Risk Preferences

Further Developments beyond Random Variables

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Motivation

The intuitive notion of risk is very recent in history but remains unclear even today.

> *The late apparition in History of circumstances indicated by means of the new term 'risk' is probably due to the fact that it accommodates a plurality of distinctions within one concept, thus constituting the unity of this plurality.*
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In the context of economic theory, KNIGHT gives a definition

> *The practical difference between the two categories, risk and uncertainty, is that in the former the distribution of the outcome in a group of instances is known (either through calculation a priori or from statistic of past experience), while in the case of uncertainty this is not true.*

But KNIGHT's idea of "risk" does not match the one expressed in the theory of **monetary risk measures** which typically address the risk of several probability models.

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But KNIGHT's idea of "risk" does not match the one expressed in the theory of **monetary risk measures** which typically address the risk of several probability models.

Rather than in a descriptive way, we try to understand "risk" in a context (setting) independent manner, focusing on some crucial invariant features. These are

- **"diversification should not increase the risk"**
- **"the better for sure, the less risky"**

Outline

Outline

Risk Preferences – Robust Representation

Definitions

Definition (Risk Order)

A total preorder $\succcurlyeq$ *on* $\mathcal{X}$ *is a* risk order *if it is*

- **Quasiconvex:** $x \succcurlyeq \lambda x + (1-\lambda) y$ *whenever* $x \succcurlyeq y$,
- **Monotone:** $x \succcurlyeq y$ *whenever* $y \trianglerighteq x$.

Definition (Risk Measure)

A function $\rho : \mathcal{X} \to [-\infty, +\infty]$ *is a* risk measure *if it is*

- **Quasiconvex:** $\rho(\lambda x + (1-\lambda) y) \leq \max\{\rho(x), \rho(y)\}$.
- **Monotone:** $\rho(x) \leq \rho(y)$ *whenever* $x \trianglerighteq y$.

Definition (Risk Acceptance Family)

A family $\mathcal{A} = (\mathcal{A}^m)_{m \in \mathbb{R}}$ *of subset of* $\mathcal{X}$ *is a* risk acceptance family *if it is*

- **Convex:** $\mathcal{A}^m$ *is convex,*
- **Monotone:** $\mathcal{A}^m \subset \mathcal{A}^n$ *and* $x \trianglerighteq y$ *for some* $y \in \mathcal{A}^m$ *implies* $x \in \mathcal{A}^m$,
- **Right-Continuous:** $\mathcal{A}^m = \bigcap_{n>m} \mathcal{A}^n$.

Risk Preferences – Robust Representation

Robust Representation of Lower Semicontinuous Risk Orders

Here, $\mathcal{X}$ is a locally convex topological vector space with dual $\mathcal{X}^*$. $\trianglerighteq$ is a vector preorder: $x \trianglerighteq y$ iff $x - y \in \mathcal{K}$ closed convex cone with polar cone $\mathcal{K}^\circ$.

Theorem (Robust Representation of l.s.c. Risk Orders)

Any lower semicontinuous risk measure $\rho : \mathcal{X} \to [-\infty, +\infty]$ has a robust representation

$$\rho(x) = \sup_{x^* \in \mathcal{K}^\circ} R\left(x^*, \langle x^*, -x\rangle\right)$$

for a unique maximal risk function $R \in \mathcal{R}^{max}$.

Definition

$\mathcal{R}^{max}$ denotes the set of maximal risk functions $R : \mathcal{K}^\circ \times \mathbb{R} \to [-\infty, +\infty]$

- *R is jointly quasiconcave*
- *nondecreasing and left-continuous in the second argument*
- *$R(\lambda x^*, s) = R(x^*, s/\lambda)$ for any $\lambda > 0$*
- *R has a uniform asymptotic minimum, $\lim_{s \to -\infty} R\left(x^*, s\right) = \lim_{s \to -\infty} R\left(y^*, s\right)$,*
- *$x^* \mapsto R^+\left(x^*, s\right) := \inf_{s' > s} R\left(x^*, s\right)$ is upper semicontinuous.*

Risk Preferences – Robust Representation

A Setting Dependant Interpretation of Risk

Possible settings by the specification of the convex set $\mathcal{X}$ and the monotonicity preorder $\trianglerighteq$.

- **Random variables** on $(\Omega, \mathscr{F}, P)$ with as preorder $\trianglerighteq$ the "$\geq$ P-almost surely".
- **Stochastic processes** modeling cumulative wealth processes $X = X_0, X_1, \ldots X_T$ with as preorder $\trianglerighteq$ the cash flow monotonicity "$X_t - X_{t-1} := \Delta X_t \geq \Delta Y_t$".
- **Probability distributions** (lotteries) $\mathcal{M}_1$ is a convex set with standard monotonicity preorders $\trianglerighteq$ either the first or second stochastic order.
- **Cumulative consumption streams** are right continuous non decreasing functions $c : [0,1] \to \mathbb{R}^+$ building a convex cone. Here $c^{(1)}$ is "better for sure" than $c^{(2)}$ if $c^{(1)} - c^{(2)}$ is still a cumulative consumption stream.
- **Stochastic kernels** are probability distributions subject to uncertainty, that is, mappings $\tilde{X} : \Omega \to \mathcal{M}_1$. Possible preorders $\trianglerighteq$ are either the P-almost sure first or second stochastic order.
- ...

