta}]$, with $0 < \underline{\theta} \leq \theta \leq \overline{\theta}$

  - Investors consider $\theta^* \in [\underline{\theta}, \overline{\theta}]$ such that it minimizes the maximized expected utility:

$$\theta^* = \underset{\hat{\theta} \in [\underline{\theta}, \overline{\theta}]}{argmin} \, J_{t_0}(\hat{\theta}). \qquad (4)$$

3.2 Model: Dynamic Optimization Problem.

The dynamic consumption-portfolio problem with stochastic precision faced by the investor that is both θ – ambiguity and risk averse can be written as:

$$\min_{\hat{\theta}\in[\underline{\theta},\overline{\theta}]}\left\{\max_{\pi,C} E_{t_0}\left[\int_{t_0}^{\infty} f\left(C_s, J_s\right) ds\right]\right\} \tag{5}$$

$$s.t.$$

$$dX_t = \left[\pi_t\left(\mu - r\right)X_t + rX_t - C_t\right]dt + \pi_t\sqrt{\frac{1}{y_t}}X_t dW_S \ ,$$

$$dy_t = \kappa\left(\hat{\theta} - y_t\right)dt + \sigma\sqrt{y_t}dW_y \ .$$

where:

- X_t - wealth (with $X_{t0} > 0$)
- π_t - fraction of wealth invested in the risky asset.

4 Problem Solution: Guidelines.

❑ For each $\hat{\theta}$, the maximization problem is a stochastic continuous-time optimal control problem with two state variables (X_t and y_t) and two control variables (C_t and π_t). The corresponding Bellman equation is:

$$
\begin{aligned}
0 \quad = \quad & \max_{\pi, C} \left\{ f\left(C_s, J_s\right) + J_X\left(\pi_t\left(\mu - r\right)X_t + rX_t - C_t\right) + J_y\kappa\left(\hat{\theta} - y_t\right) + \right. \\
& \left. + \frac{1}{2}J_{XX}\pi_t^2\frac{1}{y_t}X_t^2 + \frac{1}{2}J_{yy}\sigma^2 y_t + J_{Xy}\pi_t\rho\sigma X_t \right\} . \qquad (6)
\end{aligned}
$$

where $J_{(\cdot)}$ are partial derivatives.

❑ Chacko and Viceira (2005) found an exact solution when $\psi = 1$ and an approximate solution for $\psi \neq 1$. We study θ – ambiguity in both scenarios.

4.1 Problem Solution: Exact Solution.

When $\psi = 1$ the value function *J* that solves (6), for any value of $\hat{\theta}$, is given by:

$$J\left(\hat{\theta}, X_t, y_t\right) = exp\left\{Ay_t + B(\hat{\theta})\right\}\frac{X_t^{1-\gamma}}{1-\gamma}, \quad (7)$$

where *A* and *B* are constants depending on parameters describing investors preferences and the investment opportunity set. Optimal consumption and portfolio rules are given by:

$$C_t = \beta X_t, \quad (8)$$

$$\pi_t = \frac{1}{\gamma}(\mu - r)\, y_t + \frac{\sigma\rho}{\gamma} A y_t. \quad (9)$$

From (8), optimal consumption choice does not depend on y_t. From (9) and considering $E(y_t) = \theta$, the mean optimal allocation in the risky asset is given by:

$$\pi_\theta = \frac{1}{\gamma}(\mu - r)\,\theta + \frac{\sigma\rho}{\gamma} A\theta \quad (10)$$

4.1 Problem Solution: Exact Solution.

- What happens with the introduction of θ - ambiguity aversion?
  - New θ value (= θ^*) in accordance with (4):

$$\theta^* = \underset{\hat{\theta}\in[\underline{\theta},\overline{\theta}]}{argmin}\, J_{t_0}(\hat{\theta}).$$

=> **Proposition 1**

4.1 Problem Solution: Exact Solution.

Proposition 1 – Solution to the ambiguity problem.

When $\psi = 1$ *and* $\gamma \geqslant \omega$, *where* $\omega = \frac{\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2+\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)} < 1$, *the solution of the ambiguity problem is:*

$$\theta^* = \underline{\theta}\,.$$

Comments on Proposition 1:

❑ Domain of the solution of the ambiguity problem depends on the relation between:

- the level of relative risk aversion (γ)
- characterization of the investment opportunity dynamics (ω).

