# Cubature on Wiener space: Pathwise convergence

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   - Cubature on Wiener space
2. A Donsker theorem for cubature on Wiener space
   - Random walks in $G^2(\mathbb{R}^d)$
   - Donsker's theorem for random walks
   - Donsker's theorem for cubature formulas
3. Pathwise convergence
   - Pathwise weak convergence of the cubature method
   - A numerical example

# Donsker's theorem

- Given an i.i.d. sequence of $d$-dimensional random variables $\xi_1, \ldots, \xi_n$ with $E[\xi_1] = 0$ and $\mathrm{cov}[\xi_1] = I_n$.
- Grid: $\Delta t := T/n$, $\lfloor t \rfloor := \sup\{k\Delta t \,|\, k\Delta t \leq t\}$, $\lceil t \rceil := \inf\{k\Delta t \,|\, k\Delta t > t\}$.
- $W^{(n)}_t := \begin{cases} \sum_{i=1}^{k} \sqrt{\Delta t}\xi_i, & t = k\Delta t, \\ \frac{t-\lfloor t \rfloor}{\Delta t} W^{(n)}_{\lceil t \rceil} + \frac{\lceil t \rceil - t}{\Delta t} W^{(n)}_{\lfloor t \rfloor}, & \text{else.} \end{cases}$
- Then $W^{(n)}_{\cdot}$ converges weakly to the Brownian motion $B_{\cdot}$ in $C([0,T];\mathbb{R}^d)$.

# Approximation of SDEs

Consider two SDEs

- $dX_t = V_0(X_t)dt + \sum_{i=1}^{d} V_i(X_t) \circ dB_t^i$
- $dX_t^{(n)} = V_0(X_t^{(n)})dt + \sum_{i=1}^{d} V_i(X_t^{(n)}) \circ dW_t^{(n)}$

## Problem

Is it true that $X_\cdot^{(n)}$ converges weakly to $X_\cdot$ on the pathspace if $W^{(n)}$ is a cubature formula on Wiener space, i.e., is it true that for (bounded, continuous) functionals $f$ on the path-space:

$$E\left[f(X^{(n)})\right] \xrightarrow{n\to\infty} E[f(X)]?$$

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# Approximation of SDEs – 2

## Program

- Show weak convergence of the truncated signature of the cubature formula to the signature of the Brownian motion. (*Donsker's theorem*)
- Continuity of the solution map of the SDE in the driving signal in rough path sense implies weak convergence on the SDE level.
- Immediate extension to convergence of the flows (more complicated when using martingale problem approach).

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# Trees

- Assume that $\xi_i$ is discrete, taking values in $\{\omega_1, \ldots, \omega_m\}$.
- $X^{(n)}_{k\Delta t} = X^{(n)}_{k\Delta t}(\omega_{i_1}, \ldots, \omega_{i_k})$ with $i_1, \ldots, i_k \in \{1, \ldots, m\}$
- Obtain a tree for $X^{(n)}$.
- In general, the tree is non-recombining.

**Question**

Can we get the price of a path-dependent option $E[f(X_\cdot)]$ as a limit for $n \to \infty$ of approximations $E[\bar{f}(X^{(n)}_{\Delta t}, X^{(n)}_{2\Delta t}, \ldots, X^{(n)}_T)]$?

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# Idea of cubature on Wiener space

Consider a stochastic process $W^{\mathcal{D}}$ with paths of bounded variation and approximate

$$E[f(X_T)] \approx E\left[f\left(X_T^{\mathcal{D}}\right)\right],$$

where

$$X_t = X_0 + \int_0^t V_0(X_s)ds + \sum_{i=1}^{d} \int_0^t V_i(X_s) \circ dB_s^i, \tag{1}$$

$$X_t^{\mathcal{D}} = X_0 + \int_0^t V_0(X_s^{\mathcal{D}})dh(s) + \sum_{i=1}^{d} \int_0^t V_i(X_s^{\mathcal{D}})dW_s^{\mathcal{D},i}. \tag{2}$$

- $X^{\mathcal{D}}$ is the solution of a random ODE that can be solved using classical ODE schemes.
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# Signature of a path

## Definition

Let $x : [0, T] \to \mathbb{R}^d$ be a continuous path. The truncated signature $S_m(x)$ is the collection of the iterated integrals

$$S_m(x)_{0,t} = \left( \int_{0 \le t_1 \le \cdots \le t_k \le t} dx^{i_1}_{t_1} \cdots dx^{i_k}_{t_k} \;\middle|\; k \le m,\ (i_1, \ldots, i_k) \in \{1, \ldots, d\}^k \right).$$

