The Continuous-Time Principal-Agent Problem with Moral Hazard and Recursive Preferences

Mark Schroder[1], Sumit Sinha[2] and Shlomo Levental[3]

May 2010

[1]Mark Schroder , Department of Finance, Michigan State University.
[2]Sumit Sinha , Department of Statistics and Probability, Michigan State University.
[3]Shlomo Levental, Department of Statistics and Probability, Michigan State University.

Some Important References

- Schattler, H., and J. Sung (1993)
- Cvitanic,J., X. Wan, and J. Zhang (2008)
- Schroder, M., and C. Skiadas (2003, 2005)
- P. Briand and F. Confortola (2008)

Contribution

- Extends existing results to recursive preferences for agent and principal
- Special cases: time-additive utility, stochastic differentiable utility, differences in beliefs, robust control and multiple-prior formulations,
- Characterize general solution as FBSDE system
- Solution simplifies considerably with translation-invariant (generalized exponential utility) and scale-invariant (homothetic) preferences.
- Solution reduces to Riccati ODE system for quadratic penalties and affine-type state variable dynamics.

Setup

- All uncertainty is generated by d-dimensional standard Brownian motion B over the finite time horizon $[0, T]$.
- The set of *consumption plans* is the extended convex set $\mathcal{C} \subset \mathcal{L}_2(\mathbb{R})$. For any $c \in \mathcal{C}$, we interpret c_t as a consumption rate for $t < T$, and c_T as lump-sum terminal consumption.

Definition

X, a collection of stochastic processes is *extended convex* if $\forall\ x_1, x_2 \in X$ there is a process $\delta = \delta(\omega, t; x_{1,} x_2) > 0$ s.t.

$$\alpha x_1 + (1 - \alpha) x_2 \in X$$

for each $\alpha(\omega, t)$ that satisfies $-\delta \leq \alpha \leq 1 + \delta$.

Setup

- The set of *effort plans* is $\mathcal{E} = \{e \in \mathcal{L}_2(\mathbf{E});\ e_t \in E_t\ ;\ \forall 0 \leq t \leq T\}$ with $e_T = 0$(no lump-sum terminal effort), where $E_t \subset \mathbf{E} \subset \mathbb{R}^d$,
- The impact of agent effort is modelled as a change of probability measure.
- Define the probability measure P^e corresponding to effort e , so by Girsanov's Theorem $dB_t^e = dB_t - e_t dt$ is standard Brownian motion under P^e.

Setup

- The agent's utility $U(c,e)$ is part of the pair $\left(U,\Sigma^U\right)$ assumed to uniquely satisfy the BSDE

$$dU_t = -F\left(t, c_t, e_t, U_t, \Sigma_t^U\right) dt + \Sigma_t^{U\prime} dB_t^e, \quad U_T = F\left(T, c_T\right). \quad (1)$$

- The principal's utility $V(c,e)$ is part of the pair $\left(V,\Sigma^V\right)$ assumed to uniquely satisfy the BSDE

$$dV_t = -G\left(t, c_t, V_t, \Sigma_t^V\right) dt + \Sigma_t^{V\prime} dB_t^e, \quad V_T = G(T, c_T), \quad (2)$$

Remark: $c \in \mathcal{C}$ is admissible for the agent if $U_0(c) \geqslant K$ (participation constraint)

Statement of the Problem

- Given any $c \in \mathcal{C}$, the agent chooses effort to maximize his\her utility:

$$U_0(c) = \sup_{e \in \mathcal{E}} U_0(c, e).$$

Letting $e(c)$ denote the optimal agent effort level induced by consumption process c, the principal's problem is:

$$\sup_{c \in \mathcal{C}} V_0(c, e(c)) \text{ subject to } U_0(c) \geq K.$$

Agent Optimality

Theorem

Fix some $c \in \mathcal{C}$ and suppose integrability Conditions holds. Then $e \in \mathcal{E}$ is optimal if and only if for any $\tilde{e} \in \mathcal{E}$

$$F\left(c_t, e_t, U_t, \Sigma_t^U\right) + \Sigma_t^{U\prime} e_t \geq F\left(c_t, \tilde{e}_t, U_t, \Sigma_t^U\right) + \Sigma_t^{U\prime} \tilde{e}_t, \quad t \in [0, T) \tag{3}$$

where $U_t = U_t\,(c, e)$ and $\Sigma_t^U = \Sigma_t^U\,(c, e)$ solve the BSDE (1).

Proof.

