Modeling of Contagious Downgrades and Its Application to Multi-Downgrade Protection

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Now an important football match between JAPAN and DENMARK is going on!

Contents

Motivation

- Not only default risk but also downgrade risk is important for risk management consistent with Basel II.
- Necessary is a new model of downgrade risk so as to recognize self-exciting effect and mutually exciting effect of downgrades among some industry sectors.
- WHY? — See the historical data on rating changes of Japanese enterprises.
  - Some clusters of downgrades are observed in the past.
  - From sector to sector, the periods of clustering seem a little different.

Data summary

- The original data consists of the records on genuine **downgrades** of Japanese private enterprises from April 1998 to December 2009 reported by R&I[1].
- We (tentatively) reclassify the Japanese enterprises whose industry type is specified by Bloomberg into the following three categories.
  - Financial (including Claim Paying Ability of insurance companies),
  - Group A (Communications, Consumer-Cyclical, Industrial, Technology)— seems more influenced by business fluctuation,
  - Group B (Basic Materials, Consumer-Non-cyclical, Energy, Utilities) — seems less influenced by business fluctuation.

[1] R&I (Rating and Investment Information, Inc.) is one of the largest rating agencies in Japan.

Data: original form

Company Name	Date	Current Ra	Last Ratin	Industry Type
Taisei Corp	1998/4/6	A *	AA– *	Building&Construct–Misc
Seiyu Ltd/The	1998/4/13	BB *	BBB	Retail–Misc/Diversified
Nissan Motor Co Ltd	1998/4/17	A+ *–	A+	Auto–Cars/Light Trucks
Kitano Construction Co	1998/4/18	NR	BBB+	Building&Construct–Misc
AC Real Estate Corp	1998/4/23	BB *–	BB	Building&Construct–Misc
Aoki Corp	1998/4/23	BB *–	BB	Building&Construct–Misc
Aoyama Kanzai Corp	1998/4/23	BB *–	BB	Building&Construct–Misc
Haseko Corp	1998/4/23	BB– *–	BB–	Bldg–Residential/Commer
JDC Corp	1998/4/23	BB– *–	BB–	Building&Construct–Misc
Kumagai Gumi Co Ltd	1998/4/23	BB+ *–	BB+	Building&Construct–Misc
Sato Kogyo Co Ltd	1998/4/23	BB– *–	BB–	Building&Construct–Misc
Cosmos Initia Co Ltd	1998/4/30	NR	BB	Real Estate Mgmnt/Servic
AC Real Estate Corp	1998/5/1	B+	BB *–	Building&Construct–Misc
Aoki Corp	1998/5/1	B+	BB *–	Building&Construct–Misc
Aoyama Kanzai Corp	1998/5/1	BB–	BB *–	Building&Construct–Misc

Figure: A sample of the original data (obtained from Bloomberg)

Data: for our analyses

date	time	DOWN_F	DOWN_A	DOWN_B
1998/4/1	0	0	0	0
1998/4/2	0.004049	0	0	0
1998/4/3	0.008097	0	0	0
1998/4/6	0.012146	0	1	0
1998/4/7	0.016194	0	0	0
1998/4/8	0.020243	0	0	0
1998/4/9	0.024291	0	0	0
1998/4/10	0.02834	0	0	0
1998/4/13	0.032389	0	1	0
1998/4/14	0.036437	0	0	0
1998/4/15	0.040486	0	0	0
1998/4/16	0.044534	0	0	0
1998/4/17	0.048583	0	0	0
1998/4/20	0.052632	0	0	0
1998/4/21	0.05668	0	0	0
1998/4/22	0.060729	0	0	0
1998/4/23	0.064777	0	0	0
1998/4/24	0.068826	0	0	0
1998/4/27	0.072874	0	0	0
1998/4/28	0.076923	0	0	0
1998/4/30	0.080972	0	0	0
1998/5/1	0.08502	0	7	0
1998/5/6	0.089069	0	0	0
1998/5/7	0.093117	0	0	0
1998/5/8	0.097166	0	0	1

In all, 1,042 downgrades.
Among them,
274 downgrades in Financial,
575 in Group A,
193 in Group B.

Figure: The data processed for our analysis. In this study, we use only four columns of "time" (1 business day $\approx$ 1/250), "DOWN_F", "DOWN_A" and "DOWN_B".