❑ Under that domain, $\gamma \geqslant \omega$, precision is always good.

4.1 Problem Solution: Exact Solution.

- Impact on Optimal Consumption and Portfolio rules?
  - None, as (8) and (9) do not depend on $\underline{\theta}$
- Is θ – ambiguity aversion irrelevant?
  - No, if ambiguity averse investor observes the instantaneous precision but…can not adjust instantaneously his portfolio (e.g. transaction costs, human limitations):
    - expectation of future precision, and not instantaneous precision, drives investor's portfolio decision.
    - mean allocation to the risky asset differs from (10).

=> **Proposition 2**

4.1 Problem Solution: Exact Solution.

Proposition 2 – Portfolio choice under "expectation-driven" scenario

When $\psi = 1$, $\gamma \geqslant \omega$, and the θ-ambiguity averse investor considers the expected precision of the risky asset return instead of the instantaneous precision, the demand for the risky asset is:

$$\pi_{\underline{\theta}} = \frac{1}{\gamma}\left(\mu - r\right)\underline{\theta} + \frac{\sigma\rho}{\gamma}A\underline{\theta}, \qquad (11)$$

which can be decomposed into three components:

$$\begin{aligned} \textit{myopic demand} &= \frac{1}{\gamma}\left(\mu - r\right)\theta & (12)\\ \textit{intertemporal hedging demand} &= \frac{\sigma\rho}{\gamma}A\theta & (13)\\ \textit{ambiguity demand} &= \left[\frac{1}{\gamma}\left(\mu - r\right) + \frac{\sigma\rho}{\gamma}A\right]\left(\underline{\theta} - \theta\right). & (14) \end{aligned}$$

- **Comment on Proposition 2:** New - introduction of the ambiguity demand component (14).

4.1 Problem Solution: Exact Solution.

Proposition 3 – θ - ambiguity aversion impact on the demand for the risky asset (expectation-driven scenario):

(i) θ-ambiguity aversion reduces the mean allocation to the risky asset;

(ii) Ambiguity demand (14) *is always negative;*

(iii) Intertemporal hedging demand is negative if $\gamma > 1$ and positive if $\omega \leqslant \gamma < 1$.

5 Simulation: Guidelines.

- In Chacko and Viceira (2005) it is found that the intertemporal hedging demand is empirically small:
    - Calibration with long-run US data: monthly excess stock returns on the CRSP value-weighted portfolio over the T-Bill rate (January 1926 – December 2000)
    - Conclusion: “risk dimension” of stochastic precision is not relevant for the portfolio decision.
- **Our question:** What happens, under the expectation-driven scenario, if ambiguity on stochastic precision is considered?

5 Simulation: Guidelines.

- The same calibration as in Chacko and Viceira (2005):

$$\begin{aligned} \mu - r &= 0.0811 \\ \kappa &= 0.3374 \\ \theta &= 27.9345 \\ \sigma &= 0.6503 \\ \rho &= 0.5241 \\ r &= 0.015 \\ \beta &= 0.06 \ . \end{aligned} \qquad (15)$$

- The long-run estimate of θ in (15) is assumed to be the reference value for the investor. θ-ambiguity averse investor builds the interval for θ values $[\underline{\theta}, \overline{\theta}]$ around it.

- Taking expectations of the second order Taylor expansion of $v_t = \frac{1}{y_t}$ around θ :

$$E[v_t] \approx \frac{1}{\theta} + \frac{1}{2}\frac{\sigma^2}{\theta^2 \kappa} = \frac{1}{\theta} + \frac{Var(y_t)}{\theta^3} \ . \qquad (16)$$