The proof is mainly the use of Comparison Theorem for BSDEs. □

If the solution is interior, then (3) is equivalent to

$$-F_e\left(t, c_t, e_t, U_t, \Sigma_t^U\right) = \Sigma_t^U.$$

We will assume that above equation can be inverted to get optimal effort $e_t = I\left(\omega, t, c, U, \Sigma^U\right)$.

Principal Optimality

Using $e_t = I\left(\omega, t, c, U, \Sigma^U\right)$, the principal's problem is

$$\sup_{c \in \mathcal{C}} V_0(c) \text{ subject to } U_0(c) \geq K \tag{4}$$

where $\left(U, \Sigma^U, V, \Sigma^V\right)$ satisfy the BSDE system

$$\begin{aligned} dU_t &= -\bar{F}\left(t, c_t, U_t, \Sigma_t^U\right) dt + \Sigma_t^{U\prime} dB_t, \quad U_T = F(T, c_T), \\ dV_t &= -\bar{G}\left(t, c_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U\right) dt + \Sigma_t^{V\prime} dB_t, \quad V_T = G(T, c_T), \end{aligned} \tag{5}$$

with modified aggregators.

Gradient and Supergradient density

Definition

Let $\nu : \mathcal{C} \rightarrow \mathbb{R}$ be a functional. For any $c \in \mathcal{C}$, the process $\pi \in \mathcal{L}_2(\mathbb{R})$ is a *supergradient density* of ν at c if

$$\nu(c+h) - \nu(c) \leq E\left[\int_0^T \pi'_t h_t dt + \pi'_T h_T\right], \quad \forall\ h \text{ such that } c+h \in \mathcal{C},$$

and $\pi \in \mathcal{L}_2(\mathbb{R})$ is a *gradient density* at c if

$$E\left[\int_0^T \pi'_t h_t dt + \pi'_T h_T\right] = \lim_{\alpha \downarrow 0} \frac{\nu(c+\alpha h) - \nu(c)}{\alpha} \quad \forall\ h \text{ s.t. } c + \alpha h \in \mathcal{C}\ .$$

Gradient and Supergradient density

The computation of π, the densities above, require the following $\mathbb{R}^2$-valued adjoint process $\varepsilon_t = \left(\varepsilon_t^V, \varepsilon_t^U\right)'$, with some initial value $\varepsilon_0 \in \mathbb{R}^2$ and dynamics

$$d\varepsilon_t = \begin{pmatrix} \bar{G}_V(t) & 0 \\ \bar{G}_U(t) & \bar{F}_U(t) \end{pmatrix} \varepsilon_t dt + \begin{pmatrix} \bar{G}_{\Sigma^V}(t)' \, dB_t & 0 \\ \bar{G}_{\Sigma^U}(t)' \, dB_t & \bar{F}_\Sigma(t)' \, dB_t \end{pmatrix} \varepsilon_t . \quad (6)$$

Gradient and Supergradient density

Lemma

Suppose $c \in \mathcal{C}$ *and that* ε *satisfies* (6) *with initial value* $\varepsilon_0 \in \mathbb{R}^2_+$.

- *Under certain Integrability Condition* $\{[\bar{G}_c(t), \bar{F}_c(t)]\,\varepsilon_t;\ \ t \in [0, T]\}$ *is a utility gradient of* $[V_0(c), U_0(c)]\,\varepsilon_0$ *at* c .
- *Under Integrability Condition (different from the previous part)* $\{[\bar{G}_c(t), \bar{F}_c(t)]\,\varepsilon_t;\ \ t \in [0, T]\}$ *is a utility supergradient of* $[V_0(c), U_0(c)]\,\varepsilon_0$ *at* c .

Proof.

One of the methods to prove the above lemma is to use derivatives of the solution of BSDE. A result in this direction can be found in Briand and Confortola(2006). □

Principal Optimality

Theorem

Let $c \in \mathcal{C}$, $\left(U, \Sigma^U, V, \Sigma^V\right)$ solve the BSDE system (5) , let ε be the adjoint process. Assume appropriate integrability condition holds. Then c solves the principal's problem iff there is some $\kappa \in \mathbb{R}_+$ such that

$$\varepsilon_0 = (1, \kappa)', \quad \left[\bar{G}_c(t), \bar{F}_c(t)\right] \varepsilon_t = 0, \quad t \in [0, T], \qquad (7)$$
$$\kappa \left\{ U_0(c) - K \right\} = 0.$$

Proof.