Transition of Monthly numbers of each event

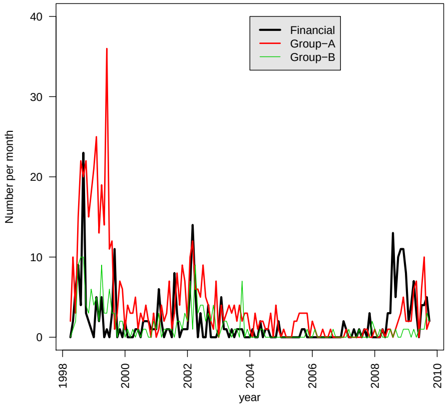

Figure: Trajectory of monthly numbers of category-by-category downgrades announced by R&I during April 1998 to September 2009. In all, 1,011 downgrades are observed. There are 263 downgrades are in Fin. category, 562 .in Gr.A and 186 in Gr.B. ◊

Methods

- For the purpose of modeling downgrade risk with self-exciting and mutually exciting effects, use intensities specified by a multivariate affine jump type process or an extension of Hawkes model.
  - Estimate the model parameters from the R&I historical data by MLE.
- Give an example to utilize the proposed model as a risk hedging tool.
  - Introduce a new product named “Multi-Downgrade Protection (MDP)”
  - Consider some efficient computation of the fair value of MDP.
  - Show some numerical illustrations related to MDP.

Related previous works

- Modeling with self-exciting / mutually exciting point processes
  - Bowsher (2007), Errais, Giesecke and Goldberg (2006), Giesecke and Goldberg (2005)[2], Hawkes (1971), Kim and Giesecke (2009)
- Maximum likelihood estimation for point processes
  - Azizpour and Giesecke (2008), Bowsher (2007), Ogata (1978)
- Results of term structure for affine-jump (diffusion) processes
  - Duffie, Pan and Singleton (2000), Errais, Giesecke and Goldberg (2006)

[2] The current version written by Giesecke, Goldberg and Ding hardly mentions the "self-exciting" property of portfolio default intensity.

General setting

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$: a filtered complete probability space
- $m (\in \mathbb{N})$ kinds of events are considered.
- $0 (\equiv \tau_0^i) < \tau_1^i < \tau_2^i < \cdots \ (i = 1, \cdots, m)$: $(\mathcal{F}_t)$-adapted point processes (i.e. an increasing sequence of stopping times)
    - τ_k^i : the time when k-th event of type i occurs
    - N_t^i : the counting process associated with $\{\tau_k^i\}_{k \in \mathbb{N}}$, that is, the cumulative number of observation times when type i events occur up to time t.
    - Suppose that $[N^i, N^j]_t = 0$ a.s. if $i \neq j$.
- L_t^i : an $(\mathcal{F}_t)$-adapted pure jump process (or marked point process)

$$L_t^i := \sum_{k=1}^{N_t^i} \eta_k^i,$$

where $\{\eta_k^i\}_{k \in \mathbb{N}}$ are i.i.d. random variables and η_k^i is $\mathcal{F}_{\tau_k^i}$-measurable. As an example, we regard η_k^i as the number of type i-events that occur coincidently at time τ_k^i.

Intensity process

- λ_t^i : the intensity process associated with N_t^i for each i

λ_t^i is defined as an $(\mathcal{F}_t)$-progressively measurable, nonnegative process such that

$$M_t^i := N_t^i - \int_0^t \lambda_s^i ds$$

is an $(\mathcal{F}_t)$-(local) martingale.

- Assume that the intensity λ_t^i is given by

$$\lambda_t^i = \Lambda_0^i(t) + \Lambda_1^i(t) \cdot \boldsymbol{X}_t,$$

where $\boldsymbol{X}_t$ is a d-dimensional stochastic state vector (given below), and $\Lambda_0^i(t) \in \mathbb{R}$ and $\Lambda_1^i(t) \in \mathbb{R}^d$ are deterministic functions. (The dot "$\cdot$" means the inner product of two vectors.)

Model specification for parameter estimation

- Assume that X_t^i itself is the intensity λ_t^i.
- The superindex 1 corresponds to downgrade in Financial category, 2 to downgrades in Group A, and 3 to downgrades in Group B
- Let $m = d = 3$ and suppose $\boldsymbol{X}_t = {}^t(X_t^1, X_t^2, X_t^3)$ follows:

(Similar to mutually exciting Hawkes model(1971))

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = \begin{pmatrix} \kappa^1(c^1 - X_t^1) \\ \kappa^2(c^2 - X_t^2) \\ \kappa^3(c^3 - X_t^3) \end{pmatrix} dt + \begin{pmatrix} \xi^{1,1} & \xi^{1,2} & \xi^{1,3} \\ \xi^{2,1} & \xi^{2,2} & \xi^{2,3} \\ \xi^{3,1} & \xi^{3,2} & \xi^{3,3} \end{pmatrix} \begin{pmatrix} dL_t^1 \\ dL_t^2 \\ dL_t^3 \end{pmatrix}, ***$$

where κ^j, c^j $(j = 1, 2, 3)$ *and* $\xi^{j,i}$ $(j, i = 1, 2, 3)$ *are all non-negative parameters.*