5.1 Simulation: Exact Solution.

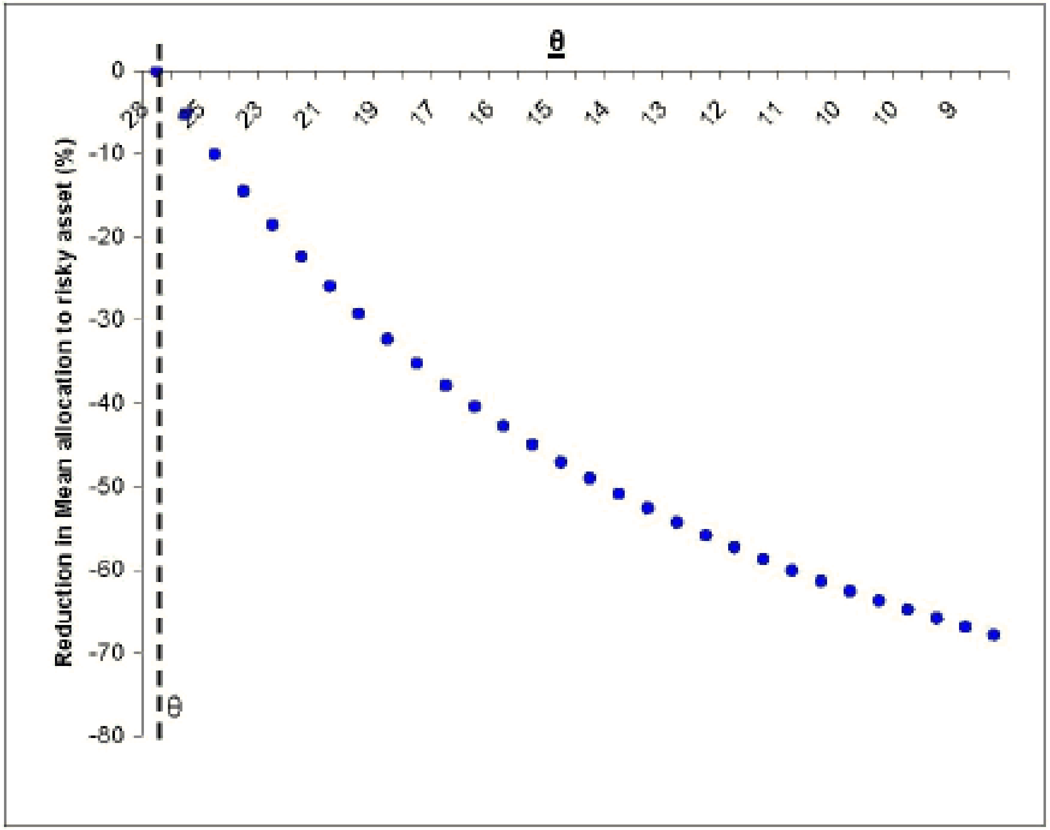
❑ Portfolio Choice

Table 1 (with ψ=1)

	Expected Annual Standard Deviation of Risky Asset Return			
	19,1314%	20%	25%	30%
implied θ	θ = 27.935	$\underline{\theta}$ = 25.612	$\underline{\theta}$ = 16.604	$\underline{\theta}$ = 11.706
implied ambiguity level	0%	8%	41%	58%
	A - Mean allocation to risky asset (%)			
R.R.A.				
0.75	305,66	280,24	181,68	128,09
2.00	111,37	102,11	66,20	46,67
4.00	55,24	50,64	32,83	23,15
20.00	10,98	10,07	6,52	4,60
	B - Ratio of hedging demand over myopic demand (%)			
R.R.A.				
0.75	1,19	1,19	1,19	1,19
2.00	-1,68	-1,68	-1,68	-1,68
4.00	-2,47	-2,47	-2,47	-2,47
20.00	-3,09	-3,09	-3,09	-3,09
	C - Ratio of Ambiguity demand over myopic demand (%)			
R.R.A.				
0.75	0,00	-8,41	-41,04	-58,79
2.00	0,00	-8,18	-39,88	-57,12
4.00	0,00	-8,11	-39,56	-56,66
20.00	0,00	-8,06	-39,31	-56,30