The proof is based on a version of Kuhn-Tucker Theorem. □

Principal Optimality

Define

$$\lambda_t = -\frac{\bar{G}_c\left(t, c_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U\right)}{\bar{F}_c\left(t, c_t, U_t, \Sigma_t^U\right)}, \quad t < T, \quad \lambda_T = -\frac{\bar{G}_c\left(T, c_T\right)}{\bar{F}_c\left(T, c_T\right)}, \tag{8}$$

Under the FOCs (7) we have

$$\lambda_t = \frac{\varepsilon_t^U}{\varepsilon_t^V}, \tag{9}$$

where $\varepsilon_0 = (1, \kappa)'$ for some $\kappa \geq 0$. From (9) we get by Ito's Lemma:

$$d\lambda_t = \left\{\lambda_t \bar{F}_U(t) - \lambda_t \bar{G}_V(t) + \bar{G}_U(t) - \bar{G}_{\Sigma^V}' \Sigma_t^\lambda\right\} dt + \Sigma_t^{\lambda\prime} dB_t, \tag{10}$$

$$\text{where } \Sigma_t^\lambda = \lambda_t \left\{\bar{F}_\Sigma(t) - \bar{G}_{\Sigma^V}\right\} + \bar{G}_{\Sigma^U}(t).$$

We will assume that (8) can be inverted to present consumption as $c_t = \phi\left(\lambda_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U\right)$:

Solution

The first-order conditions for the problem is a FBSDE system for $(U, \Sigma^U, V, \Sigma_t^V, \lambda)$:

$$
\begin{aligned}
dU_t &= -\bar{F}\left(t, c_t, U_t, \Sigma_t^U\right) dt + \Sigma_t^{U\prime} dB_t, \quad U_T = F\left(T, c_T\right), \\
dV_t &= -\bar{G}\left(t, c_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U\right) dt + \Sigma_t^{V\prime} dB_t, \quad V_T = G\left(T, c_T\right) \\
d\lambda_t &= \left\{\lambda_t \bar{F}_U\left(t\right) - \lambda_t \bar{G}_V\left(t\right) + \bar{G}_U\left(t\right) - \bar{G}_{\Sigma^V}' \Sigma_t^\lambda\right\} dt + \Sigma_t^{\lambda\prime} dB_t, \\
\text{where } \Sigma_t^\lambda &= \lambda_t \left\{\bar{F}_\Sigma\left(t\right) - \bar{G}_{\Sigma^V}\right\} + \bar{G}_{\Sigma^U}\left(t\right), \ \lambda_0 = \kappa \geq 0, \\
U_0 &\geq K, \quad \kappa\left(U_0 - K\right) = 0, \\
c_t &= \phi\left(t, \lambda_t, V_t, \Sigma_t^V, U_t, \Sigma_t^U\right), \quad c_T = \phi\left(t, \lambda_T\right).
\end{aligned}
$$

Another approach to solving Principal's problem

- The function $J : \Omega \times [0, T] \times \mathbf{C} \times \mathbf{E} \times \mathbb{R} \to \mathbb{R}^d$ is well-defined implicitly by

$$e = I\left(\omega, t, c, U, J\left(\omega, t, c, e, U\right)\right), \quad e \in \mathbf{E}. \tag{11}$$

- With this invertibility condition, the agent's optimal effort $e_t = I\left(t, c_t, U_t, \Sigma_t^U\right)$ is equivalent to $\Sigma_t^U = J\left(t, c_t, e_t, U_t\right)$.
- Substituting $\Sigma_t^U = J\left(t, c_t, e_t, U_t\right)$ into agent's utility function and assuming that participation constraint is binding we get:

$$dU_t = -\left\{F\left(t, c_t, e_t, U_t, J\left(t\right)\right) + J\left(t\right)' e_t\right\} dt + J\left(t\right)' dB_t, \quad U_0 = K. \tag{12}$$

The lump-sum terminal consumption implied by effort plan e is

$$c_T = F^{-1}\left(T, U_T\right), \tag{13}$$

The principal's problem is equivalent to choosing e and $\{c_t; t \in [0, T)\}$ to maximize $V_0(c, e)$ subject to the initial value of the agent utility (now a forward equation) satisfying the participation constraint:

$$\sup_{c,e \in \mathcal{C} \times \mathcal{E}} V_0(c, e) \text{ subject to}$$

$$\begin{aligned} dU_t &= -\{F(t, c_t, e_t, U_t, J(t))\}\, dt + J(t)'\, dB_t^e, \quad U_0 = K, \\ dV_t &= -G\left(t, c_t, V_t, \Sigma_t^V\right) dt + \Sigma_t^{V\prime} dB_t^e, \quad V_T = G\left(T, F^{-1}(T, U_T)\right) \end{aligned}$$