- Then, for each j, X_t^j can be represented as

$$X_t^j = c^j + e^{-\kappa^j t}(X_0^j - c^j) + \int_0^t e^{-\kappa^j(t-s)} \sum_{i=1}^{3} \xi^{j,i} dL_s^i \qquad (1)$$

Likelihood function

- Refer to Azizpour and Giesecke for MLE for point processes.
- $(\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\eta}}) := [\{(\tilde{\tau}^i_k, \tilde{\eta}^i_k)\}_{k=1,\cdots,\tilde{N}^i_T}]_{i=1,2,3}$: the observations during the period $[0, T]$ for parameter estimation
  - $\tilde{\tau}^i_k$: the k-th time of type i event observed during $[0, T]$.
  - $\tilde{\eta}^i_k$: the number of type i events which happen simultaneously at time $\tilde{\tau}^i_k$
- $\Theta^j := (X^j_0, \kappa^j, c^j, \{\xi^{j,i}\}_{i=1,2,3})$ $(j = 1, 2, 3)$: the set of parameters
- The likelihood function can be represented:

$$\mathcal{L}\Big(\{\Theta^j\}_{j=1,2,3} | (\tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\eta}})\Big) = \prod_{j=1}^{3} \exp\Big(\int_0^T \log(\tilde{X}^{j,\Theta^j}_{s-}) d\tilde{N}^i_s - \int_0^T \tilde{X}^{j,\Theta^j}_s ds\Big),$$

  where $\tilde{X}^{j,\Theta^j}_t$ is the actual path of state process X^j_t achieved by the observation with Θ^j.
- Note: Θ^j can be estimated separately for each j.

Log-likelihood function

- At last, we have the log-likelihood function of Θ^j as follows.

$$\ell\left(\Theta^j|(\tilde{\tau},\tilde{\eta})\right) = \sum_{k=1}^{\tilde{N}_T^j} \log\Big\{c^j + e^{-\kappa^j\tilde{\tau}_k^j}(X_0^j - c^j) + \sum_{i=1}^{3}\xi^{j,i}\sum_{\tilde{\tau}_p^i<\tilde{\tau}_k^j}\tilde{\eta}_p^i e^{-\kappa^j(\tilde{\tau}_k^j-\tilde{\tau}_p^i)}\Big\}$$

$$- c^jT - \frac{X_0^j - c^j}{\kappa^j}\left(1 - e^{-\kappa^jT}\right) - \frac{1}{\kappa^j}\sum_{i=1}^{3}\xi^{j,i}\sum_{k=1}^{\tilde{N}_T^i}\tilde{\eta}_k^i\left(1 - e^{-\kappa^j(T-\tilde{\tau}_k^i)}\right)$$

- We use the free software R for maximization of the above function, specifically the function `optim` as below.

```
optim(initial_values, obj_fun, method = "L-BFGS-B", lower =
numeric(6), control=list(fnscale=-1), hessian=TRUE)
```

- As for initial values of the parameters, we try 12 kinds of sets in total, and finally choose the estimates that maximize the objective function.

Estimation Result

Table: The maximum likelihood estimates of the parameters. The standard errors are given in parentheses.

Financial	X_0^1	κ^1	c^1	$\xi^{1,1}$	$\xi^{1,2}$	$\xi^{1,3}$
	19.17	3.96	3.18	1.46	0.00	0.00
	(12.56)	(3.39)	(1.07)	(0.73)	(0.21)	(0.77)
Group A	X_0^2	κ^2	c^2	$\xi^{2,1}$	$\xi^{2,2}$	$\xi^{2,3}$
	42.67	3.17	3.13	1.17	0.97	0.76
	(18.68)	(0.94)	(1.79)	(0.31)	(0.42)	(0.82)
Group B	X_0^3	κ^3	c^3	$\xi^{3,1}$	$\xi^{3,2}$	$\xi^{3,3}$
	23.39	4.39	0.96	0.47	0.48	1.09
	(19.15)	(1.76)	(0.87)	(0.26)	(0.40)	(0.65)

The red fonts mean that the absolute value of the estimate is more than twice the standard error. $\diamond$

Consideration of the MLE

- **Self-exciting effect**:
  - Judging from the estimates of $\xi^{1,1}, \xi^{2,2}, \xi^{3,3}$, we can recognize that self-exciting effect is significant in Group A, but less significant in Financial and Group B.
- **Mutually exciting effect**:
  - The estimate of $\xi^{2,1}$ implies that downgrades in Financial can significantly make impacts upon the downgrades in Group A.
  - The other mutually exciting effects are less clear.
- As for goodness-of-fit tests, Kolmogorov-Smirnov test implies the model is not so bad while Prahl's test do not give good suggestion.