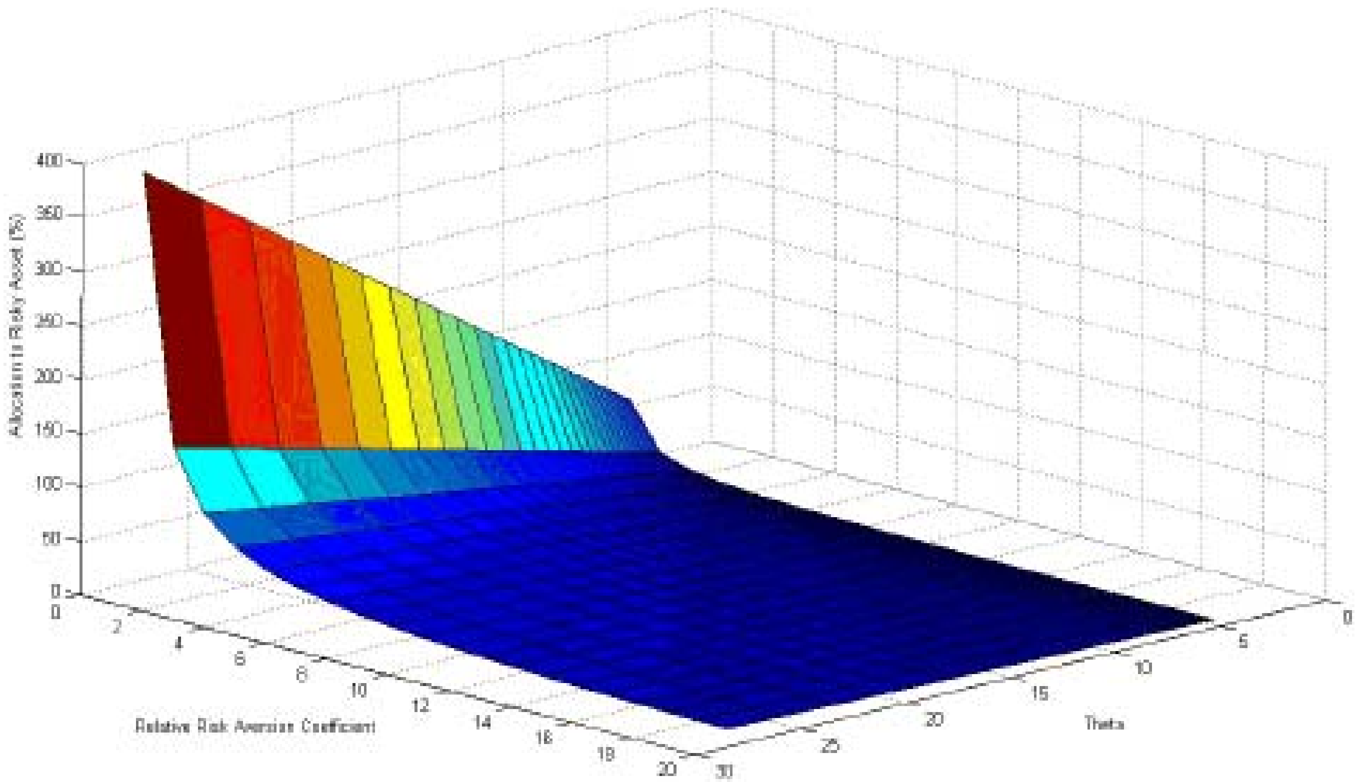
5.1 Simulation: Exact Solution.

- Comments on **Table 1**:
  - Portfolio choice strongly reacts to θ – ambiguity.



5.1 Simulation: Exact Solution.

- Comments on **Table 1** (cont.):
  - θ **-** ambiguity has the same impact (direction) of risk aversion on the portfolio choice.



6 Conclusions.

- The solution of the ambiguity problem depends on the combination between investors risk preferences and the characterization of the investment opportunity set dynamics. In our setting, precision is always good.

- θ -Ambiguity aversion is relevant if investor can not update continuously his portfolio. Expectation of future precision drives the risky asset demand.

- In this latter case, the risky asset demand is decomposed in three components: myopic and intertemporal hedging demand and ambiguity demand (novelty).

- It is found that ambiguity demand has a relevant empirical dimension, much higher than that of intertemporal hedging demand.

- Stochastic Precision of the risky return can be very relevant for the portfolio choice, essentially because of its ambiguity and not because of its risk.