## Remark

- *If $x$ has bounded variation, the integral is the Stieltjes integral, if $x$ is a Brownian motion, it is the Stratonovich integral.*
- *$S_m(x)$ has values in the free step-$m$ nilpot. Lie group $G_m(\mathbb{R}^d)$.*
- *Scaling: $S_m(B)_{0,t} \sim \delta_{\sqrt{t}}(S_m(B)_{0,1}) :=$ $\left( \sqrt{t}^k \int_{0 \le t_1 \le \cdots \le t_k \le 1} \circ dB^{i_1}_{t_1} \cdots \circ dB^{i_k}_{t_k} \right)$.*

# Cubature formulas on Wiener space

## Definition

A cubature formula on Wiener space of degree $m$ is a continuous process $W$ with paths of bounded variation such that $E[S_m(W)_{0,1}] = E[S_m(B)_{0,1}]$.

## Example

- Classical cubature in the sense of Lyons and Victoir: $W$ takes finitely many values $\omega_i$ with probability $\lambda_i$, $i = 1, \dots, k$.
- $W_t := tB_1$, $0 \le t \le 1$. (Order $m = 3$, *Wong-Zakai*)

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# The Ninomiya-Victoir scheme

- $Z_k^j$ independent $\mathcal{N}(0,1)$
- $X_{t_k}^{\mathcal{D}} = \exp\left(\frac{\Delta t_k}{2} V_0\right) \exp\left(\sqrt{\Delta t_k} Z_k^1 V_1\right) \cdots$
  $\cdots \exp\left(\sqrt{\Delta t_k} Z_k^d V_d\right) \exp\left(\frac{\Delta t_k}{2} V_0\right) X_{t_{k-1}}^{\mathcal{D}}$
- Reversed order with probability $1/2$
- Idea: Cubature path always moves parallel to the axes
- $\dot{W}_s^i = \begin{cases} 1/\epsilon, & s \in [0, \epsilon/2], i = 0, \\ Z^i/\epsilon, & s \in ]\epsilon/2 + (i-1)\epsilon, \epsilon/2 + i\epsilon], i > 0, \\ 1/\epsilon, & s \in ]1 - \epsilon/2, 1], i = 0, \\ 0, & \text{else.} \end{cases}$
- Cubature method of degree $m = 5$
- Reversed order with probability $1/2$, $\epsilon := 1/(1+d)$
- Individual ODEs can often be computed explicitly

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# Weak approximation with cubature on Wiener space

- Given a grid $\mathcal{D} = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ with mesh size $|\mathcal{D}|$, $\Delta t_k := t_k - t_{k-1}$.
- $W_{(1)}, \ldots, W_{(n)}$ independent copies of $W$, scaled to form cubature formulas on $[0, \Delta t_k]$, and concatenated gives a process $W_t^{\mathcal{D}}$, $0 \leq t \leq T$.
- Solve the random ODE $dX_t^{\mathcal{D}} = \sum_{i=1}^{d} V_i(X_t^{\mathcal{D}})dW_t^{\mathcal{D},i}$.

**Theorem (Lyons and Victoir, Kusuoka 2004)**

*$E[f(X_T)] = E\left[f(X_T^{\mathcal{D}})\right] + O\left(|\mathcal{D}|^{(m-1)/2}\right)$ provided that the Vector fields are smooth and one of the following conditions is satisfied:*

- *$f$ is smooth,*
- *$f$ is Lipschitz, the vector fields satisfy the uniform Hörmander condition and $\mathcal{D}$ is of a certain non-uniform form.*

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# Donsker theorem for cubature formulas

Given a cubature formula on Wiener space $W$ of degree $m$.

## Definition

Donsker's theorem in rough path topology holds for $W$ if for any sequence $\mathcal{D}_n$ of grids on $[0,T]$ with $|\mathcal{D}_n| \to 0$, any $1/3 < \alpha \leq 1/2$ and any continuous functional $f : C^{0,\alpha-\text{Höl}}\left([0,T]; G_2(\mathbb{R}^d)\right) \to \mathbb{R}$ we have:

$$E\left[f\left(S_2(W^{\mathcal{D}_n})_{0,\cdot}\right)\right] \xrightarrow[n\to\infty]{} E\left[f(S_2(B)_{0,\cdot})\right].$$

## Remark

- *Non-uniform grids required by Kusuoka's results.*
- $S_N(W^{\mathcal{D}})$ *is usually not piecewise geodesic.*
- *Continuity of the Lyons lift* $S_2(B) \mapsto S_N(B)$ *(and similarly for* $W^{\mathcal{D}}$*) implies result for* $S_N$, $N \geq 2$.

