Pricing of Options Exposed to Cross-Currency Rates

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Examples of FX exposed options

- Demand on non-quanto FX exposed options is growing

- Interlisted stocks with options also traded in both exchanges. For example RY.N and RY.TO traded in NYSE and TSX respectively and both have options: TSE options are issued on stocks in CAD while NYSE options have both the underlier and strike in USD. It is beneficial to price less liquid options as options on a product of more liquid stock price and FX rate – with a view towards arbitrage or hedging vega with a more liquid alternative.

		RY.N	RY.TO
Stock		910,505	9,376,234
Call	17-Jul-10	16	519
Put	17-Jul-10	25	37
Call	16-Oct-10	341	93
Put	16-Oct-10	21	250
Call	22-Jan-11	10	92
Put	22-Jan-11	1	70

- Consider a producer of a commodity who may wish to hedge their exposure to price changes. For example, Canadian producer of crude oil wants to buy a put on crude in CAD with the strike set also in CAD. Crude is mostly traded in NYMEX in USD; implied vol in CAD does not exist.

Model set up

- Equity and commodity options with a given maturity can be easily calibrated using jump-diffusion models. Adequate calibration can be achieved with normally or double exponentially distributed jumps or using variance gamma model.

$$\frac{dF_t(T)}{F_t(T)} = \sigma dW_t^F + \int_{-\infty}^{\infty} z\left(\mu(dz,dt) - \nu(dz,dt)\right)$$

- $\mu(dz, dt)$ is the Poisson random jump measure and $\nu(dz, dt)$ is its Q_d-compensator (under the domestic risk neutral measure Q_d). σ can be time dependent.

- Jump models are not widely used for rates. With this in mind, we introduce a five-parameter model inspired by the Heston (1993) stochastic variance process to model the FX rate. In departure from the Heston model, we assume that the forward process is two-factor; the first factor being a pure Brownian motion, while the second factor is another uncorrelated Brownian motion with a stochastic variance.

$$\frac{dX_t(T)}{X_t(T)} = \eta\left(dW_t^{X_1} + \gamma\sqrt{V_t}\,dW_t^{X_2}\right)$$

$$dV_t = \kappa(1 - V_t)dt + \alpha\sqrt{V_t}\,dW_t^V$$

Model set up

- The two equations for FX rate are correlated to account for the volatility skew:

$$\left[dW_t^{X_2} dW_t^{V}\right] = \rho_V dt$$

- We assume that only the first factor in the FX equation is correlated with the Brownian motion in the underlier (equity/commodity) process:

$$\left[dW_t^{X_1} dW_t^{F}\right] = \rho\, dt$$

- Correlation of forward prices and exchange rates are induced through the Brownian motions $W_t^{X_1}$ and W_t^{F}. They are not affected by terms responsible for heavy tails (jumps in the case of forward prices and stochastic volatility in the case of exchange rates). Thus, correlation can be easily extracted from historical covariance:

$$d\left[\ln F, \ln X\right]_t = \sigma\eta\rho\, dt$$

- Historical daily observations can be used. Calibrations for FX and underlier models are fully disconnected.

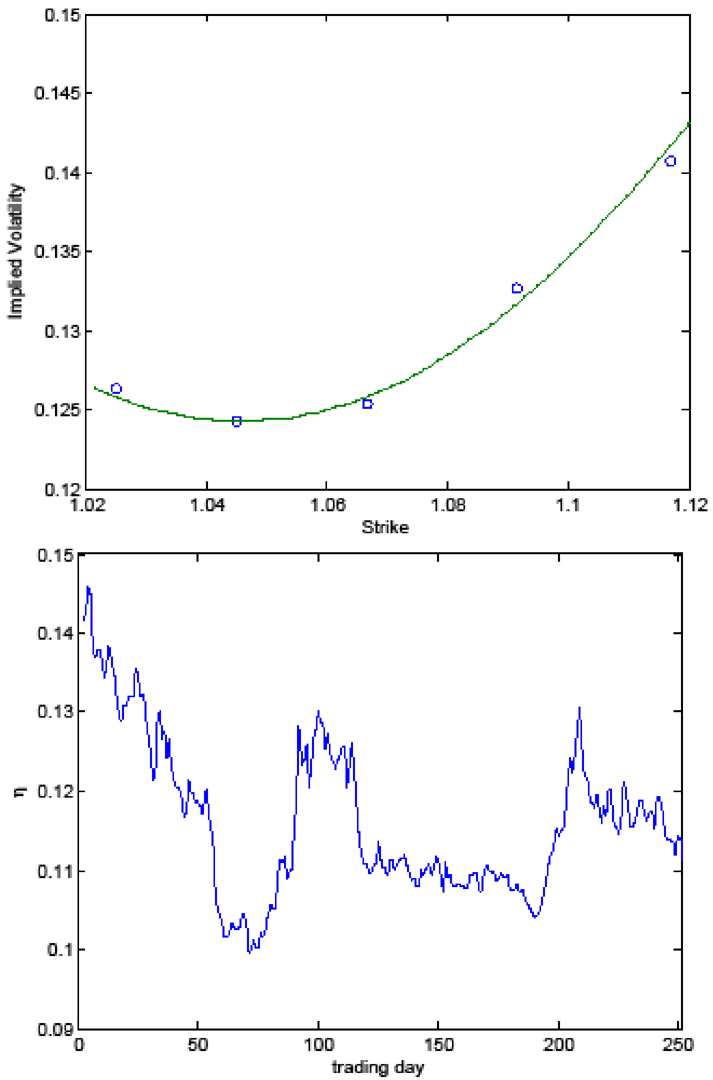
- **Applicability is limited to the case when correlations between the undelier and FX is considerably lower than 1.**

