Optimal Contracting and Nash Equilibria in the Continuous-Time Principal-Agent Problem with Multiple Principals

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Some Important References

- Schatttler, H., and Sung, J.(1993),
- Koo, H., Shim, G. and Sung, J. (2006)
- Davis, M.(1979),   Li, Q. and Karatzas, I.(2009)
- Schroder, M. and Skiadas, C. (2003, 2005)
- Schroder, M., Sinha, S. and Levental, S. (2010)

Basic Elements of The Problem

We use the superscripts $\{a, b\}$ to denote the two principals and agents respectively.

- There are 2 principals, with utility processes V^a, V^b
- Each principal of $\{a, b\}$ has 1 agent with utility processes U^a, U^b respectively.

Consumption Process

- All uncertainty is generated by d-dimensional standard Brownian motion B over the finite time horizon $[0, T]$.
- The set of *consumption plans* is an extended convex set $\mathcal{C} \subseteq \mathcal{H}_1$. For any $c = \{c^a, c^b\} \in \mathcal{C} \times \mathcal{C}$, we interpret c_t^i as consumption rate at $t < T$, and c_T^i as lump-sum terminal consumption for agent i, $i = \{a, b\}$.

Effort Process

- We define the set of *effort plans* as the convex set $\mathcal{E} \subseteq \mathcal{L}_2(\mathbf{E})$ for some set $\mathbf{E} \subset \mathbb{R}^d$. $e = \{e^a, e^b\} \in \mathcal{E} \times \mathcal{E}$ are agent a and b's effort plans respectively
- As in Koo, Shim and Sung (2006) and Li and Karatzas(2009), Define function $\Phi(e) : E \times E \to \mathbb{R}^d$, where $e = \{e^a, e^b\} \in \mathcal{E} \times \mathcal{E}$.

Define $dB_t^e = dB_t - \Phi(e_t)\, dt$. and probability measure P^e such that B_t^e is a standard Brownian motion under the new measure P^e.

- **Agent Utility**

The i'th agent's utility U^i is part of the pair $\left(U_t^i, \Sigma_t^{U^i}\right) \in \mathbb{R} \times \mathbb{R}^d$, $0 \leq t \leq T$, assumed to uniquely satisfy the BSDE

$$\begin{aligned} dU_t^i\left(c_t^i, e_t^i, e_t^j\right) &= -F^i\left(\omega, t, c_t^i, e_t^i, U_t^i, \Sigma_t^{U^i}\right) ds + \Sigma_t^{U^i\prime} dB_t^{(e^i, e^j)} \\ U_T^i &= F^i\left(T, c_T^i\right), \; U_0^i \geq K^i, \; i, j = \{a, b\}, i \neq j \end{aligned}$$

$dB_t^{(e^i, e^j)} = dB_t - \Phi\left(e_t^i, e_t^j\right) dt$ is the mechanism by which agent and principal utilities are interlinked.

- **Principal Utility**

For any agent's effort choice $e^i \in \mathcal{E}$, principal $i's$ utility V^i is part of the pair $\left(V_t^i, \Sigma_t^{V_i}\right)$ assumed to uniquely satisfy the BSDE

$$\begin{aligned} dV_t^i\left(c_t^i, e_t^i, e_t^j\right) &= -G^i\left(\omega, t, c_t^i, V_t^i, \Sigma_t^{V^i}\right)dt + \Sigma_t^{V^i\prime} dB_t^{(e^i,e^j)} \\ V_T^i &= G(T, c_T^i), i,j = \{a,b\}, i \neq j \end{aligned} \quad (1)$$

The effort is not contractable, but principal i knows both agents' aggregator functions(and therefore the agents' optimality condition) and chooses agent consumption process c^i to maximize utility.

Nash Equilibrium

- **Agent Equilibrium**

Definition

The set of Nash equilibrium efforts $\{e^{a*}, e^{b*}\}$ in our setting satisfies,

$$\begin{aligned} U_0^i(c^i, e^{i*}, e^{j*}) &\geq U_0^i(c^i, e^i, e^{j*}) \ \forall e^i \in \mathcal{E} \\ i,j &= \{a,b\}, i \neq j \end{aligned} \tag{2}$$

Agent i does not observe the other agent's effort plan, but knows both agents' consumptions $\{c^a, c^b\}$ and the other agent's aggregator, and will choose optimal effort plan e^{i*} to maximize utility

