Comparative Analysis of VaR and Some Distortion Risk Measures

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June 24, 2010

6th BFS World Congress, Toronto

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1. Distortion Risk Measures (DRM)

For a rv X representing loss, put

- df of X: $F_X(x) := \mathrm{P}(X \leq x)$
- quantile of X: $F_X^{-1}(u) := \inf\{x \in \mathbb{R} \colon F_X(x) \geq u\}, \quad 0 < u < 1$

<u>Def</u>: A functional $\rho \colon L^\infty \to \mathbb{R}$ is called *coherent* if it satisfies

[**PO**] (positivity): $\quad X \leq 0$ a.s. $\implies \rho(X) \leq 0$

[**PH**] (positive homogeneity): $\quad \forall \lambda > 0, \; \rho(\lambda X) = \lambda \rho(X)$

[**TE**] (translation equivariance): $\quad \forall c > 0, \; \rho(X + c) = \rho(X) + c$

[**SA**] (subadditivity): $\quad \rho(X + Y) \leq \rho(X) + \rho(Y)$

Add two more axioms:

[**LI**] (law invariance): $X \stackrel{\mathcal{L}}{=} Y \implies \rho(X) = \rho(Y)$

[**CA**] (comonotonic additivity):

$$X \text{ and } Y \text{ are comonotone} \implies \rho(X+Y) = \rho(X) + \rho(Y)$$

$X_1, \ldots, X_d$ are *comonotone* $\Leftrightarrow$ There exist a rv Z and increasing func's $f_1, \ldots, f_d$ s.t. $(X_1, \ldots, X_d) \stackrel{\mathcal{L}}{=} (f_1(Z), \ldots, f_d(Z))$

►► Kusuoka: The class of DRMs coincides with the set of coherent risk measures satisfying law invariance and comonotonic additivity

Distortion function

Any distribution function (df) D on $[0,1]$;

i.e., right-continuous, increasing on $[0,1]$, $D(0)=0$, $D(1)=1$

For a distortion D, a *distortion risk measure (DRM)* is defined by

$$\rho_D(X) := \int_{[0,1]} F_X^{-1}(u)\,\mathrm{d}D(u) = \int_{\mathbb{R}} x\,\mathrm{d}D \circ F_X(x).$$

[a.k.a. spectral risk measure (Acerbi), weighted V@R (Cherny)]

★ $D_\alpha^{\mathsf{VaR}}(u) = \mathbf{1}_{\{u \geq 1-\alpha\}}$ yields $\mathsf{VaR}_\alpha(X) = F_X^{-1}(1-\alpha)$, $0<\alpha<1$, but this D_α^{VaR} is not convex.

Example: *Expected Shortfall (ES)*

The expected loss that is incurred when VaR is exceeded

$$\mathsf{ES}_\alpha(X) := \frac{1}{\alpha}\int_{1-\alpha}^{1} F_X^{-1}(u)\,\mathrm{d}u$$

$$\doteq \mathrm{E}(X \mid X \geq \mathsf{VaR}_\alpha(X))$$

Taking distortion of the form

$$D_\alpha^{\mathsf{ES}}(u) = \frac{1}{\alpha}\big[u - (1-\alpha)\big]_+, \quad 0 < \alpha < 1$$

yields ES as a DRM

Other Examples:

- *Proportional Hazards*: $D_\theta^{\mathsf{PH}}(u) = 1 - (1-u)^\theta$

- *Proportional Odds*: $D_\theta^{\mathsf{PO}}(u) = \dfrac{\theta u}{1-(1-\theta)u}$

- *Gaussian* (Wang transform): $D_\theta^{\mathsf{GA}}(u) = \Phi(\Phi^{-1}(u) + \log\theta)$

- *Proportional* γ*-Odds*: $D_\theta^{\mathsf{PGO}}(u) = 1 - \left[\dfrac{(1-u)^\gamma}{\theta - \theta(1-u)^\gamma + (1-u)^\gamma}\right]^{1/\gamma}$

- *Positive Poisson Mixture*: $D_\lambda^{\mathsf{PPM}}(u) = \dfrac{\mathrm{e}^{\lambda u} - 1}{\mathrm{e}^{\lambda} - 1}$

