A class of stochastic volatility models and the q-optimal martingale measure

Sotirios Sabanis

School of Mathematics
University of Edinburgh

23 June, 2010

Outline

1. Introduction
2. q-Optimal Martingale Measure
   - A relevant Cauchy problem
   - Main Theorem
3. Further example
   - Non-degenerate case

Outline

Outline

Introduction

Let $Y := \{Y_t\}_{0 \leq t \leq T}$ denote the process that drives the volatility of the (discounted) traded asset S and consider the following setting

$$
\begin{aligned}
dS_t &= \mu(Y_t, t)S_t dt + \sigma(Y_t, t)S_t dB_t, \\
dY_t &= \alpha(Y_t, t)dt + \beta(Y_t, t)dW_t
\end{aligned} \tag{2.1}
$$

where $B := \{B_t\}_{0 \leq t \leq T}$ and $W := \{W_t\}_{0 \leq t \leq T}$ are $\mathbb{P}$-Brownian motions with instantaneous correlation ρ. In other words, we assume that $dW_t = \rho dB_t + \sqrt{1 - \rho^2} dZ_t$, where B and $Z := \{Z_t\}_{0 \leq t \leq T}$ are independent $\mathbb{P}$-Brownian motions. The filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is generated by B and Z, made complete and right continuous.

Let $T > 0$ be the fixed termination date for the economy and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ be our filtered probability space that satisfies the usual conditions of right-continuity and completeness and

$$\mathcal{M}^e(\mathbb{P}) = \{\mathbb{Q} \sim \mathbb{P} : \mathbb{Q} \text{ is a probability measure and } S \text{ is a local } \mathbb{Q} - \text{martingale}\}$$

be the set of all EMMs (Equivalent Martingale Measures). The general version of the *Fundamental Theorem of Asset Pricing* answers the question whether or not $\mathcal{M}^e(\mathbb{P}) = \emptyset$.

Let us consider the case where $q > 1$ and recall that the q-optimal measure $\mathbb{Q}^*$ minimises the qth moment of the Radon-Nikodym derivative

$$
\begin{aligned}
V_T^{\text{opt}} := \frac{d\mathbb{Q}^*}{d\mathbb{P}} &= \mathcal{E}(-\lambda \cdot B - \xi \cdot Z)_T \\
&= \exp(-\int_0^T \lambda_t dB_t - \frac{1}{2}\int_0^T \lambda_t^2 d_t - \int_0^T \xi_t dZ_t - \frac{1}{2}\int_0^T \xi_t^2 d_t)
\end{aligned}
$$

where the adapted processes $\lambda := \{\lambda_t\}_{0 \le t \le T}$ and $\xi := \{\xi_t\}_{0 \le t \le T}$ are such that

$$
\lambda_t := \lambda(Y_t, t) = \frac{\mu(Y_t, t)}{\sigma(Y_t, t)} \quad \text{and} \quad \int_0^T \xi_t^2 dt < \infty \quad \mathbb{P}-\text{a.s.}. \qquad (2.2)
$$

It is not difficult to observe that for the case where $\mu(y,t) \equiv c_0$, and indeed this is the case for many empirical studies and time series modelling of historical data, it is generally either difficult or impossible to prove that the Novikov condition (or even $\mathbb{E}[V_T^{\text{opt}}] = 1$) holds when $\sigma(y,t) \equiv y$, since $\lambda(Y_t,t) = c_0/Y_t$ and Y_t can get very close or even take the value zero, e.g. Stein & Stein model. As a result, the change of measure may become problematic.

Another approach is to consider a class of models such that the "mean-variance trade-off" process is uniformly bounded, which is favoured by practitioners and which is a common assumption in applications to mathematical finance as one can observe from papers such as Grandits & Rheinländer (2002), and Hobson (2004).

