An optimal stopping problem related to cash-flows of investments under uncertainty

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Position of the problem

Let Y^1 and Y^2 denote the expected profit and cost yields respectively. The constituants of the cash flows are:

- The profit yield per unit time dt is ψ^1 and the cost yield is ψ^2;
- When exiting/abandoning the project at time t, the incurred cost is $a(t)$ and the incurred profit is $b(t)$ (usually $a \neq b$ but often non-negative).

Exit/abandonment strategy:

The decision to exit the project at time t, depends on whether

$$Y_t^1 \leq Y_t^2 - a(t) \text{ or } Y_t^2 \geq Y_t^1 + b(t).$$

A Snell envelop formulation

If $\mathcal{F}_t$ denotes the history of the project up to time t, the expected profit yield, at time t, is

$$Y_t^1 = \text{ess sup}_{\tau \geq t} E\left[\int_t^\tau \psi^1(s, Y_s^1)ds + \left(Y_\tau^2 - a(\tau)\right) 1_{[\tau<T]} + \xi^1 1_{[\tau=T]} | \mathcal{F}_t\right]$$

where, the sup is taken over all exit times τ from the project.

The optimal exit time related to the incurred cost $Y^2 - a$ should be

$$\tau_t^* = \inf\{s \geq t,\ \ Y_s^1 = Y_s^2 - a(s)\} \wedge T.$$

The expected cost yield at time t, is

$$Y_t^2 = \text{ess inf}_{\sigma \geq t} E\left[\int_t^\sigma \psi^2(s, Y_s^2)ds + \left(Y_\sigma^1 + b(\sigma)\right) 1_{[\sigma < T]} + \xi^2 1_{[\sigma = T]} | \mathcal{F}_t\right]$$

where, the inf is taken over all exit times σ from the project.

The optimal exit time related to the incurred profit $Y^1 + b$ should be

$$\sigma_t^* = \inf\{s \geq t,\ Y_s^2 = Y_s^1 + b(s)\} \wedge T.$$

Problem formulation

Establish existence and uniqueness of (Y^1, Y^2) which solves the coupled system of Snell envelops

$$Y_t^1 = \text{ess}\ \sup_{\tau \geq t} E\left[\int_t^\tau \psi^1(s, Y_s^1)ds + \left(Y_\tau^2 - a(\tau)\right) 1_{[\tau < T]} + \xi^1 1_{[\tau = T]} |\mathcal{F}_t\right]$$

$$Y_t^2 = \text{ess}\ \inf_{\sigma \geq t} E\left[\int_t^\sigma \psi^2(s, Y_s^2)ds + \left(Y_\sigma^1 + b(\sigma)\right) 1_{[\sigma < T]} + \xi^2 1_{[\sigma = T]} |\mathcal{F}_t\right]$$

Related problems

- One-sided obstacles: The switching problem;
- Fully two-sided obstacles: The switching games problem;
- The multiple-phases membrane problem.

Set up

- $B := (B_t)_{0\le t\le T}$ a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$.
- $(\mathcal{F}_t)_{0\le t\le T}$ the completed natural filtration of B.
- $X := (X_t)_{0\le t\le T}$ a diffusion process which stands for factors which determine the price of the underlying commodity we wish to control such as e.g. the price of electricity in the energy market.

The Snell envelop versus reflected BSDEs

- $\mathcal{S}^2$ denotes the set of all right-continuous with left limits processes Y satisfying $E\left(\sup_{t\in[0,T]}|Y_t^2|\right) < \infty$.
- $\mathcal{M}^{d,2}$ denotes the set of $\mathcal{F}$-adapted and d-dimensional processes Z such that $E\left(\int_0^T |Z_s|^2 ds\right) < \infty$.
- $\mathcal{A}^+$ denotes the set of right-continuous with left limits and increasing processes K.
- $\mathcal{A}^{+,2}$ the subset of $\mathcal{A}^+$ consisting of all the processes K satisfying, in addition, $E(K_T^2) < \infty$.