Lemma

Under Integrability conditions the principal's optimality conditions are equivalent to

$$
\begin{aligned}
0 &= G_c(t) + \lambda_t F_c(t) + J_c(t)' \left\{ \lambda_t F_\Sigma(t) - \lambda_t G_{\Sigma^V}(t) - \Sigma_t^\lambda \right\}, \quad (14) \\
0 &= \Sigma_t^V + J_e(t)' \left\{ \lambda_t F_\Sigma(t) - \lambda_t G_{\Sigma^V}(t) - \Sigma_t^\lambda \right\}.
\end{aligned}
$$

and

$$
\begin{aligned}
d\lambda_t = \{ & \lambda_t F_U(t) - \lambda_t G_V(t) - J_U(t)' \left[\Sigma_t^\lambda - \lambda_t F_\Sigma(t) + \lambda_t G_{\Sigma^V}(t) \right] \\
& - G_\Sigma' \Sigma_t^\lambda \} dt + \Sigma_t^{\lambda\prime} dB_t^e, \qquad \lambda_T = -\frac{G_c\left(T, F^{-1}(T, U_T)\right)}{F_c\left(T, F^{-1}(T, U_T)\right)}.
\end{aligned}
$$

Proof.

The prove is based on the fact $e = I(\omega, t, c, U, J(\omega, t, c, e, U))$ and the previous principal optimality conditions(See (8) and (10)). □

Example (Cvitanic, Wan & Zhang, 2008)

Suppose there is no intermediate consumption and the penalty for agent effort is quadratic:

$$\begin{aligned} F(t, c, e, \Sigma) &= -\frac{1}{2} q e' e, \quad G(t, c, V, \Sigma) = 0, \quad t < T, \\ F(T, c_T) &= f(c_T), \quad G(T, c_T) = g(X_T - c_T), \end{aligned}$$

for some $q > 0$ and cash-flow X_T.

- Agent optimality implies $\Sigma_t^U = J(t, e_t) = q e_t$, and

$$d\lambda_t = \Sigma_t^{\lambda\prime} dB_t^e, \quad \lambda_T = \frac{g'(X_T - c_T)}{f'(c_T)}.$$

Example Contd.

- The principal's optimality condition reduces to $\Sigma_t^V = q\Sigma_t^\lambda$ which implies the key simplification $dV_t = qd\lambda_t$. So for some constant β

$$V_t - q\lambda_t = \beta \quad \forall t \in [0, T]$$

$$\beta = g\left(X_T - c_T\right) - q\left(\frac{g'\left(X_T - c_T\right)}{f'\left(c_T\right)}\right) \tag{15}$$

  which can be used to solve implicitly for c_T as a function of β and X_T.

- To solve for β, observe that $u_t = \exp\left(U_t/q\right)$ is a Martingale (By Ito Lemma, $du_t = u_t e_t' dB_t$). Since $u_0 = e^K$)

$$\exp\left(K\right) = E(u_T) = E\exp\left(\frac{f\left(c_T\right)}{q}\right).$$

- The martingale representation theorem gives the optimal effort e.

Translation Invariant(TI) Preferences

The agent's and principal's aggregators are of the form

$$\begin{aligned} F(\omega, t, c, e, U, \Sigma) &= f\left(\omega, t, \frac{c}{\gamma^U} - U, e, \Sigma\right), \quad F(T, c) = \frac{c}{\gamma^U}, \\ G(\omega, t, c, V, \Sigma) &= g\left(\omega, t, \frac{X(\omega, t) - c}{\gamma^V} - V, \Sigma\right), \\ G(\omega, T, c) &= \frac{X(\omega, T) - c}{\gamma^V}, \end{aligned}$$

for some constants $\gamma^U, \gamma^V \in \mathbb{R}_{++}$ and some functions $f : \Omega \times [0, T] \times \mathbb{R}^{1+2d} \to \mathbb{R}$ and $g : \Omega \times [0, T] \times \mathbb{R}^{1+d} \to \mathbb{R}$, which we refer to as absolute aggregators. $X(\omega, t)$ is the cashflow process.

Translation Invariant Example

Lemma

Under the TI preferences, at the optimum (c, e), $\lambda_t = \frac{\gamma^U}{\gamma^V}$, $t \in [0, T]$.