Estimated intensities

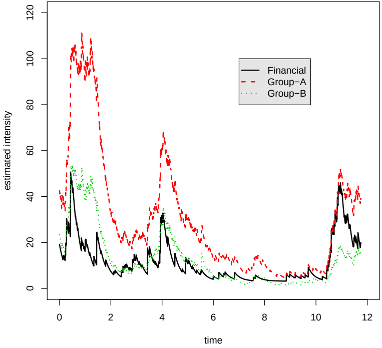

Figure: The estimated paths of the downgrade intensity for the three categories obtained by substituting the estimates in Table 1 and the observation into (1). See the previous figure again.

Multi-Downgrade Protection

- L_t^* : the counting process of a special kind of target events such as downgrades from the investment grade to the speculative grade in some industrial category.
- C_t^T : the payoff process of the protection (a continuous adapted process)
- Q : a P-equivalent martingale measure

(The premium at time t for MDP with maturity T)

$$V_t^T := E^Q\left[\int_t^T \exp\left(-\int_t^s r_u du\right) C_s^T dL_s^* \middle| \mathcal{F}_t\right].$$

Assumption

1. *r_t and L_t^* are independent.*
2. *For a fixed T, C_t^T is (approximately) given by $Z(t,T)\int_t^T E^Q[\bar{h}_u|\mathcal{F}_t]du$, where $Z(t,T)$ is the price of default-free discount bond. $\bar{h}_t$ follows a Vasicek-type model which is independent of r_t, L_t^* and $Z(t,T)$ under Q.*

Premium for Multi-Downgrade Protection

As a result of simple calculation,

$$V_t^T = Z(t,T)\int_t^T \left\{E^Q[L_s^*|\mathcal{F}_t] - L_t^*\right\} E^Q[\bar{h}_s|\mathcal{F}_t]ds.$$

In short, essential is to compute $E^Q[L_s^*|\mathcal{F}_t]$ for $s \in [t,T]$.

Remark that $Z(t,T)\int_t^T E^Q[\bar{h}_u|\mathcal{F}_t]du$ is a naive approximation of the difference of the price of corporate zero-coupon bond between before and after downgrade. (Regard $\bar{h}_t := h_t^2 - h_t^1$ as the difference of credit spreads between the current rating and the last rating.)

$$E^Q\left[\exp\left(-\int_t^T \{r_u + h_u^1\}du\right)\middle|\mathcal{F}_t\right] - E^Q\left[\exp\left(-\int_t^T \{r_u + h_u^2\}du\right)\middle|\mathcal{F}_t\right]$$

$$= Z(t,T)E^Q\left[\exp\left(-\int_t^T h_u^1 du\right) - \exp\left(-\int_t^T h_u^2 du\right)\middle|\mathcal{F}_t\right] \approx Z(t,T)E^Q\left[\int_t^T \{h_u^2 - h_u^1\}du\middle|\mathcal{F}_t\right]$$

$$= Z(t,T)\int_t^T E^Q[h_u^2 - h_u^1|\mathcal{F}_t]du = Z(t,T)\int_t^T E^Q[\bar{h}_u|\mathcal{F}_t]du.$$

A numerical illustration

- As another numerical illustration, we compute the expected cumulative number $E[L_t^2]$ of downgrades in Group A category. Remark that the expectation is w.r.t. not the pricing measure Q but the physical measure P in this illustration.
- Such computation is essentially used to value the credit derivatives whose payoffs depend on the number of downgrades in some specific category like "Multi-Downgrade Protection" introduced before.
- In our multivariate affine-jump type model, $E[L_T^i|\mathcal{F}_t]$ $(T \geq t)$ can be computed without Monte Carlo simulation.