2. Statistical Estimation

$(X_n)_{n\in\mathbb{N}}$: strictly stationary process with $X_n \sim F$

$\mathbb{F}_n$: empirical df based on the sample $X_1, \ldots, X_n$

A natural estimator of $\rho(X)$ is

$$\begin{aligned}\widehat{\rho}_n(X) &= \int_0^1 \mathbb{F}_n^{-1}(u)\, dD(u) \\ &= \sum_{i=1}^{n} c_{ni} X_{n:i}, \qquad c_{ni} := D\left(\frac{i-1}{n}, \frac{i}{n}\right]\end{aligned}$$

Strong consistency

Let $d(u) = \frac{\mathrm{d}}{\mathrm{d}u} D(u)$ for a convex distortion D, and $1 \leq p \leq \infty$, $1/p + 1/q = 1$. Suppose that $(X_n)_{n \in \mathbb{N}}$ is an ergodic stationary sequence, and that $d \in L^p(0,1)$ and $F^{-1} \in L^q(0,1)$. Then

$$\widehat{\rho}_n(X) \longrightarrow \rho(X), \quad \text{a.s.}$$

For a proof, see van Zwet (1980, AP)

[All we need is SLLN and Glivenko-Cantelli Theorem].

Assumptions:

- $(X_n)_{n\in\mathbb{N}}$ is strongly mixing with rate

$$\alpha(n) = O(n^{-\theta-\eta}) \quad \text{for some } \theta \geq 1+\sqrt{2},\ \eta > 0$$

- For F^{-1}-almost all u, d is continuous at u

- $|d| \leq B$, $\quad B(u) := Mu^{-b_1}(1-u)^{-b_2}$,

- $|F^{-1}| \leq H, \quad H(u) := Mu^{-d_1}(1-u)^{-d_2}$

Assume b_i, d_i & θ satisfy $b_i + d_i + \frac{2b_i+1}{2\theta} < \frac{1}{2}, \quad i = 1, 2$

Set

$$\sigma(u,v) := [u \wedge v - uv] + \sum_{j=1}^{\infty}[C_j(u,v) - uv] + \sum_{j=1}^{\infty}[C_j(v,u) - uv],$$

$$C_j(u,v) := \mathrm{P}(X_1 \le F^{-1}(u), X_{j+1} \le F^{-1}(v))$$

Theorem (Asymptotic Normality)

Under the above assumptions, we have

$$\sqrt{n}(\widehat{\rho}_n(X) - \rho(X)) \xrightarrow{\mathcal{L}} N(0, \sigma^2),$$

where

$$\sigma^2 := \int_0^1 \int_0^1 \sigma(u,v) d(u) d(v)\, \mathrm{d}F^{-1}(u) \mathrm{d}F^{-1}(v) < \infty$$

- GARCH model:

$X_n = \sigma_n Z_n, \quad (Z_n):$ i.i.d.

$\sigma_n^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$

▶▶ If the stationary distribution has a positive density around 0, then GARCH is strongly mixing with exponentially decaying $\alpha(n)$

- Stochastic Volatility model:

$X_n = \sigma_n Z_n, \quad (Z_n):$ i.i.d., $\quad (\sigma_n):$ strictly stationary positive

(Z_n) and (σ_n) are assumed to be independent

▶▶ The mixing rate of (X_n) is the same as that of $(\log \sigma_n)$

Simulation example: inverse-gamma SV model

$$X_t = \sigma_t Z_t$$

Z_t i.i.d. N(0,1) and $V_t = 1/\sigma_t^2$ satisfies

$$V_t = \rho V_{t-1} + \varepsilon_t,$$

where $V_t \sim$ Gamma(a, b) for each t, (ε_t) i.i.d. rv's, and $0 \leq \rho < 1$

$\Rightarrow X_t$ has scaled t-distribution with $\nu = 2a$, $\sigma^2 = b/a$

►► Lawrance (1982): the distribution of ε_t is compound Poisson

►► Can be shown that (X_t) is geometrically ergodic

Simulation results for estimating VaR, ES & PO risk measures with inverse-gamma SV observations ($n = 500$, # of replication $= 1000$)