Let $\xi \in L^2(F_T, P)$, $f(t, \omega, y, z)$ be uniformly Lipschitz in (y, z) and is such that $f(t.\omega, 0, 0) \in \mathcal{M}^{1,2}$, and $S := (S_t)_{t \leq T}$ an R-valued, continuous and uniformly square integrable s.t. $S_T \leq \xi$. Assume $\mathcal{F}_t$-adaptation. Then

Theorem (El-Karoui *et al.*, '97) There exists a unique triple $(Y_t, Z_t, K_t)_{t \leq T}$, valued in R^{1+d+1} and F_t-adapted (K continuous and increasing) such that

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \ t \leq T; \\ Y_t \geq S_t \text{ and } \int_0^T (Y_t - S_t) dK_t = 0. \end{cases}$$

In addition, Y satisfies

$$Y_t = \text{ess sup}_{\tau \geq t} E[\int_t^\tau f(s, \omega, Y_s, Z_s)ds + S_\tau 1_{[\tau < T]} + \xi 1_{[\tau = T]} | \mathcal{F}_t].$$

The Markovian framework: Connection with systems of PDEs

Let $(t,x) \in [0,T] \times R^k$ and let $(X_s^{t,x})_{s \le T}$ be the solution of the following standard SDE.

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})du + \int_t^s \sigma(u, X_u^{t,x})dB_u, \; s \in [t,T] \\ X_s^{t,x} = x, \quad \text{if } s \le t. \end{cases}$$

Assume

- $f(s,\omega,y,z) = f(s, X_s^{t,x}(\omega), y, z)$
- $\xi = g(X_T^{t,x})$
- $S_s = h(s, X_s^{t,x})$.

Then, again by a result in El-Karoui *et al.*, ('97), there exists a continuous deterministic function $v(t,x)$ such that, for any $s \in [t, T]$, $Y_s = v(s, X_s^{t,x})$. Moreover v is the unique viscosity solution of

$$\begin{array}{l} \min\{v - h, -\mathcal{G}v - f(t,x,v,\sigma(t,x)D_x v)\} = 0; \\ v(T,x) = g(x), \end{array}$$

where,

$$\mathcal{G} = \partial_t + \mathcal{L},$$

and $\mathcal{L}$ is the infinitesimal generator of $X^{t,x}$.

Cash-flows: A system of reflected BSDEs formulation

By El-Karoui *et al.* '97, (Y^1, Y^2) should solve the following system of RBSDEs:

$$
\begin{cases}
Y_t^1 = \xi^1 + \int_t^T \psi^1(s, Y_s^1)ds + (K_T^1 - K_t^1) - \int_t^T Z_s^1 dB_s, \\
Y_t^2 = \xi^2 + \int_t^T \psi^2(s, Y_s^2)ds - (K_T^2 - K_t^2) - \int_t^T Z_s^2 dB_s, \\
Y_t^1 \ \leq Y_t^2 - a(t), \quad Y_t^2 \geq Y_t^1 + b(t), \ 0 \leq t \leq T, \\
\int_0^T \left(Y_t^1 - (Y_t^2 - a(t))\right) dK_t^1 = 0, \ \int_0^T (Y_t^1 + b(t) - Y_t^2) dK_t^2 = 0.
\end{cases}
$$

Minimal and maximal solutions

We make the following assumptions:

(B1) For each $i = 1, 2$, the process ψ^i depends explicitly on (t, Y_t^i). Moreover, $(t, y) \to \psi^i(t, y)$'s are Lipschitz continuous with respect to y and satisfy,

$$E\left(\int_0^T |\psi^i(t, 0)|^2 ds\right) < \infty.$$

(B2) The obstacles a and b are continuous and in $\mathcal{S}^2$. Moreover, they admit a semimartingale decomposition:

$$a(t) = a(0) + \int_0^t U_s^1 ds + \int_0^t V_s^1 dB_s,$$

$$b(t) = b(0) + \int_0^t U_s^2 ds + \int_0^t V_s^2 dB_s,$$

for some $\mathcal{F}$-prog. meas. processes U^1, V^1, U^2 and V^2.

(B3) ξ^i's are in $L^2(\mathcal{F}_T)$ and satisfy

$$\xi^1 - \xi^2 \geq max\{-a(T), -b(T)\}, \quad P - a.s.$$

The main result

Let the coefficients $(\psi^1, \psi^2, a, b, \xi^1, \xi^2)$ satisfy Assumptions (**B1**)-(**B3**). Then the system of RBSDEs admits a minimal and a maximal $\mathcal{F}$-prog. meas. solutions $(Y^1, Y^2, Z^1, Z^2, K^1, K^2)$ and $(\bar{Y}^1, \bar{Y}^2, \bar{Z}^1, \bar{Z}^2, \bar{K}^1, \bar{K}^2)$, respectively, which are in $(\mathcal{S}^2)^2 \times (\mathcal{M}^{d,2})^2 \times (\mathcal{A}^{+,2})^2$.