The useful result for affine jump processes

Proposition (A simple version of Corollary A.3. in Errais et al)

Let $\boldsymbol{Y}_t := {}^t(\boldsymbol{X}_t, \boldsymbol{L}_t)$. *For any* $i \in \{1, \cdots, m\}$ *and* $T \geq t$, *we have*

$$E[L_T^i|\mathcal{F}_t] = A_{L^i}(t,T) + B_{L^i}(t,T) \cdot \boldsymbol{Y}_t,$$

where

$$B_{L^i}(t,T) = \exp\left(\int_t^T \left[\begin{pmatrix} {}^tK_1(s) & \boldsymbol{0}_{d\times m} \\ \boldsymbol{0}_{m\times d} & \boldsymbol{0}_{m\times m} \end{pmatrix} + \sum_{i=1}^m \left\{\begin{pmatrix} \Lambda_1^i(s) \\ \boldsymbol{0}_m \end{pmatrix} {}^t\boldsymbol{\eta}_{\mathrm{mean}}^i\right\} \begin{pmatrix} \Xi^i & \boldsymbol{0}_{d\times m} \\ \boldsymbol{0}_{m\times d} & U_{m\times m}^i \end{pmatrix}\right] ds\right) \boldsymbol{e}_{d+i},$$

$$A_{L^i}(t,T) = \int_t^T \left\{\begin{pmatrix} K_0(s) \\ \boldsymbol{0}_m \end{pmatrix} + \sum_{i=1}^m \Lambda_0^i(s) \begin{pmatrix} \Xi^i & \boldsymbol{0}_{d\times m} \\ \boldsymbol{0}_{m\times d} & U_{m\times m}^i \end{pmatrix} \boldsymbol{\eta}_{\mathrm{mean}}^i\right\} \cdot B_{L^i}(s,T) ds.$$

$U_{n\times n}^i$ *is an* n*-dim. matrix s.t. the only diagonal component corresponding to* L_t^i *is 1,*
$\boldsymbol{0}_n$ *(resp.* $\boldsymbol{0}_{n\times n'}$*) is an* n*-dim. zero vector (resp.* $n \times n'$*-zero matrix),*
$\boldsymbol{e}_k$ *is a* $(d+m)$*-dimensional vector such that only* k*-th element is 1,*
$\boldsymbol{\eta}_{\mathrm{mean}}^i$ *is a* $(d+m)$*-dimensional vector s.t. every component is the average of* η^i.

Specification for numerical work

For our specific model, let $d = m = 3$ and

- $K_0(t) \equiv K_0 = {}^t(\kappa^1 c^1, \kappa^2 c^2, \kappa^3 c^3)$, $K_1(t) \equiv K_1 = \mathrm{diag}(-\kappa^1, -\kappa^2, -\kappa^3)$
- $\Lambda^1 = {}^t(1,0,0), \Lambda^2 = {}^t(0,1,0), \Lambda^3 = {}^t(0,0,1)$ (In short, $\lambda_t^j = X_t^j$)
- $\Xi^1 = \mathrm{diag}(\xi^{1,1}, \xi^{2,1}, \xi^{3,1}), \Xi^2 = \mathrm{diag}(\xi^{1,2}, \xi^{2,2}, \xi^{3,2}), \Xi^3 = \mathrm{diag}(\xi^{1,3}, \xi^{2,3}, \xi^{3,3})$
- $\overline{\eta}^1, \overline{\eta}^2, \overline{\eta}^3$ are estimated as the sample averages from the historical data of R&I.

The last proposition implies $E[L_t^2] = A(0,t) + B(0,t) \cdot {}^t(\boldsymbol{X}_0, 0, 0, 0)$, where $A(0,t)$ and $B(0,t)$ are specified in the next slide.

Specification for numerical work

$B(0,t)$ is given by the product of the exponential mapping $\exp(tH)$ of the following matrix H and $\boldsymbol{e}_5 := {}^t(0,0,0,0,1,0)$.

$$H = \begin{pmatrix} -\kappa^1 + \overline{\eta}^1 \xi^{1,1} & \overline{\eta}^1 \xi^{2,1} & \overline{\eta}^1 \xi^{3,1} & 1 & 0 & 0 \\ \overline{\eta}^2 \xi^{1,2} & -\kappa^2 + \overline{\eta}^2 \xi^{2,2} & \overline{\eta}^2 \xi^{3,2} & 0 & 1 & 0 \\ \overline{\eta}^3 \xi^{1,3} & \overline{\eta}^3 \xi^{2,3} & -\kappa^3 + \overline{\eta}^3 \xi^{3,3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

This exponential mapping $\exp(tH)$ can be numerically computed with Runge-Kutta method.
In addition, the integral

$$A(0,t) = \int_0^t (\kappa^1 c^1, \kappa^2 c^2, \kappa^3 c^3, 0, 0, 0) \cdot \{\exp(uH)\boldsymbol{e}_5\} du.$$

is approximately calculated as some finite sum by discretization of time.