A further suggestion that leads to the desired uniform boundedness is the introduction of a new but not restrictive condition about $\sigma(y,t)$. It is a suggestion closely related to the approach adopted in the influential paper by Hobson & Rogers (1998):

$$\sigma(y,t) = \sqrt{\sigma_0 + y^2}$$

where σ_0 is an infinitesimally small positive constant. One then observes that (especially for typical volatility inputs - e.g. mean reverting type of SV model), the values for the two different σ functions follow each other very closely . Moreover,

$$\mathbb{E}[\ln S_t] = \ln S_0 + \int_0^t \mathbb{E}[\mu(Y_s,s) - \frac{Y_s^2}{2}]ds - \frac{\sigma_0 t}{2}$$

$$Var[\ln S_t] = \int_0^t \mathbb{E}[Y_s^2]ds + \sigma_0 t \tag{2.3}$$

Typically, there are two steps in the process of finding the q-optimal martingale measure.

1. One obtains a candidate measure (usually through a solution of a PDE problem which is closely related to the so-called fundamental representation equation)
2. One verifies that the above candidate measure is indeed the q-optimal martingale measure .

Outline

1. Introduction
2. q-Optimal Martingale Measure
   - A relevant Cauchy problem
   - Main Theorem
3. Further example
   - Non-degenerate case

Hobson (2004) presents the sufficiency of the so-called fundamental representation equation (3.1) in obtaining a unique q-optimal equivalent martingale measure. The search for two predictable processes $\eta := \{\eta_t\}_{0 \leq t \leq T}$ and $\xi := \{\xi_t\}_{0 \leq t \leq T}$ and a finite constant c such that the so-called fundamental representation equation is satisfied, i.e.

$$\frac{q}{2}\int_0^T \lambda_t^2 dt = \int_0^T \eta_t (dB_t + q\lambda_t dt) - \frac{q-1}{2}\int_0^T \eta_t^2 dt + \int_0^T \xi_t dZ_t + \frac{1}{2}\int_0^T \xi_t^2 dt + c \tag{3.1}$$

is reduced (by assuming that η_t and ξ_t are connected to the same process θ_t), to finding a unique solution to either

$$\begin{cases} -\dot{f} = \frac{q}{2}\lambda^2(y,t) + [\alpha(y,t) - q\rho\beta(y,t)\lambda(y,t)]f' + \frac{1}{2}\beta^2(y,t)f'', \\ f(y,T) = 0, \end{cases} \tag{3.2}$$

when $R := 1 - q\rho^2 = 0$, or

$$\begin{cases} -\dot{g} + \frac{q}{2}R\lambda^2(y,t)g = [\alpha(y,t) - q\rho\beta(y,t)\lambda(y,t)]g' + \frac{1}{2}\beta^2(y,t)g'' \\ g(y,T) = 1, \end{cases} \tag{3.3}$$

when $R \neq 0$ (and $g = e^{-Rf}$), since it can be shown that

$$\xi(Y_t,t) = \sqrt{1-\rho^2} f'(Y_t,t)\beta(Y_t,t)$$
$$\eta(Y_t,t) = \rho f'(Y_t,t)\beta(Y_t,t). \tag{3.4}$$

Each equation, (3.2) or (3.3), presents a Cauchy problem. Moreover, it is well known that in order to obtain a unique solution that admits a stochastic representation, it suffices to show that the coefficients satisfy suitable regularity conditions (Feynman-Kac theorem, for more details see Karatzas and Shreve (1988), Thm 5.7.6). However, the aforementioned sufficient conditions are too restrictive for stochastic volatility models .

One then observes that the findings of Gyöngy & Krylov (1990) for stochastic partial differential equations (SPDEs) in well-weighted Sobolev spaces can be applied to similar Cauchy problems for PDEs, e.g. problems (3.2) and (3.3).

Let us therefore briefly recall here that a well-weighted Sobolev space $W_p^m(\mathbb{R};\delta)$ consists of all functions $u:\mathbb{R}\to\mathbb{R}$ such that for each index i with $i\le m$, the derivative $D^i u$ exists (even in the weak sense) and

$$||u||^p_{W_p^m} = \sum_{i\le m} ||(D^i u)\delta^i||^p_{L^p} < \infty \qquad (p<\infty),$$

where i is a positive integer and $\delta:\mathbb{R}\to\mathbb{R}$ is such that

$$\sum_{i\le m} |\delta^{i-1}(D^i\delta)| < C \tag{3.5}$$

where C is a positive constant.