- We demonstrate an example with TI preferences with quadratic volatility and effort penalties ,where we get explicit expression for optimal (c, e).

$$
\begin{aligned}
f\left(\omega, t, x^U, e, \Sigma\right) &= h^U\left(\omega, t, x^U\right) + p^U(\omega, t)'\Sigma \\
&\quad -\frac{1}{2}q^U(\omega, t)\Sigma'\Sigma - \frac{1}{2}q^e(\omega, t)e'e, \\
g\left(\omega, t, x^V, \Sigma\right) &= h^V\left(\omega, t, x^V\right) + p^V(\omega, t)'\Sigma - \frac{1}{2}q^V(\omega, t)\Sigma'\Sigma
\end{aligned}
\qquad (16)
$$

- where $x_t^U = c_t/\gamma^U - U_t$, $x_t^U = \frac{X_t - c_t}{\gamma^V} - V_t$,and $q^e, q^U, q^V \in \mathcal{L}(\mathbb{R}_+)$ represent the effort and risk-aversion penalties, and $p^U, p^V \in \mathcal{L}(\mathbb{R}^d)$ can be interpreted as differences in beliefs of the agent and principal from the true probability measure

Translation Invariant Example Contd.

Let

$$w_t = \frac{1 + \lambda q_t^e q_t^V}{1 + \lambda q_t^e q_t^V + q_t^e q_t^U}, \tag{17}$$

then $J(t, e_t) = q_t^e e_t$ and the optimal effort satisfies

$$e_t = \frac{w_t}{\lambda q_t^e} \Sigma_t^Y + \frac{1 - w_t}{q_t^e q_t^U} \left(p_t^U - p_t^V \right). \tag{18}$$

and optimal $x_t^U = \phi \left(t, \frac{X_t}{\gamma^V} - Y_t \right)$, which we get from Principal's optimality equation.

Translation Invariant Example Contd.

Define $Y_t = V_t + \lambda U_t$

The BSDE for Y is

$$dY_t = -\left\{ H\left(t, \frac{X_t}{\gamma^V} - Y_t\right) + \mu_t^Y + p_t^{Y\prime}\Sigma_t^Y - \frac{1}{2} q_t^Y \Sigma_t^{Y\prime}\Sigma_t^Y \right\} dt + \Sigma_t^{Y\prime} dB_t \tag{19}$$

$$Y_T = \frac{X_T}{\gamma^V}$$

where

$$H(\omega, t, x) = h^V(\omega, t, -\lambda\phi(\omega, t, x) + x) + \lambda h^U(\omega, t, \phi(\omega, t, x)),$$

$$\mu_t^Y = \frac{1}{2}\frac{\lambda(1 - w_t)}{q_t^U}\left\| p_t^U - p_t^V \right\|^2,$$

$$p_t^Y = w_t p_t^U + (1 - w_t) p_t^V,$$

$$q_t^Y = \frac{1}{\lambda}\left(q_t^U w_t - \frac{1}{q_t^e}\right),$$

Translation Invariant Example Contd.

In the case of constant q^e, q^U and q^V, we can rearrange (19) to obtain an expression for $\Sigma_t^{Y\prime} dB_t$, and substitute into FSDE for U. The terminal consumption is given by

$$
\begin{aligned}
c_T &= wX_T + \gamma^U (1-w)\left(K - \int_0^T h^U\left(t, x_t^U\right) dt\right) \\
&- \gamma^V w \left(V_0 - \int_0^T h^V\left(t, x_t^V\right) dt\right) + \\
&\frac{1}{2}\gamma^U \frac{w(1-w)}{\lambda^2 q^e} \int_0^T \left\|\Sigma_t^Y\right\|^2 dt - \gamma^V \frac{w(1-w)}{q^U q^e} \int_0^T \left(p_t^U - p_t^V\right)' \Sigma_t^Y dt \\
&+ \frac{\gamma^U}{2}\left(\frac{1-w}{q^U}\right) \int_0^T \left(\left\|p_t^V\right\|^2 - \left\|p_t^U\right\|^2 - \left(\frac{1-w}{q^e q^U}\right)\left\|p_t^U - p_t^V\right\|^2\right) dt \\
&+ \gamma^U \frac{1-w}{q^U} \int_0^T \left(p_t^U - p_t^V\right)' dB_t .
\end{aligned}
$$

THANK YOU

The working paper is available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1573246.