Numerical example (1)

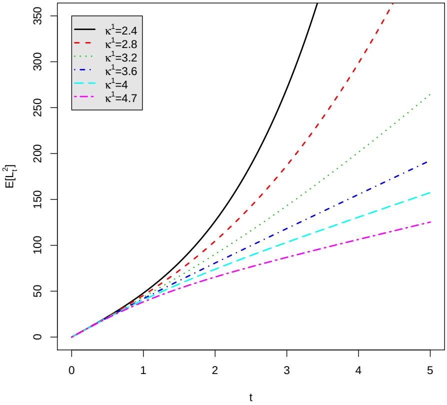

Figure: $E[L_t^2]$ for different values of the reversion speed κ^1 of Financial. ($\hat{\kappa}^1 = 3.96$)

Numerical example (2)

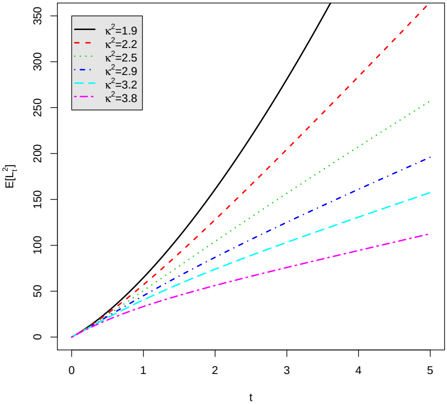

Figure: $E[L_t^2]$ for different values of the reversion speed κ^2 of Group A. ($\hat{\kappa}^2 = 3.17$)

Numerical example (3)

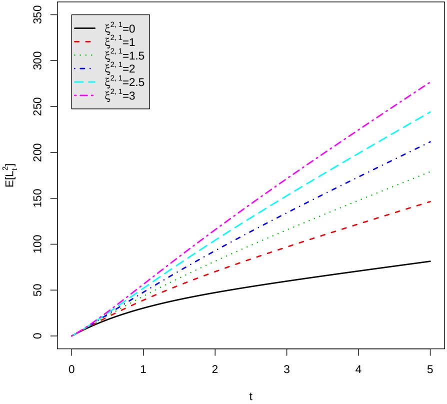

Figure: $E[L_t^2]$ for different values of the mutually exciting component $\xi^{2,1}$ from Financial to Group A. ($\hat{\xi}^{2,1} = 1.17$)

Numerical example (4)

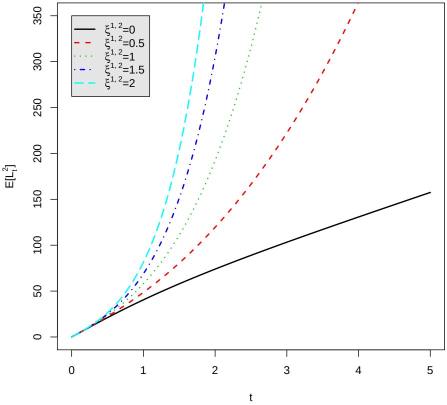

Figure: $E[L_t^2]$ for different values of the mutually exciting component $\xi^{1,2}$ from Group A to Financial. ($\hat{\xi}^{1,2} = 0$)

Consideration of the numerical illustration

- Small κ^1 means that the downgrade intensity of Financial remains relatively high even though time passes, so we consider that if downgrades are likely to happen in Financial, then downgrade risk of Fin. category is contagious to Group A due to the positive mutually exciting effect of $\xi^{2,1}$.
- Large $\xi^{1,2}$ means that each downgrade in Group A causes a larger jump of the intensity of Fin. category. As a consequence, downgrades are more likely to occur in Financial and after all downgrade risk is contagious to Group A because of the positive mutually exciting effect.
- On the whole, $E[L_t^2]$ (hence V_0^t) must be quite sensitive to the parameters $\kappa^1, \kappa^2, \xi^{2,1}$ and $\xi^{1,2}$.

Concluding remarks

- The mutually exciting intensity model specified by a multivariate affine jump type process is introduced so as to see whether there are some mutually contagious downgrades among some categories.
- Based on the specific model, we use actual data on rating migrations of Japanese enterprises to display some numerical illustraions.
    - The estimation result implies that not only some self-exciting effects but some mutually exciting effects exist for downgrades.
- Although we think we could verify applicability of our model to some extent via this tentative analysis, we still have a lot of assignments...

Selected references

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Mutually exciting intensity model

- $\boldsymbol{X}_t := {}^t(X_t^1, \cdots, X_t^d)$ is specified as follows:

(Multivariate affine-jump model / an extension of mutually exciting Hawkes model)

$$d\boldsymbol{X}_t = (K_0(t) + K_1(t) \cdot \boldsymbol{X}_t)dt + \sum_{i=1}^{m} \Xi^i d\boldsymbol{Z}_t^i, \quad \boldsymbol{X}_0 \in \mathbb{R}^d,$$

- $K_0(t) \in \mathbb{R}^d$, $K_1(t) \in \mathbb{R}^{d \times d}$: *deterministic*
- Ξ^i := *diag*$(\xi^{1,i}, \cdots, \xi^{d,i})$, *where* $\xi^{j,i} \geq 0$ *for* $\forall\ j, i$.
- $\boldsymbol{Z}_t^i := {}^t(Z_t^i, \cdots, Z_t^i)$: *d-dimensional vector. Either* $Z_t^i = N_t^i$ *or* $Z_t^i = L_t^i$.