Outline

Model Risk

Set of Random Variables

- $\mathcal{X}$ is the convex set $(A, B) := \{X \in \mathbb{L}^\infty | a < \operatorname{ess\,inf} X \geq \operatorname{ess\,sup} X < b\}$.
- $\trianglerighteq$: relation "greater than P-almost surely" corresponding to the cone $\mathcal{K} = \mathbb{L}^\infty_+$.
- Under the good topology $\sigma\left(\mathbb{L}^\infty, \mathbb{L}^1\right)$, then $\mathcal{K}^\circ = \mathbb{L}^1_+$ and $\mathcal{K}^\circ_1 = \left\{Z \in \mathbb{L}^1_+ | E[Z] = 1\right\} =: \mathcal{M}_1\left(P\right)$ is a set of probability measures.

Proposition (Random Variables $\rightsquigarrow$ Modell Risk)

Any $\|\cdot\|_\infty$-l.s.c. risk measure $\rho : (A, B) \to [-\infty, +\infty]$ with the fatou property has a robust representation

$$\rho\left(X\right) = \sup_{Q \in \mathcal{M}_1(P)} R\left(Q, E_Q\left[-X\right]\right)$$

- Certainty equivalents of expected losses.
- Monotone Versions of Mean variance preferences (Markowitz)
- Coherent and convex monetary risk measures (Artzner et al. and Föllmer and Schied)
- Performance measures such as the Sharpe ratio and their monotone versions (Cherny and Madan)
- Economic index of riskiness (Aumann and Serrano)
- ...

Outline

Distributional Risk

Set of Probability Distributions

- $\mathcal{X}$ is the convex set $\mathcal{M}_{1,c}$ of prob. dist. with compact support, i.e., $\mu([-c, c]) = 1$ for some $c > 0$
- Diversification has a different meaning as for random variables: randomization. In general $P_{\lambda X + (1-\lambda) Y} \neq \lambda P_X + (1-\lambda) P_Y$.
- $\mu \trianglerighteq \nu$: first stocastic order, $\int l d\mu \geq \int l d\nu$ for any continuous increasing function l, or equivalently,

$$F_\nu(x) := \nu(]-\infty, x]) \geq= \mu(]-\infty, x]) =: F_\mu(x)$$

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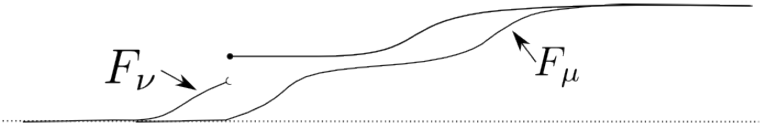

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Proposition (Probability Distributions $\rightsquigarrow$ Distributional Risk)

Any $\sigma(ca_c, C)$-l.s.c. risk measure $\rho : \mathcal{M}_{1,c} \to [-\infty, +\infty]$ monotone w.r.t. the first stochastic order has a robust representation

$$\rho(\mu) = \sup_{l \text{ continuous nondecreasing}} R\left(l, -\int l(x)\, \mu(dx)\right)$$

Distributional Risk

Automatic Continuity: A von Neuman and Morgenstern representation without weak continuity assumptions

The assumption of $\sigma\left(\mathcal{M}_{1,c}, C\right)$ lower semicontinuity is far from being negligible. We wish some automatic continuity result à la Borwein where this lower semicontinuity might be a consequence of the monotonicity, but...

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$\mathcal{M}_{1,c}$ is a convex set and not a vector space and it is moreover not metrizable for this weak topology.

However, in a recent paper with Freddy Delaben and Michael Kupper, we show that the monotonicity with respect to the first stochastic order allows to give a so called von Neumann and Morgenstern representation

Theorem (DDK 2010)

Any affine risk measure ρ of a risk order $\succcurlyeq$ on $\mathcal{M}_1(\mathbb{R})$ (satisfies the archimedian and independance axiom) is $\sigma(\mathcal{M}_1, B_b)$ continuous and can be represented by

$$\rho(\mu) = -\int l(x)\,\mu(dx)$$

for some nondecreasing bounded function l.

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Distributional Risk

Example: Value at Risk

Probability Distributions: Value at Risk

$$V@R_q(X) = \sup\left\{s \in \mathbb{R} \,\middle|\, P[X + s \leq 0] \geq q\right\}$$

monotone and cash additive but not quasiconvex! Might even penalize diversification!