- **Principal Equilibrium**

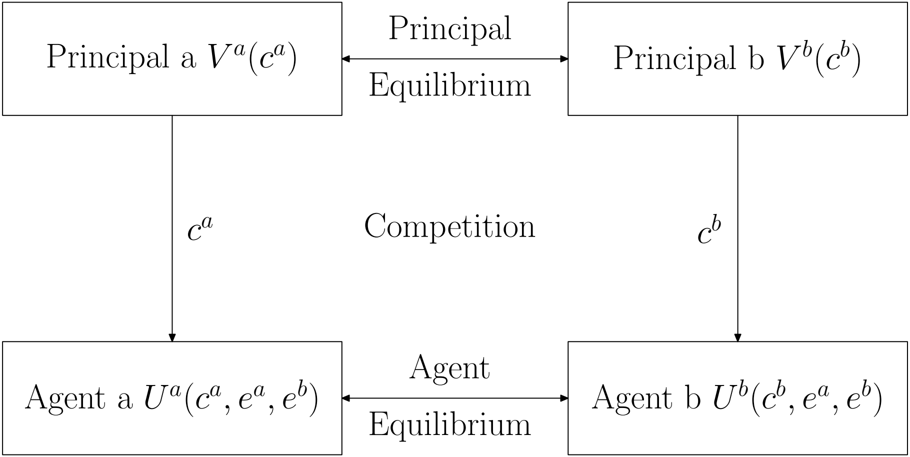
Definition

. The set of Nash equilibrium consumptions $\{c^{a*}, c^{b*}\}$ satisfies

$$\begin{aligned} V_0^i(c^{i*}, e^*(c^{i*}, c^{j*})) &\geq V_0^i(c^i, e^*(c^i, c^{j*})) \quad (3)\\ \text{subject to } U_0^i(c^i, e^{i*}, e^{j*}) &\geq U_0^i(c^i, e^i, e^{j*}) \\ U_0^i(c^{i*}, e^{i*}, e^{j*}) &\geq K^i \\ \forall c^i &\in \mathcal{C},\ i,j = \{a,b\}, i \neq j \end{aligned}$$

The effort process is not contractable, but principal i knows both agents' aggregator functions(and therefore agents' equilibrium condition) and chooses agent i's consumption process c^i to maximize utility.

Nash Equilibrium



A Necessary and Sufficient Condition for General Recursive Utility

The result is obtained by comparison theory for BSDE. Given a set of consumption plans $\{c^a, c^b\}$, a necessary and sufficient condition for Agent equilibrium is:

$$F^i\left(t, c_t^i, e_t^{i*}, U_t^i, \Sigma_t^{U^i}\right) + \Sigma_t^{U^i\prime}\Phi\left(e_t^{i*}, e_t^{j*}\right) \quad \geq \quad F^i\left(t, c_t^i, e_t^i, U_t^i, \Sigma_t^{U^i}\right) + \Sigma_t^{U^i\prime}\Phi\left(e_t^i, e_t^{j*}\right)$$

$$\forall e^i \in \mathcal{E},\ t \in [0, T),\ i,\ j \in \{a, b\}, i \neq j$$

We assume that there exists a unique maximum of the above function and denote it by $I^i : \Omega \times [0, T] \times \mathbf{C} \times \mathbb{R}^{2+2d} \rightarrow = \mathbb{R}^d$. ,such that

$$\begin{aligned} e_t^{i*} &= I^i(t, c_t^i, U_t^i, \Sigma_t^{U^i}, e_t^{j*}) \\ &= \arg\max_{e^i \in \mathbf{E}} \left\{F^i\left(t, c_t^i, e_t^i, U_t^i, \Sigma_t^{U^i}\right) + \Sigma_t^{U^i\prime}\Phi\left(e_t^i, e_t^{j*}\right)\right\}, t \in [0, T) \end{aligned}$$

The Case of Time-Additive Utility

Varaiya(1976), Uchida(1978) and Davis(1979) gave a necessary and sufficient condition for equilibrium in the case when utility is defined as:

$$
\begin{aligned}
U_t^i &= E_t^{(e^i,e^j)}\left[\int_t^T F^i(s,X_s,e_s^i,e_s^j)ds + U_T^i\right] \\
U_T^i &= F^i(T,X_T),\ i,\ j \in \{a,b\}, \quad i \neq j
\end{aligned}
$$