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# Random walks in $G^2(\mathbb{R}^d)$

- Grid $\mathcal{D}_n = \{0 = t_0 < \cdots < t_n = T\}$, cubature formula $W$.
- $G^2(\mathbb{R}^d) \subset \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2}$, $\xi = \exp\left(\sum_{i=1}^d X^i e_i + \sum_{i<j} A^{i,j}[e_i, e_j]\right)$ a random variable in $G^2(\mathbb{R}^d)$
- $X^i$, $A^{i,j}$ have finite moments of all orders, $E[X^i] = 0$.
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- Connection to cubature: choose $\xi = S_2(W)_{0,1}$

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**Theorem (Breuillard, Friz, Huessman)**

*Assume that the grids $\mathcal{D}_n$ are uniform. Construct stochastic processes $\Xi_t^n$ with values in $G^2(\mathbb{R}^d)$ by $\Xi_{t_k}^n = \Xi_k^n$ and by geodesic interpolation between the grid points. Then Donsker's theorem in rough path topology holds for $\Xi_\cdot^n$.*

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# Donsker's theorem for random walks

**Lemma (Pap 1993)**

*The sequence $\Xi^n_k$, $k = 0, \ldots, n$, $n \in \mathbb{N}$, satisfies the central limit theorem, i.e., $\Xi^n_n$ converges weakly to the Gaussian measure on $G^2(\mathbb{R}^d)$ with infinitesimal generator*

$$\sum_{i<j} E[A^{i,j}] \frac{\partial}{\partial x^{i,j}} + \frac{1}{2} \sum_{i \leq j} \operatorname{cov}(X^i, X^j) \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}.$$

- Tightness in Hölder norm for the geodesically interpolated random walk $\Xi$.
- Norm on $G^2(\mathbb{R}^d)$ is the Carnot-Caratheodory norm $\|\cdot\|$.
- Use Kolmogorov's criterion, i.e., need moment bound.

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# Donsker's theorem for random walks – 2

**Lemma**

*For every $p \in \mathbb{N}$ there is a constant $C$ independent of $k$ and $n$ such that* $E\left[\left\|\Xi_k^n\right\|^{4p}\right] = E\left[\left\|\xi_1^n \otimes \cdots \otimes \xi_k^n\right\|^{4p}\right] \leq C t_k^{2p}$.

**Proof.**

- $\|x\| \leq C\left(\left|\pi_1(\log(x))\right| + \sqrt{\left|\pi_2(\log(x))\right|}\right)$
- $E\left[\left|\pi_1(\log(\Xi_k^n))\right|^{4p}\right] \leq C t_k^{2p}$ by *Rosenthal's inequality* (on $\mathbb{R}^d$)
- $E\left[\left|\pi_2(\log(\Xi_k^n))\right|^{2p}\right] \leq C t_k^{2p}$ uses the fact that $\left|\pi_2(\log(\Xi_k^n))\right|^{2p}$ is a homogeneous polynomial and iterates through the random walk using the Markov property

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## Proof.

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# Donsker's theorem for random walks – 3

**Theorem**

*Define $\Xi^n_t$, $0 \le t \le T$, by geodesical interpolation of $\Xi^n_k$, $k = 0, \dots, n$. Then $\Xi^n$ converges weakly in $\alpha$-Hölder topology (for every $1/3 < \alpha \le 1/2$) to $S_2(B)$ provided that $|\mathcal{D}_n| \to 0$.*

**Remark**

*This removes the requirement of uniform grids.*

# An assumption

## Assumption

*The cubature measure is supported on finite element paths, i.e., $W$ takes values in the Cameron-Martin space $\mathcal{H}$ and the Cameron-Martin norm has finite moments of all orders, i.e., $\forall k \in \mathbb{N}$*

$$E\left[\|W\|_{\mathcal{H}}^k\right] = E\left[\left(\int_0^1 \left|\dot{W}(s)\right|^2 ds\right)^{k/2}\right] < \infty.$$

## Remark

*This assumption is (up to reparametrization) satisfied for all classical cubature formulas in the sense of Lyons and Victoir and holds for all reasonable variations, like the Ninomiya-Victoir scheme.*

# Donsker's theorem for cubature formulas

## Theorem

*The signature $S_2(W^{\mathcal{D}_n})_{0,\cdot}$ converges weakly in $\alpha$-Hölder topology to $S_2(B)_{0,\cdot}$ for any $1/3 < \alpha \leq 1/2$ provided that $|\mathcal{D}_n| \to 0$ and $W$ satisfies the Assumption.*

## Corollary

*Let ${}^hW$ denote an $\mathbb{R}^{d+1}$-valued process given by ${}^hW_t = (h(t), W_t)$, where $h : [0,1] \to \mathbb{R}$ denotes a deterministic Lipschitz function with $h(0) = 0$ and $h(1) = 1$. Again, let ${}^hW^{\mathcal{D}_n}$ (a process defined on $[0,T]$ with values in $\mathbb{R}^{d+1}$) be defined by proper re-scaling and concatenating independent copies of ${}^hW$. Then $S_2\left({}^hW^{\mathcal{D}_n}\right)_{0,\cdot} \xrightarrow[n\to\infty]{} S_2\left(\widetilde{B}\right)_{0,\cdot}$ weakly in rough path topology, where $\widetilde{B}_t := (t, B_t)$.*