$X_t = \sigma_t Z_t$, where $V_t = 1/\sigma_t^2$ follows AR(1) with gamma(2,16000) marginal & $\rho = 0.5$, Z_t i.i.d. N(0,1)

		VaR		ES		PO	
	$\theta = \alpha$	bias	RMSE	bias	RMSE	bias	RMSE
SV	0.1	0.0692	10.9303	−2.2629	22.1361	−1.7739	17.5522
	0.05	2.5666	17.6755	−1.2168	37.2719	−2.0200	28.5053
	0.01	14.9577	61.2290	−11.9600	103.9269	−15.7888	73.7147
i.i.d.	0.1	0.7976	10.5893	−1.2914	19.5756	−1.3574	15.3271
	0.05	0.7974	16.1815	−2.6346	31.3166	−2.8342	23.9933
	0.01	10.6838	53.2567	−12.9355	95.9070	−15.8086	69.5425

Simulation results for estimating VaR, ES & PO risk measures with GARCH observations ($n = 500$, # of replication $= 1000$)

$$X_t = 0.0009 + \varepsilon_t, \quad \sigma_t^2 = 0.5 + 0.85\sigma_{t-1}^2 + 0.1\varepsilon_{t-1}^2$$

	VaR		ES		PO		PH	
$\theta = \alpha$	mean	std	mean	std	mean	std	mean	std
0.5	0.0077	0.1679	2.4590	0.2687	1.2134	0.1854	2.2206	0.3119
0.05	5.1429	0.5488	6.6250	0.8048	5.0339	0.5959	8.9421	1.8604
0.01	7.7766	1.1182	8.8885	1.4658	7.3829	1.0806	10.2292	2.2618

- Estimation of Asymptotic Variance

$$\sigma^2 = \iint \sigma(F(x), F(y)) d(F(x)) d(F(y)) \,\mathrm{d}x\mathrm{d}y$$

where

$$\sigma(F(x), F(y)) = [F(x) \wedge F(y) - F(x)F(y)] + \sum_{j=1}^{\infty} [F_j(u, v) - F(x)F(y)] + \sum_{j=1}^{\infty} [F_j(y, x) - F(x)F(y)],$$

and

$$F_j(x, y) = \mathrm{P}(X_1 \le x, X_{j+1} \le y)$$

►► How to estimate this? (to construct confidence intervals)

3. Capital Allocation

d investment opportunities (e.g., business units, subportfolios, assets)
X_i: loss associated with the ith investments

1. Compute the overall risk capital $\rho(X)$, where $X = \sum_{i=1}^{d} X_i$ and ρ is a particular risk measure.

2. Allocate the capital $\rho(X)$ to the individual investment possibilities according to some mathematical *capital allocation principle* such that, if κ_i denotes the capital allocated to the investment opportunity with potential loss X_i, we have $\sum_{i=1}^{d} \kappa_i = \rho(X)$.

►► Find $\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{R}^d$ s.t. $\sum_{i=1}^{d} \kappa_i = \rho(X)$ according to some criterion

Setup

It is convenient to introduce 'weights' $\lambda = (\lambda_1, \ldots, \lambda_d)$
(to be interpreted as amount of money invested in each opportunity)

Put $X(\lambda) := \sum_{i=1}^{d} \lambda_i X_i$ and

$$r_\rho(\lambda) := \rho(X(\lambda)) \quad \text{risk measure function}$$

If ρ is positive homogeneous, then, for $h > 0$

$$r_\rho(h\lambda) = h r_\rho(\lambda)$$

i.e., r_ρ is positive homogeneous of degree 1

Euler's rule: If r_ρ is positive homogeneous and differentiable,

$$r_\rho(\lambda) = \sum_{i=1}^{d} \lambda_i \frac{\partial r_\rho}{\partial \lambda_i}(\lambda)$$

