Optimal execution in limit order books with stochastic liquidity

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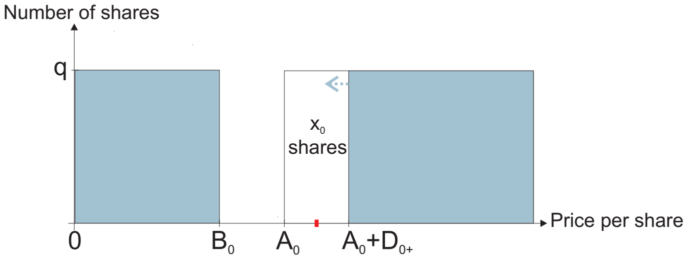
Outline

- Problem: Minimize impact on execution prices (as in Predoiu, Shaikhet, Shreve)
- Limit order book model with *stochastic liquidity*
- Structure of optimal strategies
- Examples and numerical implementation

Block order book model

- Market buy order of x_0 shares at $t = 0$ has linear price impact



- Ask price A_t martingale and bid $B_t < A_t$
  $\rightsquigarrow$ effect of A can be neglected for risk neutral investor
- Dynamic of price displacement D with resilience speed $\rho > 0$

$$dD_t = \frac{1}{q_t} d\Theta_t - \rho D_t dt$$

- Impact cost at t: $\left(D_t + \frac{1}{2q_t} x_t\right) x_t$

Model with stochastic liquidity

- Dynamic order book height: $K_t := \frac{1}{q_t}$ e.g. positive diffusion
- Risk-neutral investor wants to purchase x shares on $[t, T]$

Singular control problem in continuous time

$$U(t, \delta, x, \kappa) := \inf_{\Theta \in \mathcal{A}(x)} J(t, \delta, \Theta, \kappa)$$

Admissible strategies $\mathcal{A}(x)$

$$\Theta : \Omega \times [t, T] \to [0, x] \text{ adapted,increasing,càglàd, } \Theta_t = 0, \Theta_{T+} = x \text{ a.s.}$$

Trading costs $(\Delta\Theta_s := \Theta_{s+} - \Theta_s)$

$$J(\Theta) := J(t, \delta, \Theta, \kappa) := \mathbb{E}\Big[\int_{[t,T]} \Big(D_s + \frac{K_s}{2} \Delta\Theta_s \Big) \, d\Theta_s \Big| D_t = \delta, K_t = \kappa \Big]$$

Intuition: Wait and Buy region

- Scaling property of value function reduces dimension:

$$
\begin{aligned}
U(t, a\delta, ax, \kappa) &= a^2 U(t, \delta, x, \kappa) \text{ for } a \in \mathbb{R}_{\geq 0} \\
\overset{a=\frac{1}{\delta}}{\Rightarrow} U(t, \delta, x, \kappa) &= \delta^2 U(t, 1, \frac{x}{\delta}, \kappa)
\end{aligned}
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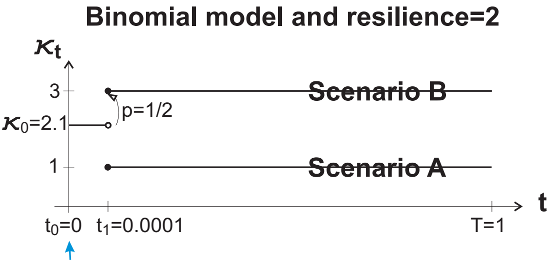
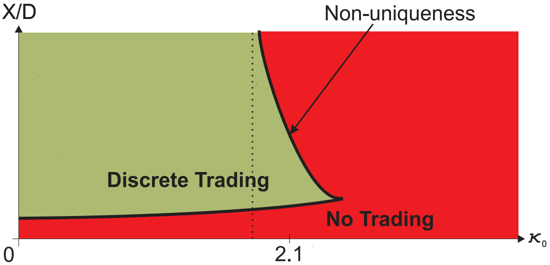
- How could optimal strategy look like for fixed t and κ?
  - Wait if $\frac{x}{\delta}$ is small, say $\frac{x}{\delta} \leq c \in (0, \infty]$
  - Otherwise buy $\xi > 0$ shares s.t. $\frac{x-\xi}{\delta+\frac{\xi}{q}} \overset{!}{=} c$

WR-BR-WR example





Unique optimal strategies

Theorem (F./Schöneborn/Urusov)

$$dK_s = \mu(s, K_s)ds + \sigma(s, K_s)dW_s$$

Let K be a positive, continuous diffusion satisfying

i) $\eta_s := \frac{2\rho}{K_s} + \frac{\mu(s,K_s)}{K_s^2} - \frac{\sigma^2(s,K_s)}{K_s^3} > 0$ for all $s \in [t, T]$

ii) $\mathbb{E}\left[\frac{\sup_{s\in[t,T]} K_s^2}{\inf_{s\in[t,T]} K_s}\right] < \infty$

iii) $\mathbb{E}\left[\left(\int_t^T |\eta_s| ds\right)\left(\sup_{s\in[t,T]} K_s^2\right)\right] < \infty$

Then $J(\Theta)$ is strictly convex and there exists a unique optimal strategy Θ^*.