- $\xi^{i,i}$: the magnitude of self-exciting effect of each type i event.
- $\xi^{j,i} (j \neq i)$: the magnitude of mutually exciting effects from type i event to j-th state component. ***
- Each event intensity can only jump upwards so as to keep $\boldsymbol{X}_t$ positive.

Hawkes(1971) model

- $N_k(t)$ $(k = 1, \cdots, K)$: point processes s.t.

$$P(N_k(t+\Delta t) - N_k(t) = 1|\mathcal{H}_t) = \Lambda_k(t)\Delta t + o(\Delta t),$$
$$P(N_k(t+\Delta t) - N_k(t) > 1|\mathcal{H}_t) = o(\Delta t),$$

where $\mathcal{H}_t := \sigma\{N_k(s); s \leq t,\ k = 1, \cdots, K\}$ and, for some $\nu_k > 0$ and nonnegative functions $g_{kj}(u)$ satisfying $g_{kj}(u) = 0$ if $u < 0$,

$$\Lambda_k(t) = \nu_k + \sum_{j=1}^{K} \int_{-\infty}^{t} g_{kj}(t-u)dN_j(u).$$

- As a special example of the functions $g_{kj}(u)$, the following is given:

$$g_{kj}(u) = \alpha_{kj} e^{-\beta_{kj} u} 1_{\{u>0\}}(u), \quad \alpha_{kj} \geq 0, \beta_{kj} > 0.$$

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Derivation of the value of MDP (1)

Remember the fair value of MDP is given by

$$V_t^T = E^Q\left[\int_t^T \exp\left(-\int_t^s r_u du\right) C_s^T dL_s^* \middle| \mathcal{F}_t\right].$$

Using the integration-by-parts formula, we have

$$\begin{aligned} V_t^T &= E^Q\left[\exp\left(-\int_t^T r_u du\right) C_T^T L_T^* \middle| \mathcal{F}_t\right] - C_t^T L_t^* - E^Q\left[\int_t^T L_s^* d\left\{\exp\left(-\int_t^s r_u du\right) C_s^T\right\} \middle| \mathcal{F}_t\right] \\ &= E^Q\left[\exp\left(-\int_t^T r_u du\right) C_T^T L_T^* \middle| \mathcal{F}_t\right] - C_t^T L_t^* \\ &\quad + E^Q\left[\int_t^T L_s^* \left\{r_s \exp\left(-\int_t^s r_u du\right) C_s^T ds - L_s^* \exp\left(-\int_t^s r_u du\right) dC_s^T\right\} \middle| \mathcal{F}_t\right]. \end{aligned}$$

Derivation of the value of MDP (2)

Note that the price of the default-free zero-coupon bond with maturity T is defined by

$$Z(t,T) := E^Q\left[\exp\left(-\int_t^T r_u du\right)\middle|\mathcal{F}_t\right].$$

For further calculation, we assume the followings.

Assumption

1. *$\{r_t\}$ and $\{L_t^*\}$ are independent under Q.*
2. *Under Q, the continuous process C_t^T follows*

$$dC_t^T = \mu^C(t,T)dt + \sigma^C(t,T)dW_t^C,$$

where $\mu^C(t,T)$ and $\sigma^C(t,T)$ are $(\mathcal{F}_t)$-adapted processes satisfying some technical conditions, and W_t^C is a $(Q,(\mathcal{F}_t))$-standard Brownian motion that is independent of $r_t, Z(t,T)$ and L_t^*.

Derivation of the value of MDP (3)

Under the above assumptions, we have

$$V_t^T = Z(t,T)E^Q\left[C_T^T|\mathcal{F}_t\right]E^Q\left[L_T^*|\mathcal{F}_t\right] - C_t^T L_t^*$$
$$+ \int_t^T E^Q\left[L_s^*|\mathcal{F}_t\right]E^Q\left[\exp\left(-\int_t^s r_u du\right)\left\{r_s C_s^T - \mu^C(s,T)\right\}\Big|\mathcal{F}_t\right]ds.$$

($E^Q\left[\int_t^T \cdots dW_s^C \Big|\mathcal{F}_t\right]$ vanishes due to its martingale property.)