Where $X \in \mathbb{R}^d$ is the state process satisfying a forward SDE with a given starting value x_0.
A necessary and sufficient condition for time-additive utility is

$$
\begin{aligned}
H_s^i(e_s^{i*},e_s^{j*}) &\geq H_s^i(e_s^i,e_s^{j*}),\ s \in [0,T),\ i,\ j \in \{a,b\}, \quad i \neq j \\
\text{Where } H_s^i(e_s^i,e_s^j) &= F^i(s,X_s,e_s^i,e_s^j) + g_s^{i\prime}\sigma_s\Phi\left(e_s^i,e_s^j\right)
\end{aligned}
$$

Implementable Effort Process

Assuming for any pair of effort plans (e^i, e^j),the function $J^i : \Omega \times [0, T] \times \mathbb{R}^{2+2d} \to \mathbb{R}^d$ is well-defined implicitly by

$$e_t^i = I^i \left(t, c_t^i, U_t^i, J^i \left(t, c_t^i, e_t^i, U_t^i, e_t^j \right), e_t^j \right) \quad t \in [0, T), \; i \neq j, \quad i \in \{a, b\}$$

(that is, we can invert the agent equilibrium condition to express it in terms of the agent's utility-diffusion term), then agent equilibrium is equivalent to

$$\Sigma_t^{Ui} = J^i \left(t, c^i, e^i, U^i, e^j \right) \;\; t \in [0, T) \quad i \neq j, \quad i \in \{a, b\} \,. \tag{4}$$

If the solution of e^i is interior(relative to E), and F^i is differentiable in e^i, then the agent equilibrium is equivalent to

$$\Sigma_t^{Ui} = - \left\{ \partial_{e^i} \Phi \left(e_t^i, e_t^j \right) \right\}^{-1} F_{e^i}^i \left(t, c_t^i, e_t^i, U_t^i, \Sigma_t^{Ui} \right)$$

A Forward-Backward SDE Characterization of Principal Equilibrium

In the spirit of Schattler and Sung(1993) and Holmstrom and Milgrom(1987), upon plugging in equation (4) and assuming participation constraint binds, the principal equilibrium (3) is characterized by a system of coupled FBSDE.

The set of Nash equilibrium efforts and consumptions $\{c^{a*}, e^{a*}, c^{b*}, e^{b*}\}$ satisfies:

$$V_0^i\left(c^{i*}, e^{i*}\right) \geq V_0^i\left(c^i, e^i\right) \quad \forall \{c^i, e^i\} \in \mathcal{C} \times \mathcal{E} \text{ subject to}$$

$$\begin{aligned}
& dU_t^i = -F^i\left(t, c_t^i, e_t^i, U_t, J^i\left(t, c_t^i, U_t^i, e_t^i, e_t^{j*}\right)\right) dt + J^i\left(t, c_t^i, U_t^i, e_t^i, e_t^{j*}\right)' dB_t^{(e^i, e^{j*})}, \\
& U_0^i = K^i \\
& dV_t^i = -G^i\left(t, c_t^i, V_t^i, \Sigma_t^{Vi}\right) dt + \Sigma_t^{Vi\prime} dB_t^{(e^i, e^{j*})}, \\
& V_T^i = G^i\left(T, \left(F^i\right)^{-1}\left(T, U_T^i\right)\right) \\
& i \neq j, \quad i \in \{a, b\}
\end{aligned} \tag{5}$$

Definition of Translation-Invariant Utility

Schroder and Skiadas(2005) first introduced translation-invariant (TI) recursive utility as a generalization of time-additive exponential utility. we say that the utility function ν is *translation-invariant* if for any $c^1, c^2 \in \mathcal{C}$ and $t \in [0, T]$,

$$\nu_t\left(c^1\right) = \nu_t\left(c^2\right) \quad \Longrightarrow \quad \nu_t\left(c^1 + k\right) = \nu_t\left(c^2 + k\right) \text{ for all } k \in \mathbb{R}.$$

If we further assume that ν is in certainty-equivalent form using some constant γ as the numeraire, then ν is *quasilinear* with respect to γ:

$$\nu_t\left(c + k\gamma\right) = \nu_t(c) + k \quad \text{for all } k \in \mathbb{R},\ c \in \mathcal{C},\ t \in [0, T].$$