Moreover,

- the processes Y^i and $\bar{Y}^i$, $i = 1, 2$ are $P-$a.s. continuous and admit the above Snell representations.
- the random times τ^* and σ^* defined above and associated with either Y^i or $\bar{Y}^i$, are optimal stopping times.

A minimal solution through the increasing sequences scheme

Start with the pair $(Y^{1,0},\ Z^{1,0})$ that solves uniquely the BSDE

$$Y_t^{1,0} = \xi^1 + \int_t^T \psi^1(s, Y_s^{1,0})ds - \int_t^T Z_s^{1,0} dB_s.$$

and introduce the following system of RBSDEs

$$\begin{cases} dY_s^{2,n+1} = \psi^2(s, Y_s^{2,n+1})ds - dK_s^{2,n+1} - Z_s^{2,n+1} dB_s, \\ dY_s^{1,n+1} = \psi^1(s, Y_s^{1,n+1})ds + dK_s^{1,n+1} - Z_s^{1,n+1} dB_s, \\ Y_s^{2,n+1} \geq Y_s^{1,n} + b(s), \quad Y_s^{1,n+1} \leq Y_s^{2,n+1} - a(s),\ 0 \leq s \leq T, \\ \int_0^T (Y_t^{1,n+1} - (Y_t^{2,n+1} - a(t))dK_t^{1,n+1} = 0, \quad Y_T^{1,n+1} = \xi^1; \\ \int_0^T (Y_t^{1,n} + b(t) - Y_t^{2,n+1})dK_t^{2,n+1} = 0, \quad Y_T^{2,n+1} = \xi^2. \end{cases}$$

This sequence of solutions satisfies the following properties:

- For any $n \geq 0$, both $(Y^{1,n}, Z^{1,n}, K^{1,n})$ and $(Y^{2,n+1}, Z^{2,n+1}, K^{2,n+1})$ exist and are in $\mathcal{S}^2 \times \mathcal{M}^{d,2} \times \mathcal{A}^{+,2}$.
- The two sequences $(Y^{1,n})_{n\geq 0}$ and $(Y^{2,n})_{n\geq 1}$ are increasing in n, meaning that for all $n \geq 0$,

$$Y_t^{1,n} \leq Y_t^{1,n+1} \quad \text{and} \quad Y_t^{2,n+1} \leq Y_t^{2,n+2} \;\; P\text{-a.s. and for all } t.$$

- the limit process (Y^1, Y^2) of $(Y_t^{1,n}, Y_t^{2,n})$ is continuous, a minimal solution of our system of RBSDEs and admits a Snell envelop representation.

A maximal solution through the decreasing sequences scheme

Start with the pair $(\bar{Y}^{2,0},\ \bar{Z}^{2,0})$ that solves the standard BSDE

$$\bar{Y}_t^{2,0} = \xi^2 + \int_t^T \psi^2(s, \bar{Y}_s^{2,0})ds - \int_t^T \bar{Z}_s^{2,0} dB_s,$$

and introduce the following system of RBSDEs

$$\begin{cases} d\bar{Y}_s^{1,n+1} = \psi^1(s, \bar{Y}_s^{1,n+1})ds + d\bar{K}_s^{1,n+1} - \bar{Z}_s^{1,n+1} dB_s, \\ d\bar{Y}_t^{2,n+1} = \psi^2(s, \bar{Y}_s^{2,n+1})ds - d\bar{K}_s^{2,n+1} - \bar{Z}_s^{2,n+1} dB_s, \\ \bar{Y}_s^{1,n+1} \le \bar{Y}_s^{2,n} - a(s), \quad \bar{Y}_s^{2,n+1} \ge \bar{Y}_s^{1,n+1} + b(s),\ 0 \le s \le T, \\ \int_0^T (\bar{Y}_t^{1,n+1} - (\bar{Y}_t^{2,n} - a(t))d\bar{K}_t^{1,n+1} = 0, \quad \bar{Y}_T^{1,n+1} = \xi^1, \\ \int_0^T (\bar{Y}_t^{1,n+1} + b(t) - \bar{Y}_t^{2,n+1})d\bar{K}_t^{2,n+1} = 0, \quad \bar{Y}_T^{2,n+1} = \xi^2. \end{cases}$$

This sequence of solutions satisfies the following properties.

