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Pricing of perpetual American options in a model with partial information

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Outline

- The hidden Markov (switching) model
- The optimal stopping problem
- The optimal exercise boundary
- The free-boundary problem I
- The case of full information
- The change of variables
- The free-boundary problem II
- The verification assertion
- The main result
- Some estimates

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The model

Let $(\Omega, \mathcal{G}, P)$ be a probability space, $B = (B_t)_{t \geq 0}$ be a standard Brownian motion,

$\Theta = (\Theta_t)_{t \geq 0}$ be a continuous Markov chain with two states 0 and 1,

initial distribution $[1 - \pi, \pi]$ for $\pi \in [0, 1]$,

transition probability matrix $[e^{-\lambda_0 t}, 1 - e^{-\lambda_0 t}; 1 - e^{-\lambda_1 t}, e^{-\lambda_1 t}]$ for $t \geq 0$,

and intensity matrix $[-\lambda_0, \lambda_0; \lambda_1, -\lambda_1]$ for some $\lambda_i \geq 0$, $i = 0, 1$.

Assume that the asset price $S = (S_t)_{t \geq 0}$ is given by:

$$S_t = s \, \exp\left(\int_0^t \left(r - \frac{\sigma^2}{2} - \delta_0 - (\delta_1 - \delta_0)\, \Theta_u \right) du + \sigma \, B_t \right)$$

where $r \geq 0$, $\sigma > 0$, $0 < \delta_i < r$, $i = 0, 1$.

The asset with price S pays dividends at the rate δ_0 when $\Theta_t = 0$,

and at the rate δ_1 when $\Theta_t = 1$.

The model

It is shown that the asset price solves the equation

$$dS_t = \big(r - \delta_0 - (\delta_1 - \delta_0)\,\Theta_t\big)\,S_t\,dt + \sigma\,S_t\,dB_t \quad (S_0 = s)$$

and thus admits the representation

$$dS_t = \big(r - \delta_0 - (\delta_1 - \delta_0)\,\Pi_t\big)\,S_t\,dt + \sigma\,S_t\,d\overline{B}_t \quad (S_0 = s)$$

where the filtering estimate $\Pi = (\Pi_t)_{t \geq 0}$ defined by $\Pi_t = E[\Theta_t \,|\, \mathcal{F}_t]$ $\equiv P(\Theta_t = 1 \,|\, \mathcal{F}_t)$ solves the equation

$$d\Pi_t = \big(\lambda_1\,(1 - \Pi_t) - \lambda_0\,\Pi_t\big)\,dt - \frac{\delta_1 - \delta_0}{\sigma}\,\Pi_t(1 - \Pi_t)\,d\overline{B}_t \quad (\Pi_0 = \pi)$$

and the process $\overline{B} = (\overline{B}_t)_{t \geq 0}$ defined by

$$\overline{B}_t = \int_0^t \frac{dS_u}{\sigma S_u} - \frac{1}{\sigma}\int_0^t \big(r - \delta_0 - (\delta_1 - \delta_0)\,\Pi_u\big)\,du$$

is the innovation Brownian motion.

Some references

[1] R.S. Liptser and A.N. Shiryaev (1977). *Statistics of Random Processes I-II.* Springer, Berlin.

[2] R.J. Elliott, L. Aggoun and J.B. Moore (1995). *Hidden Markov Models: Estimation and Control.* Springer, Berlin.

[3] A.N. Shiryaev (1999). *Essentials of Stochastic Finance.* World Scientific.

[4] R.J. Elliott, C.A. Wilson (2007). The term structure of interest rates in a hidden Markov setting. In the Volume *Hidden Markov Models in Finance.* R.S. Mamon and R.J. Elliott eds. Springer, New York (15–30).

[5] D.C. Brody, L.P. Hughston, and A. Macrina (2008). Information-based asset pricing. *International Journal of Theoretical and Applied Finance* **11** (107–142).

[6] P.G. and M. Jeanblanc (2010). Pricing and filtering in a two-dimensional models with random dividends. To appear in *International Journal of Theoretical and Applied Finance.*

The optimal stopping problem

The problem is to compute the value

$$V_* = \sup_\tau E\big[e^{-r\tau}\,(S_\tau - K)^+\big]$$

where the supremum is taken over τ with respect to $\mathcal{F}_t = \sigma(S_u \,|\, 0 \le u \le t)$.