Remark that there exists an $(\mathcal{F}_t)$-adapted positive process σ_t^T such that

$$dZ(t,T) = Z(t,T)\{r_t dt + \sigma_t^T dW_t^Z\}, \quad Z(T,T) = 1,$$

where W_t^Z is another $(Q,(\mathcal{F}_t))$-standard Brownian motion that is independent of r_t and L_t^*.

Derivation of the value of MDP (4)

Moreover we specify the form of C_t^T as follows

- For given T, C_t^T is given by $Z(t,T)\varphi(t,T)$, where $\varphi(t,T)$ is an $(\mathcal{F}_t)$-adapted process defined by

$$\varphi(t,T) := \int_t^T E^Q[\overline{h}_u|\mathcal{F}_t]du,$$

and the process $\overline{h}_t$ follows under Q

$$d\overline{h}_u = \alpha(\beta - \overline{h}_t)dt + \sigma_h dW_t^h, \quad \overline{h}_0 > 0$$

where α, β and σ_h are positive constants and W_t^h is another $(Q,(\mathcal{F}_t))$-standard Brownian motion that is independent of r_t, L_t^* and W_t^Z.

It is easy to see that for $s \geq t$

$$\overline{h}_s = \overline{h}_t e^{-\alpha(s-t)} + \beta\left(1 - e^{-\alpha(s-t)}\right) + \sigma_h e^{-\alpha(s-t)} \int_0^{s-t} e^{\alpha u} dW_u^h.$$

Derivation of the value of MDP (5)

Therefore we can obtain

$$E^Q[\bar{h}_s|\mathcal{F}_t] = (\bar{h}_t - \beta)e^{-\alpha(s-t)} + \beta.$$

Hence

$$\varphi(t,T) = \int_t^T \{(\bar{h}_t - \beta)e^{-\alpha(u-t)} + \beta\}du = \frac{\bar{h}_t - \beta}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) + \beta(T-t).$$

In addition, we can make sure the dynamics of $\varphi(t,T)$ is given by $\varphi(T,T) = 0$ and

$$d\varphi(t,T) = -\bar{h}_t dt + \frac{\sigma_h\left(1 - e^{-\alpha(T-t)}\right)}{\alpha} dW_t^h.$$

At last, we achieve

$$\begin{aligned} dC_t^T &= \varphi(t,T)dZ(t,T) + Z(t,T)d\varphi(t,T) \\ &= \{r_t C_t^T - Z(t,T)\bar{h}_t\}dt + C_t^T \sigma_t^Z dW_t^Z + Z(t,T)\frac{\sigma_h\left(1 - e^{-\alpha(T-t)}\right)}{\alpha} dW_t^h. \end{aligned}$$

Derivation of the value of MDP (6)

Since $C_T^T \equiv 0$ and $\mu^C(s,T) = r_s C_s^T - Z(s,T)\bar{h}_s$, we can see

$$
\begin{aligned}
V_t^T &= -C_t^T L_t^* + \int_t^T E^Q\left[L_s^* | \mathcal{F}_t\right] E^Q\left[\exp\left(-\int_t^s r_u du\right)\left\{r_s C_s^T - r_s C_s^T + Z(s,T)\bar{h}_s\right\} \middle| \mathcal{F}_t\right] ds \\
&= -C_t^T L_t^* + \int_t^T E^Q\left[L_s^* | \mathcal{F}_t\right] E^Q\left[\exp\left(-\int_t^s r_u du\right) E^Q\left[\exp\left(-\int_s^T r_u du\right) \middle| \mathcal{F}_s\right] \bar{h}_s \middle| \mathcal{F}_t\right] ds \\
&= -C_t^T L_t^* + \int_t^T E^Q\left[L_s^* | \mathcal{F}_t\right] E^Q\left[E^Q\left[\exp\left(-\int_t^T r_u du\right) \bar{h}_s \middle| \mathcal{F}_s\right] \middle| \mathcal{F}_t\right] ds \\
&= -C_t^T L_t^* + \int_t^T E^Q\left[L_s^* | \mathcal{F}_t\right] E^Q\left[\exp\left(-\int_t^T r_u du\right) \middle| \mathcal{F}_t\right] E^Q\left[\bar{h}_s | \mathcal{F}_t\right] ds \\
&= -C_t^T + Z(t,T) \int_t^T E^Q[L_s^* | \mathcal{F}_t] E^Q[\bar{h}_s | \mathcal{F}_t] ds \\
&= Z(t,T) \int_t^T \left\{E^Q[L_s^* | \mathcal{F}_t] - L_t^*\right\} E^Q[\bar{h}_s | \mathcal{F}_t] ds.
\end{aligned}
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