The Aggreagator Functions of Translation-Invariant Utility

The translation-invariant aggregator is defined as:

$$F^i\left(t, c, e, U^i, \Sigma^{Ui}\right) = f^i\left(t, \frac{c^i}{\gamma^{Ui}} - U^i, e^i, \Sigma^{Ui}\right), \quad F^i\left(T, c^i\right) = \frac{c_T^i}{\gamma^{Ui}},$$
$$G^i\left(t, c^i, V^i, \Sigma^i\right) = g^i\left(t, \frac{X^i - c^i}{\gamma^{Vi}} - V^i, \Sigma^{Vi}\right), \quad G^i\left(T, c\right) = \frac{X_T^i - c_T^i}{\gamma^{Vi}},$$

where X^i is the cash-flow process received by principal i, it is easy to confirm the following quasilinear relationships:

$$U_t^i\left(c^i + k\gamma^{Ui}, e^i, e^j\right) = U_t^i\left(c^i, e^i, e^j\right) + k,$$
$$V_t\left(c^i + k\gamma^{V}, e^i, e^j\right) = V_t\left(c^i, e^i, e^j\right) - k, \quad \text{for all } c^i \in \mathcal{C},\ k \in \mathbb{R}.$$

and define

$$x_t^{Ui} = \frac{c_t^i}{\gamma^{Ui}} - U_t^i, \quad x_t^{Vi} = \frac{X_t^i - c_t^i}{\gamma^{Vi}} - V_t^i.$$

Principal Equilibrium under Translation-Invariant Preference

Under Translation-Invariant preference, the set of Nash equilibrium strategies $\{x^{Ua*}, e^{a*}, x^{Ub*}, e^{b*}\}$ satisfies

$$V_0^i\left(x^{Ui*}, e^{i*}\right) \geq V_0^i\left(x^{Ui}, e^i\right) \text{ subject to}$$

$$\begin{aligned}
dU_t^i &= -f^i\left(t, x_t^{Ui}, J\left(t, x_t^{Ui}, e_t^i, e_t^j\right), e_t^i\right) dt + J^i\left(t, x_t^{Ui}, e_t^i, e_t^j\right)'\left(dB_t - \Phi\left(e_t^i, e_t^j\right) dt\right. \\
U_0^i &= K^i \\
dV_t^i &= -g^i\left(t, \frac{X_t^i}{\gamma^{Vi}} - \lambda^i\left(x_t^{Ui} + U_t^i\right) - V_t^i, \Sigma_t^{Vi}\right) dt + \Sigma_t^{Vi\prime}\left(dB_t - \Phi\left(e_t^i, e_t^j\right) dt\right) \\
V_T &= \frac{X_T^i - \gamma^{Ui} U_T^i}{\gamma^{Vi}}
\end{aligned}$$

The main goal of this method is to obtain a solution under weaker conditions on the principal aggregator function than that required by utility gradient approach.

FOC for Translation-Invariant Utility

Under translation-invariant preferences, the system of 2 linked FBSDEs (5) uncouples and simplifies to the solution of 2 single backward SDEs (7).

Define $Y_t^i = V_t^i\left(c^i, e^i\right) + \frac{\gamma^{Ui}}{\gamma^{Vi}} U_t\left(c_t^i, e_t^i\right),\ i \in \{a, b\},\ t \in [0, T]$. By dynamic programming approach, we proved a necessary and sufficient condition for principal equilibrium is

$$\max_{\left(x_t^{Ui}, e_t^i\right) \in \mathbb{R} \times \mathbf{E}} g^i\left(t, -\lambda^i x_t^{Ui} + \frac{X_t^i}{\gamma^{Vi}} - Y_t^i, \Sigma_t^{Yi} - \lambda^i J^i\left(t, x_t^{Ui}, e_t^i, e_t^j\right)\right) + \Sigma_t^{Yi\prime}\Phi\left(e_t^i, e_t^j\right)$$
$$+ u_t^{Yi} + \lambda^i f^i\left(x_t^{Ui}, J^i\left(t, x_t^{Ui}, e_t^i, e_t^j\right), e_t^i, e_t^j\right) = 0. \quad t \in [0, T),\ i \in \{a, b\}$$

The FOCs for an interior solution to the above equation are:

$$\begin{aligned} 0 &= f_x^i - g_x^i + \left(J_x^i\right)'\left(f_\Sigma^i - g_\Sigma^i\right), \\ 0 &= \left\{\partial_{e^i}\Phi\left(e_t^i, e_t^j\right)\right\}\Sigma^{Yi} + \lambda^i\left\{\left(J_{e^i}^i\right)'\left(f_\Sigma^i - g_\Sigma^i\right) + f_{e^i}^i\right\}. \end{aligned} \qquad (6)$$