Euler allocation principle

If r_ρ is a positive homogeneous risk measure function, which is differentiable on the set Λ, then the (per-unit) Euler capital allocation principle associated with r_ρ is

$$\kappa_i(\lambda) = \frac{\partial r_\rho}{\partial \lambda_i}(\lambda)$$

Justification

- Tasche: RORAC compatibility

  r_ρ: differentiable risk measure function

  κ: capital allocation principle

  κ is called *suitable for performance measurement* if for all λ we have

$$\frac{\partial}{\partial \lambda_i}\left(\frac{-\mathrm{E}(X(\lambda))}{r_\rho(\lambda)}\right)\begin{cases} > 0 & \text{if } \frac{-\mathrm{E}(X_i)}{\kappa_i(\lambda)} > \frac{-\mathrm{E}(X(\lambda))}{r_\rho(\lambda)}, \\ < 0 & \text{if } \frac{-\mathrm{E}(X_i)}{\kappa_i(\lambda)} < \frac{-\mathrm{E}(X(\lambda))}{r_\rho(\lambda)}. \end{cases}$$

  ►► The only per-unit capital allocation principle suitable for performance measurement is the Euler principle.

- Denault: Coorperative game theory

  d investment opportunities $=$ d players
  If ρ is subadditive, then $\rho(X(\lambda)) \leq \sum_{i=1}^{d} \rho(\lambda_i X_i)$.

  A fuzzy core (Aubin, 1981) is given by

$$\mathscr{C} = \left\{ \kappa \in \mathbb{R}^d \colon r_\rho(\mathbf{1}) = \sum_{i=1}^{d} \kappa_i \ \& \ r_\rho(\lambda) \geq \sum_{i=1}^{d} \lambda_i \kappa_i \ \forall \lambda \in [0,1]^d \right\}$$

  ►► If r_ρ is differentiable at $\lambda = \mathbf{1}$, then $\mathscr{C}$ consists only of the gradient vector of r_ρ at $\lambda = \mathbf{1}$:

$$\kappa_i = \frac{\partial r_\rho(\lambda)}{\partial \lambda_i} \Bigg|_{\lambda = \mathbf{1}}$$

Examples

- Covariance principle:

$$r_\rho(\lambda) = \sqrt{\text{var}(X(\lambda))} = \sqrt{\lambda' \Sigma \lambda}$$

where Σ is the covariance matrix of $(X_1, \dots, X_d)$. Then

$$\kappa_i(\lambda) = \frac{\partial r_\rho(\lambda)}{\partial \lambda_i} = \frac{\text{cov}(X_i, X(\lambda))}{\sqrt{\text{var}(X(\lambda))}}$$

In particular, the capital allocated to the ith investment opportunity is

$$\kappa_i = \frac{\text{cov}(X_i, X)}{\sqrt{\text{var}(X)}}$$

- VaR contributions:

$$r_\rho(\lambda) = \mathsf{VaR}_\alpha(X(\lambda))$$

Then (Tasche, 1999)

$$\kappa_i(\lambda) = \frac{\partial r_\rho(\lambda)}{\partial \lambda_i} = \mathrm{E}(X_i \,|\, X(\lambda) = \mathsf{VaR}_\alpha(X(\lambda)))$$

In particular, the capital allocated to the ith investment opportunity is given by

$$\kappa_i = \mathrm{E}(X_i \,|\, X = \mathsf{VaR}_\alpha(X))$$

(It is hard to compute, though)

- ES contributions:

$$r_\rho(\lambda) = \mathsf{ES}_\alpha(X(\lambda)) = \frac{1}{\alpha}\int_{1-\alpha}^{1} F^{-1}_{X(\lambda)}(u)\,du$$

Then

$$\kappa_i(\lambda) = \frac{\partial r_\rho(\lambda)}{\partial \lambda_i} = \mathrm{E}(X_i \mid X(\lambda) \geq \mathsf{VaR}_\alpha(X(\lambda)))$$

In particular, the capital allocated to the ith investment opportunity is given by

$$\kappa_i = \mathrm{E}(X_i \mid X \geq \mathsf{VaR}_\alpha(X))$$

Capital Allocation with DRM

$$r_\rho(\lambda) = \rho_D(X(\lambda)) = \int_{[0,1]} F_{X(\lambda)}{}^{-1}(u)\,\mathrm{d}D(u)$$