FX model calibration

- Both models have known characteristic functions and therefore European options can be priced by integrating respective characteristic exponents:

$$C_t = -Ke^{-r_d T} \int_{0-ia}^{\infty - ia} e^{\Psi(\omega) - i\omega k} \left(\frac{1}{1 - i\omega} + \frac{1}{i\omega} \right) \frac{d\omega}{\pi}$$

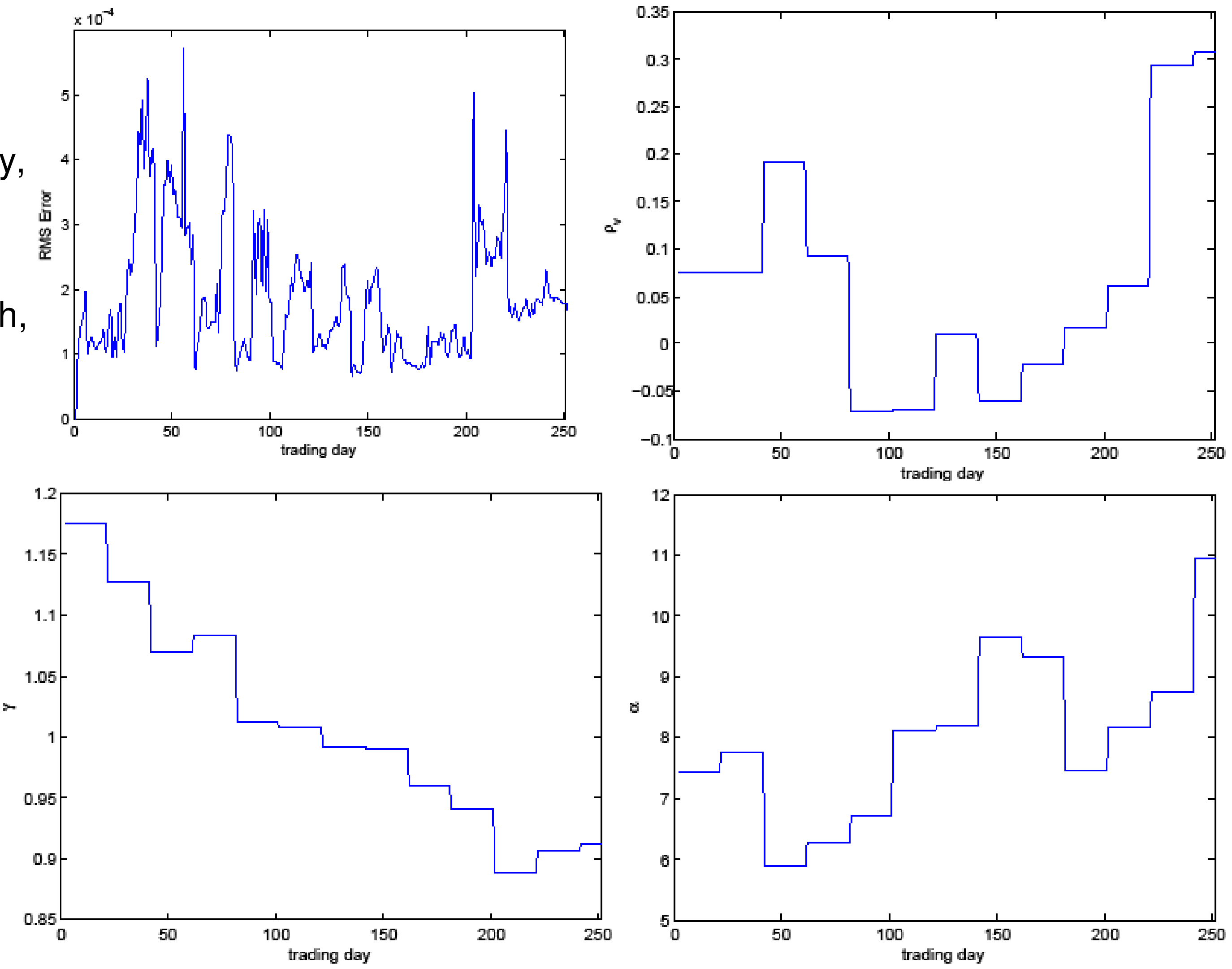
- Calibration to CAD/USD rate.
- We reduce the model to 4 parameters by keeping κ constant. Parameter κ should be fixed if we calibrate for one maturity.
- We want to fix all model parameters except for one for as long as we can and only adjust η on a daily basis.
- Other parameters are recalibrated only when model does not calculate vols within the bid/ask spread. Preferably not more often than once a month.
- One year recalibrations of fixed term (3M) options.