Let us consider the following extended optimal stopping problem

$$V_*(s,\pi) = \sup_\tau E_{s,\pi}\big[e^{-r\tau}\,(S_\tau - K)^+\big]$$

where $P_{s,\pi}$ is a measure of (S, Π) started at some $(s,\pi) \in (0,\infty) \times [0,1]$.

The optimal stopping time is given by

$$\tau_* = \inf\{t \ge 0 \,|\, V_*(S_t, \Pi_t) \le (S_t - K)^+\}$$

so that the continuation region has the form

$$C_* = \{(s,\pi) \in (0,\infty) \times [0,1] \,|\, V_*(s,\pi) > (s-K)^+\}.$$

The optimal stopping boundary

By means of a generalized Itô's formula, we get:

$$e^{-rt}\,(S_t-K)^+=(s-K)^+ + M_t^K$$
$$+\int_0^t e^{-ru}\,\Delta(S_u,\Pi_u)\,I(S_u>K)\,du+\frac{1}{2}\int_0^t e^{-ru}\,I(S_u\neq K)\,d\ell_u^K(S)$$

where $\Delta(s,\pi)=rK-(\delta_0+(\delta_1-\delta_0)\pi)s$ and

$$\ell_t^K(S)=\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t I(K-\varepsilon<S_u<K+\varepsilon)\,\sigma^2 S_u^2\,du$$

exists as a limit in probability. Here, the process $M^K=(M_t^K)_{t\geq 0}$ defined by:

$$M_t^K=\int_0^t e^{-ru}\,I(S_u>K)\,\sigma S_u\,d\overline{B}_u$$

is a continuous (uniformly integrable) martingale under $P_{s,\pi}$.

The optimal stopping boundary

Applying Doob's optional sampling theorem, we get:

$$E_{s,\pi}\left[e^{-r\tau}\,(S_\tau - K)^+\right] = (s-K)^+$$
$$+\,E_{s,\pi}\left[\int_0^\tau e^{-ru}\,\Delta(S_u,\Pi_u)\,I(S_u>K)\,du + \frac{1}{2}\int_0^\tau e^{-ru}\,I(S_u\neq K)\,d\ell_u^K(S)\right]$$

for any τ and all $(s,\pi)\in(0,\infty)\times[0,1]$.

It is seen that it is never optimal to stop when

$$\Delta(S_t,\Pi_t)\equiv rK-(\delta_0+(\delta_1-\delta_0)\Pi_t)S_t\le 0 \quad \text{and} \quad S_t>K$$

and thus, all the points (s,π) such that

$$K<s\le \underline{b}(\pi) \quad \text{with} \quad \underline{b}(\pi)=rK/(\delta_0+(\delta_1-\delta_0)\pi)$$

belong to C_* clearly containing the rectangle $\{(s,\pi)\in(0,K]\times[0,1]\}$.

The optimal stopping boundary

For some $(s, \pi) \in C_*$ and $\tau_* = \tau_*(s, \pi)$, we have:

$$V_*(s,\pi) - (s-K)^+$$
$$= E_{s,\pi}\left[\int_0^{\tau_*} e^{-ru}\,\Delta(S_u,\Pi_u)\,I(S_u > K)\,du + \frac{1}{2}\int_0^{\tau_*} e^{-ru}\,I(S_u \neq K)\,d\ell_u^K(S)\right] > 0$$

Hence, taking $K < \underline{b}(\pi) < s' < s$, we get:

$$V_*(s',\pi) - (s'-K)^+$$
$$\geq E_{s',\pi}\left[\int_0^{\tau_*} e^{-ru}\,\Delta(S_u,\Pi_u)\,I(S_u > K)\,du + \frac{1}{2}\int_0^{\tau_*} e^{-ru}\,I(S_u \neq K)\,d\ell_u^K(S)\right]$$
$$\geq E_{s,\pi}\left[\int_0^{\tau_*} e^{-ru}\,\Delta(S_u,\Pi_u)\,I(S_u > K)\,du + \frac{1}{2}\int_0^{\tau_*} e^{-ru}\,I(S_u \neq K)\,d\ell_u^K(S)\right]$$

and taking into account $0 < \delta_i < r$, we see that $(s', \pi) \in C_*$.