Lemma

Let m be a positive integer. Consider the Cauchy problem (3.2), if $R = 0$*, or (3.3), if* $R \neq 0$*, and suppose that* C *is a positive constant,* δ *satisfies condition (3.5),* $\lambda : \mathbb{R} \times [0, T] \to \mathbb{R}$ *is such that*

($\tilde{H}1$) $\int_0^T ||\lambda^2(y,t)||^2_{W_2^m} dt < \infty$ *and moreover, for all* $0 \leq i \leq m$*,* $t \in [0, T]$ *and* $y \in \mathbb{R}$*,*

($\tilde{H}2$) *the function* $a(y,t) := \frac{1}{2}\beta^2(y,t)$ *is such that* $|D^i a(y,t)| \leq C\delta^{2-i}$*;*

($\tilde{H}3$) *the function* $b(y,t) := \alpha(y,t) - q\delta\beta(y,t)\lambda(y,t)$ *is such that* $|D^i b(y,t)| \leq C\delta^{1-i}$ *;*

($\tilde{H}4$) *the function* $c(y,t) := \frac{q}{2}R\lambda^2(y,t)$ *is such that* $|D^i c(y,t)| \leq C\delta^{-i}$*;*

($\tilde{H}5$) *either* $a(y,t)$ *satisfies the strong ellipticity condition or*

Lemma

($\tilde{H}5$) *there exists a function* $\sigma : \mathbb{R} \times [0,T] \to \mathbb{R}$ *such that* $\sigma^2(y,t) = a(y,t)$ *and* $D\sigma(y,t)$ *is bounded for all* $(y,t) \in \mathbb{R} \times [0,T]$.

Then, there exists a unique solution in $C([0,T]; W_2^m(\mathbb{R};\delta))$. *Furthermore, for* $m \geq 2$ *and* $\delta = (1+y^2)^{\frac{1}{2}}$ *either* $(Df)\delta \in C_b(\mathbb{R})$, *if* $R = 0$, *or* $(Dg)\delta \in C_b(\mathbb{R})$, *if* $R \neq 0$, *for every* $t \in [0,T]$.

A set of parameters which satisfy the above conditions:

$$\mu(y,t) \equiv \mu_0, \quad \sigma(y,t) = \sqrt{\sigma_0 + y^2}, \quad \alpha(y,t) = \kappa(m-y),$$

$$\beta(y,t) \equiv \beta_0 \text{ and } \delta = (1+y^2)^{\frac{1}{2}}$$

where μ_0, σ_0, κ, m and β_0 are positive constants. [Models of similar form as the Stein & Stein approach].

Outline

Theorem

Suppose that S and Y are given by equation (2.1) and $\mu(y,t)$, $\sigma(y,t)$, $\alpha(y,t)$ and $\beta(y,t)$ are such that the conditions of Lemma 1 are satisfied for $m \geq 3$ and are continuous in t. Then, the q-optimal martingale measure $\mathbb{Q}^*$, for $q > 1$, exists for any finite T and is given by

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \mathcal{E}(-\lambda \cdot B - \xi \cdot Z)_T \tag{3.6}$$

where $\xi_t := \xi(Y_t, t) = \sqrt{1-\rho^2} f'(Y_t, t)\beta(Y_t, t)$ and the function $f \in C^{2,1}(\mathbb{R} \times [0, T])$ is bounded and is given by

$$f(y,t) = \begin{cases} -\frac{1}{R} \ln \mathbb{E}^{\tilde{\mathbb{P}}}\left[\exp\left(-\frac{q}{2}R\int_t^T \lambda^2(Y_u, u)du\right)\Big| Y_t = y\right], \\ \text{for } R \neq 0, \\ \mathbb{E}^{\tilde{\mathbb{P}}}\left[\frac{q}{2}\int_t^T \lambda^2(Y_u, u)du \Big| Y_t = y\right], \text{ for } R = 0 \end{cases}$$

Theorem

where the expectation is taken with respect to $\tilde{\mathbb{P}}$, Y is described by

$$dY_t = [\alpha(Y_t, t) - q\rho\beta(Y_t, t)\lambda(Y_t, t)]dt + \beta(Y_t, t)d\tilde{W}_t$$

and $\tilde{W}$ is a $\tilde{\mathbb{P}}$-Brownian motion.