- For any $n \geq 0$, both $(\bar{Y}^{2,n}, \bar{Z}^{2,n}, \bar{K}^{2,n})$ and $(\bar{Y}^{1,n+1}, \bar{Z}^{1,n+1}, \bar{K}^{1,n+1})$ exist and are in $\mathcal{S}^2 \times \mathcal{M}^{d,2} \times \mathcal{A}^{+,2}$.
- The two sequences $(\bar{Y}^{1,n})_{n\geq 1}$ and $(Y^{2,n})_{n\geq 0}$ are decreasing in n, meaning that for all $n \geq 0$,

$$\bar{Y}_t^{1,n} \geq \bar{Y}_t^{1,n+1} \quad \text{and} \quad \bar{Y}_t^{2,n+1} \geq \bar{Y}_t^{2,n+2} \; P\text{-a.s. and for all } t.$$

- the limit process $(\bar{Y}^1, \bar{Y}^2)$ of $(\bar{Y}_t^{1,n}, \bar{Y}_t^{2,n})$ is continuous, a maximal solution of our system of RBSDEs and admits a Snell envelop representation.

Non-uniqueness: A counter example

Assume

- $\psi^1(t, \omega, y) = y$ and $\psi^2(t, \omega, y) = 2y$,
- $a = b = 0$ and $\xi^1 = \xi^2 = 1$.

The corresponding system of BSDEs is

$$\begin{cases} Y_t^1 = 1 + \int_t^T Y_s^1 ds - \int_t^T Z_s^1 dB_s + \left(K_T^1 - K_t^1\right), \\ Y_t^2 = 1 + 2\int_t^T Y_s^2 ds - \int_t^T Z_s^2 dB_s - \left(K_T^2 - K_t^2\right), \\ Y_t^1 \geq Y_t^2, \quad t \leq T, \\ \int_0^T \left(Y_s^1 - Y_s^2\right) d(K_s^1 + K_s^2) = 0. \end{cases}$$

It can be ckecked that

$$\left(e^{T-t}, e^{T-t}, 0, 0, 0, e^{T}(1-e^{-t})\right)$$

and

$$\left(e^{2(T-t)}, e^{2(T-t)}, 0, 0, \frac{1}{2}e^{2T}(1-e^{-2t}), 0)\right)$$

are solutions of the system of BSDEs.

A uniqueness result

Theorem. Assume that

(i) the mappings ψ^1 and ψ^2 do not depend on y, i.e., $\psi_i := (\psi_i(t,\omega))$, $i = 1, 2$,

(ii) the barriers a and b satisfy

$$P - a.s. \quad \int_0^T 1_{[a(s)=b(s)]} ds = 0.$$

Then, the solution of the system of BSDE's is unique.

The Markovian framework. A PDE formulation

When the dependence of (Y^1, Y^2) on the sources of uncertainty (the diffusion process $X^{t,x}$) is explicit, we can show that there exists two deterministic functions u^1 and u^2 such that

$$Y_s^1 = u^1(s, X_s^{t,x}), \quad Y_s^2 = u^2(s, X_s^{t,x}),$$

and are viscosity solutions of the following system of variational inequalities:

$$\begin{cases} \min\{u^1(t,x) - u^2(t,x) + a(t), -\mathcal{G}u^1(t,x) - \psi^1(t,x,u^1(t,x))\} = 0, \\ \max\{u^1(t,x) + b(t) - u^2(t,x), \mathcal{G}u^2(t,x) + \psi^2(t,x,u^2(t,x))\} = 0, \\ u^1(T,x) = g^1(x), \quad u^2(T,x) = g^2(x). \end{cases}$$

Through a counter-example, we can show that the system may have infinitely many solutions.

Some references

- BD, S. Hamadène and M-E. Morlais (2009): Optimal stopping of expected profit and cost yields in an investment under uncertainty (*Preprint*).
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- T. Arnarsson, BD, M. Poghosyan, H. Shahgholian (2009): A PDE approach to regularity of solutions to finite horizon optimal switching problems. Nonlinear Analysis: Theory, Methods & Applications.