Then, under some regularity conditions (Tsanakas),

$$\begin{aligned}\kappa_i(\lambda) &= \frac{\partial r_\rho(\lambda)}{\partial \lambda_i} = \int_{[0,1]} \frac{\partial}{\partial \lambda_i} F_{X(\lambda)}{}^{-1}(u)\,\mathrm{d}D(u) \\ &= \int_{[0,1]} \mathrm{E}[X_i \,|\, X(\lambda) = F_{X(\lambda)}{}^{-1}(u)]\,\mathrm{d}D(u) \\ &= \int_{\mathbb{R}} \mathrm{E}[X_i \,|\, X(\lambda) = x] d(F_{X(\lambda)}(x))\,\mathrm{d}F_{X(\lambda)}(x) \\ &= \mathrm{E}[X_i d(F_{X(\lambda)}(X(\lambda)))]\end{aligned}$$

Thus, the capital allocated to the ith investment opportunity is

$$\kappa_i = \mathrm{E}[X_i d(F_X(X))]$$

▶▶ We can think of $d(F_X(X))$ as a Radon-Nikodym density:
$\mathrm{E}(d(F_X(X)) = 1$ trivially

$$\frac{\mathrm{dQ}}{\mathrm{dP}} = d(F_X(X)) \quad \Longrightarrow \quad \kappa_i = \mathrm{E}^{\mathrm{Q}}(X_i)$$

Even when we know the joint df of $(X_1, \ldots, X_d)$, it is still difficult to compute κ_i since the joint df of X_i and X is needed (The only exception is a Gaussian case).

$\Rightarrow$ Resort to Monte Carlo

Given a random sample $(X_1^k, \ldots, X_d^k)$, $k = 1, \ldots, n$, put

$$X^k = X_1^k + \cdots + X_d^k, \qquad \mathbb{F}_X(x) = \frac{1}{n+1} \sum_{k=1}^{n} \mathbf{1}_{\{X_k \le x\}}$$

Then we can estimate κ_i by

$$\begin{aligned} \widehat{\kappa}_i &= \frac{1}{n} \sum_{k=1}^{n} X_i^k d(\mathbb{F}_X(X^k)) \\ &= \iint x_i d(\mathbb{F}_X(x)) \, \mathrm{d}\mathbb{F}_{X_i,X}(x_i, x) \end{aligned}$$

where

$$\mathbb{F}_{X_i,X}(x_i, x) = \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{X_{k,i} \le x_i,\ X^k \le x\}}$$

The error $\widehat{\kappa}_i - \kappa_i$ can be asymptotically evaluated by proving asymptotic normality: Under certain regularity conditions,

$$\sqrt{n}(\widehat{\kappa}_i - \kappa_i) \xrightarrow{\mathcal{L}} N(0, \sigma^2)$$

where

$$\sigma^2 = \mathrm{var}\left(F_{X_i}^{-1}(\xi_i) d(\xi) + \iint F_{X_i}^{-1}(u_i) d'(u) \mathbf{1}_{\{\xi \leq u\}} \, \mathrm{d}C_i(u_i, u) \right)$$

$$C_i(F_{X_i}(x_i), F_X(x)) = \mathrm{P}(X_i \leq x_i, X \leq x) \text{ and } (\xi_i, \xi) \sim C_i$$

(Needs to be modified for ES)

Numerical Experiments: Take distortion densities

- Expected Shortfall: $$d_\theta(u) = \frac{1}{\theta}\mathbf{1}_{\{u \geq 1-\theta\}}$$

- Proportional Odds: $$d_\theta(u) = \frac{\theta}{(1-u+\theta u)^2}$$

- Proportional Hazards: $$d_\theta(u) = \theta(1-u)^{\theta-1}$$

- Gaussian: $$d_\theta(u) = \frac{\phi(\Phi^{-1}(u) + \log\theta)}{\phi(\Phi^{-1}(u))}$$

Elliptical loss distribution: $E_d(\mu, \Sigma, \psi)$

μ: location vector, Σ: dispersion matrix, ψ: characteristic generator

Assume r_ρ is the risk measure function of a positive homogeneous, law invariant risk measure ρ. Let $(X_1, \ldots, X_d) \sim E_d(\mathbf{0}, \Sigma, \psi)$. Then under an Euler allocation, the relative capital allocation is given by

$$\frac{\kappa_i}{\kappa_j} = \frac{\kappa_i(\mathbf{1})}{\kappa_j(\mathbf{1})} = \frac{\sum_{k=1}^{d} \Sigma_{ik}}{\sum_{k=1}^{d} \Sigma_{jk}}, \quad 1 \leq i, j \leq d.$$

►► The relative amounts of capital allocated to each investment opportunity are the same as long as we use a positive homogeneous, law invariant risk measure.