Proof (sketch of).

For $m \geq 3$, due to Lemma 1 and the Sobolev embedding theorem, $f \in C^{2,0}(\mathbb{R} \times [0, T])$. Moreover, the continuity in t of the parameters of the corresponding Cauchy problems guarantees that in fact $f \in C^{2,1}(\mathbb{R} \times [0, T])$. It is then well-known that f has a stochastic representation which is given in equation (3.7). □

Proof (continues).

Furthermore, the existence of a unique solution for the corresponding Cauchy problem, i.e. either (3.2) or (3.3), implies that there is a solution for the so-called fundamental representation equation (3.1) and thus the density of the candidate measure $\mathbb{Q}^*$ is given by

$$V_T^{\text{opt}} := \frac{d\mathbb{Q}^*}{d\mathbb{P}} = \mathcal{E}(-\lambda \cdot B - \xi \cdot Z)_T = e^c(\mathcal{E}((q-1)\frac{\eta - \lambda}{\sigma S} \cdot S)_T)^{\frac{1}{q-1}},$$

where $\xi_t := \xi(Y_t, t) = \sqrt{1-\rho^2} f'(Y_t, t)\beta(Y_t, t)$,
$\eta_t := \eta(Y_t, t) = \rho f'(Y_t, t)\beta(Y_t, t)$, and f is given by equation (3.7).
Therefore

$$(\frac{d\mathbb{Q}^*}{d\mathbb{P}})^{q-1} = e^{c(q-1)}\mathcal{E}((q-1)\frac{\eta - \lambda}{\sigma S} \cdot S)_T$$

Proof (continues).

Furthermore, we know that λ is uniformly bounded but also $\eta(y,t)$ and $\xi(y,t)$ are too uniformly bounded due to Lemma 1, since $f'(y,t)\beta(y,t) \leq Cf'(y,t)(1+y^2)^{1/2} \leq K$ (where C is a positive constant and β is assumed to behave as in typical stochastic volatility models) and thus

$$\mathbb{E}[\mathcal{E}(-\lambda \cdot B - \xi \cdot Z)_T] = 1,$$

which implies $\mathbb{Q}^* \in \mathcal{M}^e(\mathbb{P})$, and for any $\mathbb{Q}^0 \in \mathcal{M}^e(\mathbb{P})$,

$$\mathbb{E}_{\mathbb{Q}^0}[\mathcal{E}((q-1)(\eta-\lambda) \cdot B^{\mathbb{Q}^0})_T] = 1 \qquad (3.8)$$

where $B^{\mathbb{Q}^0}$ is a standard $\mathbb{Q}^0$-Brownian motion. Equation (3.8) is the key difference with the proof appearing in Hobson (2004)and it is in agreement with the findings of Delbaen and Schachermayer (1996).

Outline

Consider the case where

$$\mu(y,t)=\mu_0, \quad \alpha(y,t)=\kappa(m-y) \text{ and } \beta(y,t)=\beta_0 y \tag{4.1}$$

for all $t\in[0,T]$. Therefore, equation (2.1) becomes

$$dS_t=\mu_0 S_t dt+\sqrt{\sigma_0+Y_t^2}S_t dB_t \quad \text{and} \quad dY_t=\kappa(m-Y_t)dt+\beta_0 Y_t dW_t \tag{4.2}$$

where μ_0, κ, m and β_0 are positive constants. Then,

$$\begin{cases} -\dot{f}=\frac{q}{2}\frac{(\mu_0-r)^2}{\sigma_0+y^2}+[\kappa(m-y)-q\rho\beta_0 y\frac{\mu_0-r}{\sqrt{\sigma_0+y^2}}]f'+\frac{1}{2}\beta_0^2 y^2 f'', \\ f(y,T)=0, \end{cases} \tag{4.3}$$

and

$$\begin{cases} -\dot{g}+\frac{q}{2}R\frac{(\mu_0-r)^2}{\sigma_0+y^2}g=[\kappa(m-y)-q\rho\beta_0 y\frac{\mu_0-r}{\sqrt{\sigma_0+y^2}}]g'+\frac{1}{2}\beta_0^2 y^2 g'', \\ g(y,T)=1. \end{cases} \tag{4.4}$$