Solution to the Principal Equilibrium

At the equilibrium, $\left(Y^a, \Sigma^{Ya}, Y^b, \Sigma^{Yb}\right)$ satisfies the following BSDE:

$$dY_t^i = -\{g^i\left(t, -\lambda^i x_t^{Ui} + \frac{X_t^i}{\gamma^{Vi}} - Y_t^i, \Sigma_t^{Yi} - \lambda^i J^i\left(t, x_t^{Ui}, e_t^i, e_t^j\right)\right) \quad (7)$$

$$+\lambda^i f^i\left(x_t^{Ui}, J^i\left(t, x_t^{Ui}, e_t^i, e_t^j\right), e_t^i, e_t^j\right)\}dt + \Sigma_t^{Yi}\left(dB_t - \Phi\left(e_t^i, e_t^j\right)dt\right),$$

$$Y_T^i = X_T^i / \gamma^{Vi},$$

$$i, j \in \{a, b\}, \quad i \neq j$$

The equilibrium consumption and effort plans are given by

$$x_t^{Ui} = \phi^i\left(t, Y_t^i, \Sigma_t^{Yi}, e_t^j\right), \; e_t^i = \psi^i\left(t, Y_t^i, \Sigma_t^{Yi}, e_t^j\right) \quad (8)$$

where $\phi^i\left(\right)$ and $\psi^i\left(\right)$ are obtained from the FOC equation (6).

TI Preference with Quadratic Penalty and No Intermediate Consumption

Let $\Phi\left(e_t^i, e_t^j\right) = e_t^i + e_t^j$ and suppose we have the TI absolute aggregators

$$
\begin{aligned}
f^i\left(\omega, t, x^U, e, \Sigma\right) &= -\frac{1}{2} q^{ei}(\omega, t)\, e' e - \frac{1}{2} q^{Ui}(\omega, t)\, \Sigma' \Sigma, \\
g^i\left(\omega, t, x^V, \Sigma\right) &= -\frac{1}{2} q^{Vi}(\omega, t)\, \Sigma' \Sigma, \quad i \in \{a, b\}
\end{aligned}
$$

where $q^{ei}, q^{Ui}, q^{Vi} \in \mathbb{R}_+$

BSDE for Y

Define

$$\lambda^i = \frac{\gamma^{Ui}}{\gamma^{Vi}}, \qquad \xi^i = \frac{1}{1+\lambda^i q^{ei} q^{Vi} + q^{ei} q^{Ui}},$$

$$w^i = \frac{1+\lambda^i q^{ei} q^{Vi}}{1+\lambda^i q^{ei} q^{Vi} + q^{ei} q^{Ui}}, \quad q^{Yi} = \frac{1}{\lambda^i}\left(q^{U^i} w^i - \frac{1}{q^{ei}}\right)$$

Upon directly applying (7), $\left(Y^a, \Sigma^{Ya}, Y^b, \Sigma^{Yb}\right)$ satisfies the following BSDE

$$\begin{aligned} dY_t^i &= -\left\{ e_t^j \Sigma_t^{Yi} - \frac{1}{2} q_t^{Yi} \Sigma_t^{Yi\prime} \Sigma_t^{Yi} \right\} dt + \Sigma_t^{Yi\prime} dB_t \\ Y_T^i &= X_T^i / \gamma^{Vi} \end{aligned} \tag{9}$$

Solution with Constant Cash-Flow Volatility

Suppose the cash flow for principal i is

$$X_T^i = \sigma^{i\prime} B_T, \quad i \in \{a, b\}.$$

Then the solution to equation (9) is $\Sigma_t^{Yi} = \sigma^i$

Given the other principal's (c^j, e^j), the equilibrium effort plan satisfies:

$$e_t^i = \frac{w^i}{\lambda^i q^{ei}} \sigma^i = \frac{1 + \lambda^i q_t^{ei} q_t^{Vi}}{(1 + \lambda^i q_t^{ei} q_t^{Vi} + q_t^{ei} q_t^{Ui}) \lambda^i q^{ei}} \sigma^i, \quad i \in \{a, b\} \tag{10}$$

The equilibrium terminal consumption satisfies:

$$c_T^i = w^i \left(X_T^i - \sigma^{i\prime} \sigma^j \frac{w^j}{\lambda^j q^{ej}} T \right) + \gamma^{Ui} K^i + \frac{\gamma^{Ui}}{2} \left(q^{Ui} - \frac{1}{q^{ei}} \right) \left\| \frac{w^i}{\gamma^{Ui}} \sigma^i \right\|^2 T, \quad i \in \{a, b\} \tag{11}$$

TI Preference Affine in Consumption

Either both pincipals' aggregators are affine in consumption, that is

$$g^i\left(\omega, t, x^V, \Sigma\right) = \beta^i(\omega, t)\, x^V + k^{Vi}(\omega, t, \Sigma)$$

or both agents' aggregators are affine in consumption, that is

$$f^i\left(\omega, t, x^U, e, \Sigma\right) = \beta^i(\omega, t)\, x^U + k^{Ui}(\omega, t, e, \Sigma)$$

The above setup implies that either both principals or agents exhibit infinite elasticity of intertemporal substitution.