FX model calibration

- On the first day of calibration η is set equal to 70% of the lowest implied volatility, and γ, α and ρ_V are calibrated to minimize RMS errors. On the first day of each month, γ, α and ρ_V are recalibrated. Every trading day within the month, only η is recalibrated.

- Keeping γ constant during monthly recalibrations would r change the result. Calibrated on one maturity, the model h effectively three parameters.



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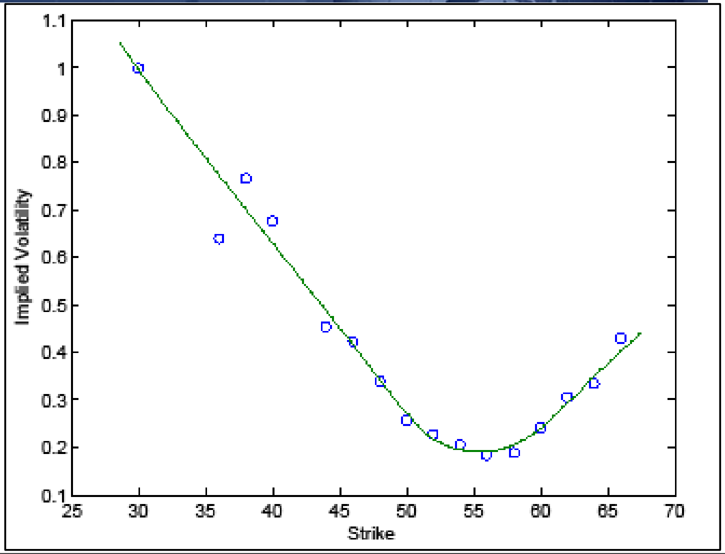
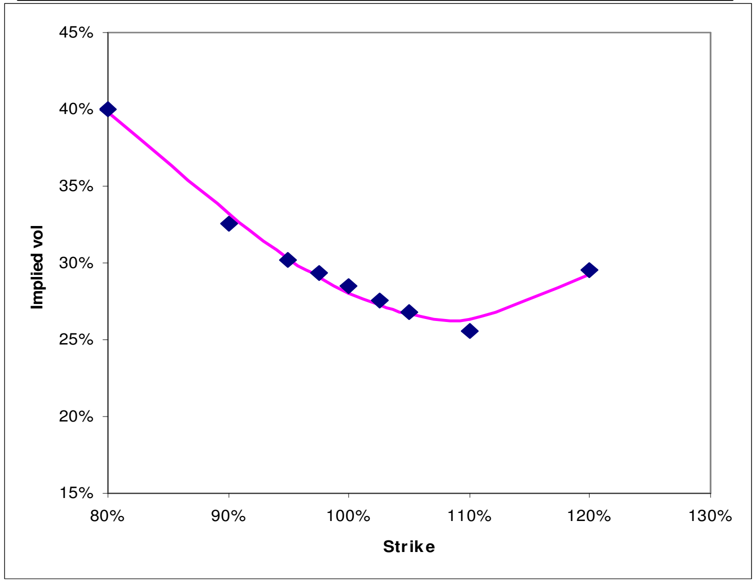
Equity calibrations

- We used Kou's double-exponential jumps to model equity forwards with jump sizes distributed as:

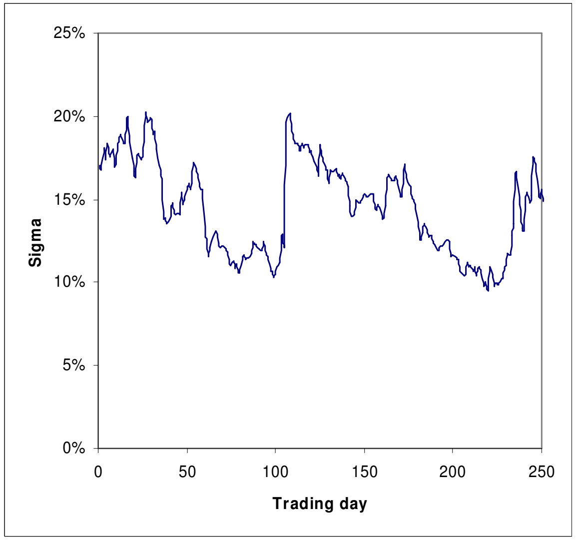
$$f_{\ln(J+1)} = p\zeta e^{-\zeta y}|_{y\geq 0} + (1-p)\xi\eta e^{\xi\eta y}|_{y<0}$$

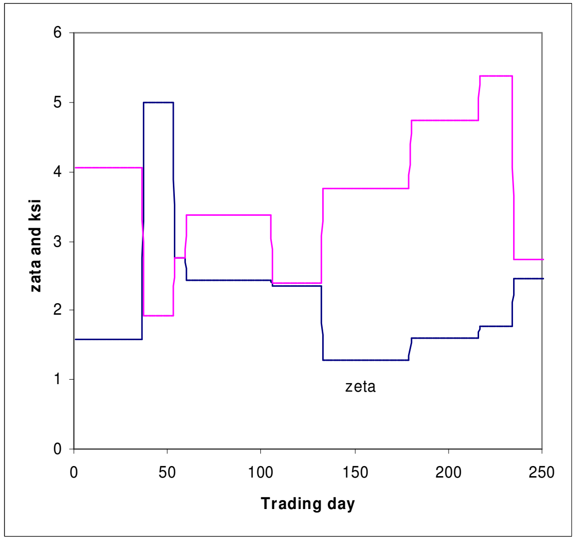
- Calibration test for equity was changed: parameter σ and jump frequency were changed daily, while other parameters were kept constant as long as vols are within bid/ask.
- The model is also over-parameterized: probability p of jumps up can be fixed. Due to a noticeable skew it was kept at 1%.
- Tests were for RY.TO 3M fixed maturity options.
- Only 8 recalibrations were required over a year.

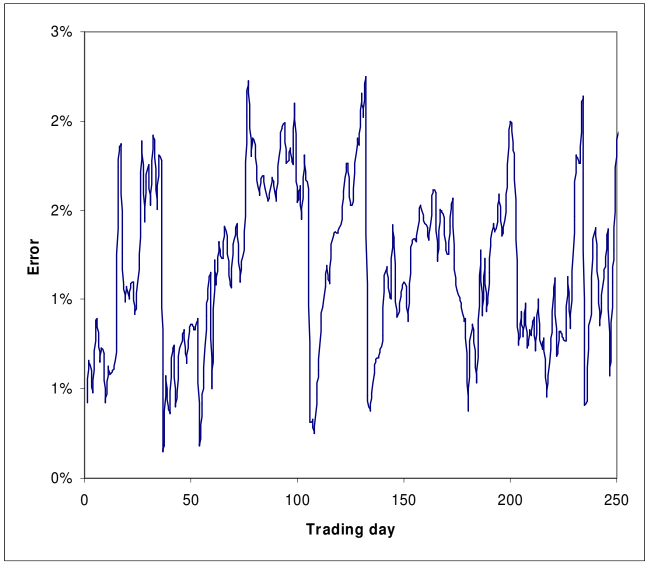


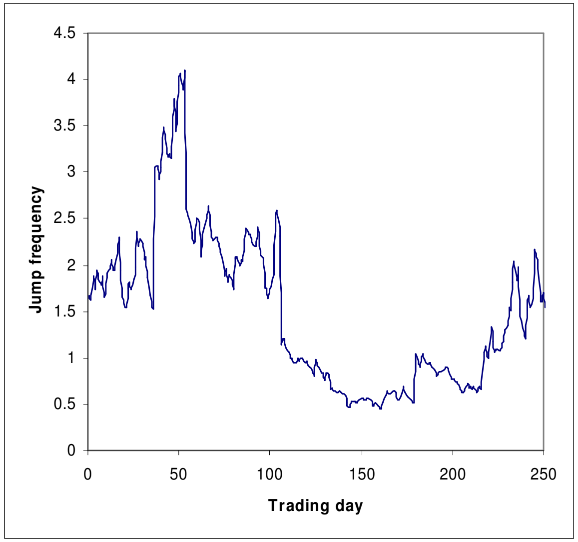


Equity calibrations









Pricing European options

- Process for $S = F / X$ is:

$$\frac{dS_t(T)}{S_{t\_}(T)} = \zeta_t dW_t^{(f,5)} - \eta\gamma\sqrt{v_t}\, dW_t^{(f,3)} + \int_{-\infty}^{\infty} z\,(\mu(dz,dt) - \nu(dz,dt))$$

$$\zeta_t^2 = (\eta^2 - 2\rho\eta\sigma_t + \sigma_t^2) \qquad dW_t^{(f,5)} = \frac{1}{\zeta_t}(\sigma_t\, dW_t^{(f,1)} - \eta\gamma\, dW_t^{(f,2)})$$

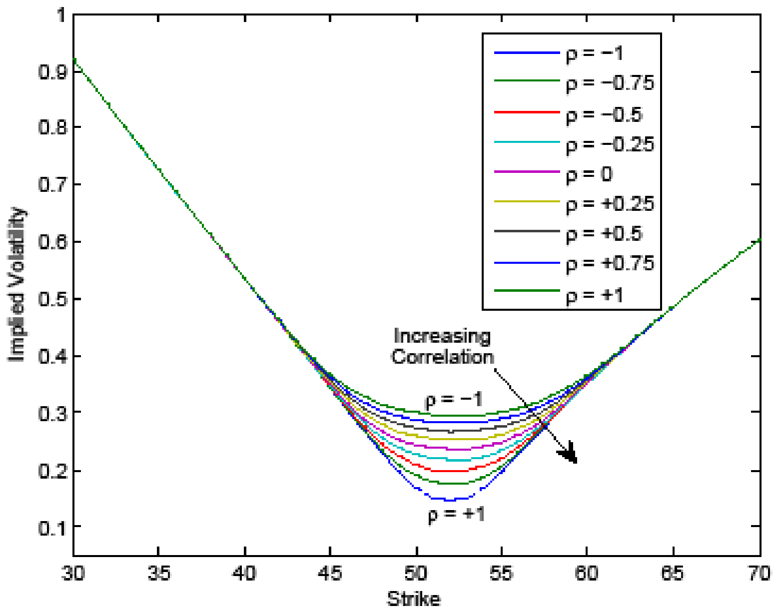
- The latter is is a standard Q_f -Brownian motion in the foreign risk neutral measure.
- The same pricing formula can be applied for European calls:

$$C_t = -Ke^{-r_d T} \int_{0-ia}^{\infty - ia} e^{\Psi(\omega) - i\omega k} \left(\frac{1}{1 - i\omega} + \frac{1}{i\omega} \right) \frac{d\omega}{\pi}$$

- But r_d is replaced by r_f. The characteristic exponent has again an explicit expression.

Pricing European options

- We use parameters previously calibrated to extract the implied Black volatility smiles of options on FX exposed stocks.
- ATM volatilities exhibit significant dependence on correlation.
- Deep out-of-the-money options are mostly unaffected by correlation. In this region, the implied volatility is mostly dominated by the jumps in the forward price process, since the stochastic volatility of the exchange rate is fairly mild in comparison.



Extension to path dependent and American cases

- Models should be calibrated on multiple maturities: processes should explain out-of-the-money volatility at each maturity.
- Five parameter models may become inadequate.
- For FX the most popular extension is the use of two-factor model.

$$\frac{dX_t(T)}{X_t(T)} = \eta\left(dW_t^{X_1} + \gamma_1\sqrt{V_{1t}}\,dW_t^{X_2} + \gamma_2\sqrt{V2_t}\,dW_t^{X_3}\right)$$

- For equities and commodities a combination of jumps and stochastic volatility will do.

$$\frac{dF_t(T)}{F_t(T)} = \sigma dW_t^{F_1} + \int_{-\infty}^{\infty} z\left(\mu(dz,dt) - \nu(dz,dt)\right) + \gamma\sqrt{V_t}\,dW_t^{F_2}$$

Conclusions

- Convenient model for options on two correlated underliers with moderate correlations. Easy to maintain observed correlations.
- Stable calibration with minimal changes of parameters over a year.
- Closed form solution for European option.
- FFT can be used for early exercise options.