Estimated ratios $\widehat{\kappa}_i/\widehat{\kappa}_{i+1}$ of capital allocation ($\theta = \alpha = 0.05$)

sample from $N\left(\mathbf{0}, \begin{pmatrix} 1 & 0.1 & 0.5 \\ 0.1 & 1 & 0.9 \\ 0.5 & 0.9 & 1 \end{pmatrix}\right)$, size $= n$, 1000 runs

n	true ratio	ES bias	ES $\sqrt{\text{MSE}}$	PO bias	PO $\sqrt{\text{MSE}}$	PH bias	PH $\sqrt{\text{MSE}}$	Gaussian bias	Gaussian $\sqrt{\text{MSE}}$
100	4/5	0.0740	0.3962	0.0352	0.2815	0.0422	0.3281	0.0587	0.3933
	5/6	−0.0081	0.1045	−0.0028	0.0793	−0.0023	0.0908	−0.0033	0.1048
250	4/5	0.0129	0.2239	0.0101	0.1669	0.0219	0.2185	0.0332	0.2660
	5/6	0.0007	0.0634	−0.0003	0.0483	−0.0017	0.0623	−0.0030	0.0740
500	4/5	0.0092	0.1441	0.0064	0.1103	0.0138	0.1594	0.0188	0.1911
	5/6	−0.0006	0.0429	−0.0007	0.0329	−0.0015	0.0465	−0.0019	0.0552
5000	4/5	0.0017	0.0459	0.0006	0.0356	10^{-5}	0.0888	0.0005	0.0931
	5/6	−0.0003	0.0139	$9\cdot10^{-6}$	0.0108	0.0008	0.0265	0.0008	0.0278

Comparison in terms of DI ($\theta = \alpha = 0.05$)

Marginal: N(0,1)

Dependence: Gaussian & t copula with correlation matrix $\begin{pmatrix} 1 & 0.1 & 0.5 \\ 0.1 & 1 & 0.9 \\ 0.5 & 0.9 & 1 \end{pmatrix}$

►► Compute diversification index: $\mathsf{DI}_\rho(X) = \dfrac{\rho(X)}{\sum \rho(X_i)}$

- Gaussian: $\mathsf{DI}_\rho(X) = 0.8165$ for all DRM ρ theoretically
- t copula: $\mathsf{DI}_{\mathsf{ES}}(X) = 0.8329$ (std$= 0.021$),
  $\mathsf{DI}_{\mathsf{PO}}(X) = 0.8285$ (std$= 0.015$),
  $\mathsf{DI}_{\mathsf{GA}}(X) = 0.7367$ (std$= 0.076$)

Estimated capital allocation with GPD & t marginals ($\theta = \alpha = 0.05$)

using Gaussian copula with correlation matrix $\begin{pmatrix} 1 & 0.1 & 0.5 \\ 0.1 & 1 & 0.9 \\ 0.5 & 0.9 & 1 \end{pmatrix}$

	ES		PO		PH		Gaussian	
	cont.	ratio	cont.	ratio	cont.	ratio	cont.	ratio
GPD(1/25)	2.60		2.21		1.58		3.25	
GPD(1/10)	4.38	(0.59)	3.45	(0.64)	4.18	(0.38)	8.30	(0.39)
GPD(1/3)	9.12	(0.48)	6.99	(0.49)	24.32	(0.17)	38.87	(0.21)
t(25)	1.28		0.99		0.74		1.60	
t(10)	2.04	(0.63)	1.54	(0.64)	1.69	(0.44)	3.44	(0.47)
t(3)	3.82	(0.53)	2.88	(0.54)	9.62	(0.18)	14.97	(0.23)

4. Concluding Remarks

- Estimation of DRMs is possible, but for some DRMs, we don't get nice asymptotic properties; proportional odds risk measure has some nice features.

- Euler capital allocation based on DRMs are easy to compute and widely applicable (more stable than VaR). Need more computational efficiency for tail-exaggerating DRMs.

- Future research: Careful study of portfolio optimization

- Future research: Extension to dynamic setting