BSDE for Y

At the equilibrium, $\left(Y^a, \Sigma^{Ya}, Y^b, \Sigma^{Yb}\right)$ satisfies:

$$\begin{aligned} dY_t^i &= -\left\{\kappa_t^{Yi}\left(\frac{X_t^i}{\gamma^{Vi}} - Y_t^i\right) + \mu_t^{Yi} + e_t^{j\prime}\Sigma_t^{Yi} - \frac{1}{2}\Sigma_t^{Yi\prime}Q_t^{Yi}\Sigma_t^{Yi}\right\} dt + \Sigma_t^{Yi\prime} dB_t, \\ Y_T^i &= X_T^i/\gamma^{Vi}, \quad i,j \in \{a,b\}, \quad i \neq j. \end{aligned} \tag{12}$$

We introduce a *state process* $Z \in \mathcal{L}(\mathbb{R}^n)$ with dynamics

$$dZ_t = \left(\mu_t^Z + \beta_t^Z Z_t\right) dt + \sigma_t^Z dB_t,$$

$$\text{where} \quad \mu^Z \in \mathcal{L}(\mathbb{R}^n),\ \beta^Z \in \mathcal{L}\left(\mathbb{R}^{n\times n}\right),\ \sigma^Z \in \mathcal{L}\left(\mathbb{R}^{d\times n}\right)$$

Assume

$$\frac{X^i}{\gamma^{Vi}} = m^i + M^{i\prime}Z \quad \text{where} \quad m^i \in \mathcal{L}(\mathbb{R}); M^i \in \mathcal{L}(\mathbb{R}^n)$$

Riccati System

We seek a solution for Y^i, $i \in \{a,b\}$ affine in the state variables:

$$Y_t^i = \theta^i(t) + \Theta^i(t)' Z_t, \quad i \in \{a,b\}, \tag{13}$$

for some processes $\theta^i \in \mathcal{L}(\mathbb{R})$ and $\Theta^i \in \mathcal{L}(\mathbb{R}^n)$, $i \in \{a,b\}$
The solution reduces to the following Riccati system such that $(\theta^a, \theta^b, \Theta^a, \Theta^b)$ solves

$$\begin{aligned}
0 &= \dot{\theta}^i + \mu^{Z\prime}\Theta^i + \kappa^{Yi}\left(m^i - \theta^i\right) + \mu^{Yi} + \left(\lambda^j\right)^{-1}\Theta^{j\prime}d^j\Theta^i - \frac{1}{2}\Theta^{i\prime}h^i\Theta^i, \quad (14)\\
0 &= \dot{\Theta}^i + \beta^{Z\prime}\Theta^i + \kappa^{Yi}\left(M^i - \Theta^i\right) + \left(\lambda^j\right)^{-1}\left(\sum_{k,l}\Theta_k^j\Theta_l^i D^{i,(k,l)}\right)\\
&\quad -\frac{1}{2}\left(\sum_{k,l}\Theta_k^i\Theta_l^i H^{i,(k,l)}\right),\\
i,j &\in \{a,b\}, \quad i \neq j,
\end{aligned}$$

Solution

Suppose the state-variable process satisfies:

$$dZ_t = \left(\mu^Z + \beta_t^Z Z_t\right) dt + \Sigma' \text{diag}\left(\sqrt{\upsilon + VZ_t}\right) dB_t.$$

Furthermore, we assume the same quadratic penalty for both principal and agent.

Through solving the Riccati system, the equilibrium terminal consumption is:

$$c_T^i = w^i X_T + \gamma^{Ui}\left(1 - w^i\right) K - \gamma^{Vi} w^i V_0^i - \gamma^{Ui} \mu^{Ui} T$$

$$+ \gamma^{Ui} \int_0^T \beta_t^{Ui\prime} Z_t dt, \quad i \in \{a, b\}.$$

THANK YOU