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Idea:

- Strict convexity: rewrite J in terms of D via $dD_s = K_s d\Theta_s - \rho D_s ds$
  $J(\Theta) \approx \mathbb{E}\left[\int_{[t,T]} \eta_s D_s^2 ds\right] \rightsquigarrow$ Assumption i)
- Existence: Komlos argument

Uniqueness ensures WR-BR structure

Theorem (F./Schöneborn/Urusov)

Under the above assumptions there exists a unique barrier function $c : [0, T] \times (0, \infty) \to (0, \infty]$ with $c(T, \kappa) \equiv 0$ such that

$$\Delta\Theta_t^*(t, \delta, x, \kappa) = \max\left\{0, \frac{x - c(t,\kappa)\delta}{1 + \kappa c(t,\kappa)}\right\}. \qquad (1)$$

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Idea:

- Trade splitting argument
- Exclude upper WR by uniqueness

Example 1/3: K deterministic

- K càglàd, bounded ensures WR-BR structure
- Obizhaeva/Wang ($dK_t = 0$) gives $c(t,\kappa) = \frac{\rho(T-t)+1}{\kappa}$
- Explicit barrier via Euler-Lagrange formalism, e.g., $K_t = K_0 e^{\nu\rho t}$ gives

$$c(t,\kappa) = \left\{ \begin{array}{cc} \infty & \text{if } \nu < -1 \\ \frac{1+\nu-e^{-\rho\nu(T-t)}}{\nu(1+\nu)\kappa} & \text{otherwise} \end{array} \right\}$$

Example 2/3: K GBM

$$dK_t = K_t(\mu_t dt + \sigma_t dW_t)$$

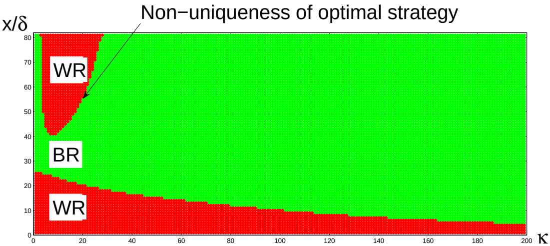
- WR-BR-WR examples exist for time-inhomogeneous GBM
- If WR-BR structure holds: $c(t,\kappa) = \frac{c(t)}{\kappa}$ via scaling property, 'bad model' due to passive in the liquidity behavior

Example 3/3: K CIR- numerical scheme

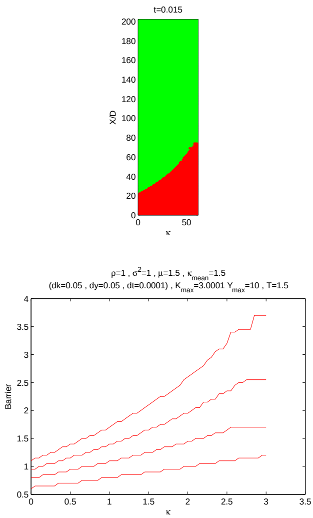
$$dK_s = \overline{\mu}(\overline{K} - K_s)ds + \overline{\sigma}\sqrt{K_s}\, dW_s$$

1. **Possible idea:**
   Implement HJB equation (QVI) by finite difference scheme
   $$\min\left\{\kappa U_D - U_X + D, U_t - \rho D U_D + \overline{\mu}(\overline{\kappa} - \kappa)U_\kappa + \frac{\overline{\sigma}^2}{2}\kappa U_{\kappa\kappa}\right\} = 0$$
2. **Here:**
   Approximate state space diffusion by a Markov chain à la Kushner
   - Code is essentially the same as in 1.
   - Convergence proof by probabilistic methods, i.e. no use of HJB eq./verification argument or convexity/smoothness/growth conditions

Example 3/3: K CIR- WR-BR-WR example (for large vola)

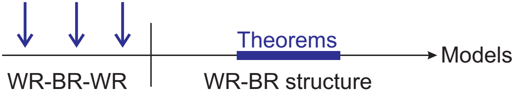


Example 3/3: K CIR- aggressive in the liquidity behavior (for high mean-reversion)



Summary

- Market microstructure model of order book to study optimal execution problem
- Stochastic liquidity $\rightsquigarrow$ differential order placement
- Wait/Buy Region structure does not always hold!



- Numerical analysis via Markov chain implementation: Aggressive/passive in the liquidity behavior

Related Literature

[1] Obizhaeva, Wang: *Optimal trading strategy and supply/demand dynamics.* Forthcoming in Journal of Financial Markets (2005)

[2] Alfonsi, F., Schied: *Optimal execution strategies in limit order books with general shape functions.* Quantitative Finance (2009)

[3] Predoiu, Shaikhet, Shreve: *Optimal execution in a general one-sided limit order book.* Preprint (2010)

[4] Budhiraja, Ross: *Convergent numerical scheme for singular stochastic control with state constraints in a portfolio selection problem.* SIAM Journal of Control and Optimization (2007)

Thank you for your attention!