On the level of probability distribution

$$V@R_q(\mu) := \sup\left\{s \in \mathbb{R} \,\middle|\, \mu(]-\infty, -s]) \geq q\right\} \quad \text{note: } V@R_q(P_X) = V@R_q(X)$$

is a lower semi continuous risk measure for probability distributions as

$$\mathcal{A}^m = \left\{\mu \,\middle|\, V@R_q(\mu) \leq m\right\} = \bigcap_{\{s > -m\}} \left\{\mu \,\middle|\, \mu(]-\infty, s]) \geq q\right\}$$

It has a robust representation (only quasiconvex, not convex!)

$$V@R_q(\mu) = \sup_{l \text{ continuous nondecreasing, } \inf l > -\infty} -l^{-1}\left(\frac{1}{1-q}\int l(x)\,\mu(dx) + \frac{q}{q-1}\inf l\right).$$

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Outline

Discounting Risk

Set of Consumption Streams

- $\mathcal{X}$ is the convex cone of (deterministic) consumptions streams $c : [0,1] \to \mathbb{R}^+$ (right continuous increasing).
- $c^1 \trianglerighteq c^2$: if $c^1 - c^2$ is still a consumption stream.
- Some Orlicz topology (economically sound as argued by Hindy, Huang, Kreps).

Proposition (Consumption Streams $\rightsquigarrow$ Discounting Risk)

Any l.s.c. risk measure ρ on the cone of consumption streams $c : [0,1] \to \mathbb{R}^+$ has a robust representation

$$\rho(c) = \sup_{\beta \in \mathcal{CS}^\circ} R\left(\beta, -\int \beta(s)\, dc_s\right)$$

Here, the β's are some positive functions $\Rightarrow$ **discounting risk**.

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Consumption Streams: Hindy-Huang-Kreps for exponential delay

$$\rho(c) := -\int_0^1 u\left(\int_0^t e^{-\gamma(t-s)} dc_s\right) = \sup_\beta \exp\left(-\int_0^1 \frac{\beta(s)}{\gamma} dc_s - \frac{g(\beta)}{\gamma}\right)$$

where

$$g(\beta) = \int_0^1 d\beta(s) - \gamma \int_0^1 \beta(s)\left[\frac{e^{-\gamma t}}{1-e^{-\gamma t}} - \ln\left(\frac{\beta(t)}{1-e^{-\gamma t}}\right) + \ln\left(\int_0^1 \frac{\beta(s)}{1-e^{-\gamma s}} ds\right)\right] ds$$

Wrap Up

Model Risk — Distributional Risk — Discounting Risk

This approach allows within one concept different interpretations of risk depending on the underlying context

Random Variables ⇝ Modell Risk

$$\rho(X) = \sup_{Q} R\left(Q, E_Q\left[-X\right]\right)$$

for probability models Q.

Probability Distributions ⇝ Distributional Risk

$$\rho(\mu) = \sup_{l \text{ continuous nondecreasing}} R\left(l, -\int l(x)\, \mu(dx)\right)$$

for test functions l

Consumption Streams ⇝ Discounting Risk

$$\rho(c) = \sup_{\beta \in \mathcal{CS}^{\circ}} R\left(\beta, -\int \beta(s)\, dc_s\right)$$

for discounting functions β.

Outline

Interplay Model Risk ↔ Distributional Risk

Stochastic kernels

Illustrate the interplay between model risk and distributional risk.

Proposition

ρ is a l.s.c. risk measure of a risk order $\succcurlyeq$ on stochastic kernels $\tilde{X}(\omega, dx)$ monotone w.r.t. the P-almost sure second stochastic order and satisfying

$$\tilde{X}(\omega, \cdot) \succcurlyeq \tilde{Y}(\omega, dx) \text{ for any } \omega \in \Omega \qquad \Longrightarrow \qquad \tilde{X} \succcurlyeq \tilde{Y}$$

Then, the risk order can be factorized into a model risk component and a distributional risk component, that is,

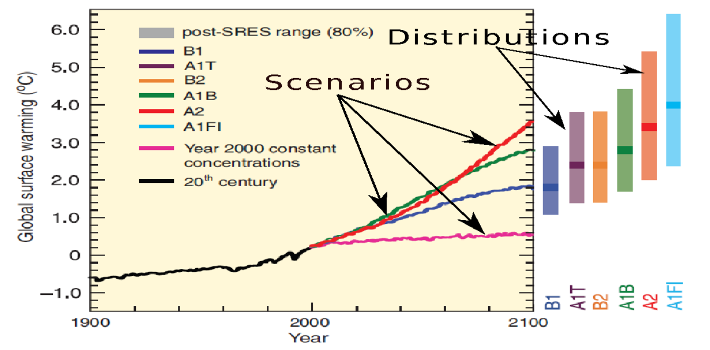
$$\rho\left(\tilde{X}\right) := \Phi\left(g\left(\tilde{X}(\omega, \cdot)\right)\right)$$

Where Φ is a l.s.c. risk measure on random variables $\mathbb{L}^\infty$ and g is a risk measure on probability distributions $\mathcal{M}_{1,c}$.

Interplay Model Risk ↔ Distributional Risk

Stochastic kernels

Temperature related long term insurance contract



Scenario dependant loss distribution of the contract: $\tilde{X}(\omega, dx) = \sum \mu_i 1_{\omega_i}(\omega)$.

$$\rho\left(\tilde{X}\right) = \sup_{P=(p_1,p_2,\dots)} \left\{\sum V@R_q(\mu_i)\, p_i - \alpha(P)\right\}$$

Conclusion

MANY THANKS FOR YOUR ATTENTION!