# Pathwise weak convergence of the cubature method

## Theorem

*Given grids $\mathcal{D}_n$ and a cubature formula ${}^hW^{\mathcal{D}_n}$ as above, define $h^{\mathcal{D}_n}(t) := {}^hW^{\mathcal{D}_n,0}$ and let $X^{\mathcal{D}_n}$ denote the solution to the random ODE*

$$dX_t^{\mathcal{D}_n} = V_0\left(X_t^{\mathcal{D}_n}\right) dh^{\mathcal{D}_n}(t) + \sum_{i=1}^{d} V_i\left(X_t^{\mathcal{D}_n}\right) dW_t^{\mathcal{D}_n,i}.$$

*Moreover, let $f : C^{0,\alpha-\text{Höl}}([0,T];\mathbb{R}^d) \to \mathbb{R}$ be bounded and continuous. Then*

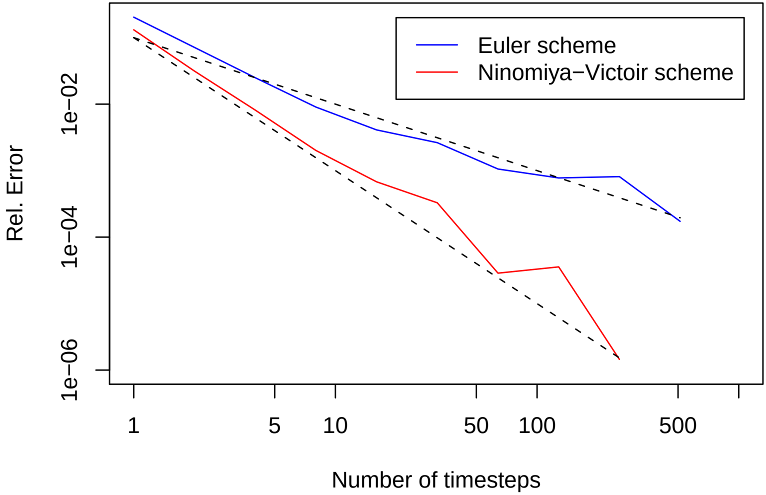
$$E\left[f\left(X_\cdot^{\mathcal{D}_n}\right)\right] \xrightarrow[n\to\infty]{} E[f(X_\cdot)].$$

# Pathwise weak convergence of the cubature method – 2

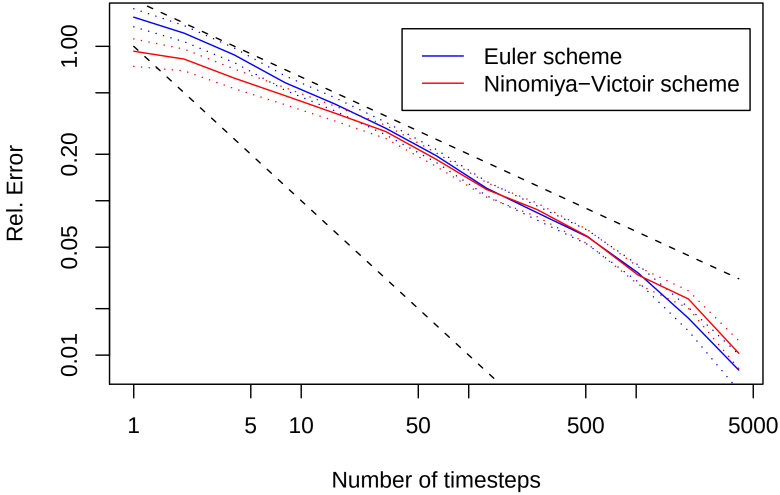
## Remark

- *In particular, the theorem holds for all bounded functionals, which are continuous with respect to the uniform topology.*
- *Convergence for unbounded continuous functionals can be obtained by uniform integrability properties (or put-call parities).*
- *In the case of barrier option, convergence holds if the payoff function is continuous except for a set with measure zero on path space.*

# An Asian option in the Heston model



# A barrier option in the Heston model



# Remarks

- For the Asian option, we recover the rate. This is not surprising, since we integrate the path by the *trapezoidal rule*, instead of using a local third order approximation of the corresponding ODE.
- For the barrier option we fall back to the convergence of the Euler method.
- Note that we do not use the full path $\left(X_t^{\mathcal{D}}\right)_{0\leq t\leq T}$ but only the points $\left(X_t^{\mathcal{D}}\right)_{t\in\mathcal{D}}$.

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