The optimal stopping boundary

These arguments together with concavity of $s \mapsto V_*(s, \pi)$ show that

there exists a function $b_*(\pi)$ such that $K < \underline{b}(\pi) \leq b_*(\pi)$ for all $\pi \in [0, 1]$, and

$$C_* = \{(s, \pi) \in (0, \infty) \times [0, 1] \,|\, s < b_*(\pi)\}$$

so that the corresponding stopping region is the closure of the set:

$$D_* = \{(s, \pi) \in (0, \infty) \times [0, 1] \,|\, s > b_*(\pi)\}.$$

Lemma 1. *The optimal exercise time has the structure:*

$$\tau_* = \inf\{t \geq 0 \,|\, S_t \geq b_*(\Pi_t)\}$$

where the function $b_*(\pi)$ satisfies the properties:

$$b_*(\pi) : [0, 1] \to (K, \infty) \quad \text{is decreasing/increasing if} \quad \delta_0 < \delta_1 / \delta_0 > \delta_1$$

$$K < \underline{b}(\pi) \leq b_*(\pi) \quad \text{with} \quad \underline{b}(\pi) = rK / (\delta_0 + (\delta_1 - \delta_0)\pi).$$

The optimal stopping boundary

For any $(s, \pi) \in C_*$, we take π' such that $\pi' < \pi$ if $\delta_0 < \delta_1$ (or $\pi < \pi'$ if $\delta_0 > \delta_1$) whenever $s > K$. Then, since $\tau_* = \tau_*(s, \pi)$ does not depend on π', we have:

$$
\begin{aligned}
& V_*(s, \pi') - (s - K)^+ \\
& \geq E_{s,\pi'} \left[\int_0^{\tau_*} e^{-ru} \, \Delta(S_u, \Pi_u) \, I(S_u > K) \, du + \frac{1}{2} \int_0^{\tau_*} e^{-ru} \, I(S_u \neq K) \, d\ell_u^K(S) \right] \\
& \geq E_{s,\pi} \left[\int_0^{\tau_*} e^{-ru} \, \Delta(S_u, \Pi_u) \, I(S_u > K) \, du + \frac{1}{2} \int_0^{\tau_*} e^{-ru} \, I(S_u \neq K) \, d\ell_u^K(S) \right] \\
& = V_*(s, \pi) - (s - K)^+ > 0
\end{aligned}
$$

and thus conclude that $(s, \pi') \in C_*$, so that the boundary $b_*(\pi)$ is decreasing (increasing) on $[0, 1]$ whenever $\delta_0 < \delta_1$ ($\delta_0 > \delta_1$).

The free-boundary problem I

The infinitesimal operator $\mathbb{L}_{(S,\Pi)}$ has the structure:

$$\mathbb{L}_{(S,\Pi)} = (r - \delta_0 - (\delta_1 - \delta_0)\,\pi)\,s\,\partial_s + \frac{1}{2}\,\sigma^2 s^2\,\partial_{ss} - (\delta_1 - \delta_0)\,s\,\pi(1-\pi)\,\partial_{s\pi}$$

$$+ \big(\lambda_1\,(1-\pi) - \lambda_0\,\pi\big)\,\partial_\pi + \frac{1}{2}\left(\frac{\delta_1 - \delta_0}{\sigma}\right)^2 \pi^2(1-\pi)^2\,\partial_{\pi\pi}$$

for all $(s, \pi) \in (0, \infty) \times [0, 1]$.

It follows from the general optimal stopping theory that the unknown value function $V_*(s, \pi)$ and the boundary $b_*(\pi)$ satisfy the free-boundary problem:

$$(\mathbb{L}_{(S,\Pi)}V - rV)(s, \pi) = 0 \quad \text{for} \quad (s, \pi) \in C$$

$$V(s, \pi)\big|_{s=b(\pi)-} = b(\pi) - K \quad (\textit{instantaneous stopping})$$

$$V(s, \pi) = (s - K)^+ \quad \text{for} \quad (s, \pi) \in D$$

$$V(s, \pi) > (s - K)^+ \quad \text{for} \quad (s, \pi) \in C.$$

The case of full information

Let us now recall the problem with full information

$$W_*(s,\pi) = \sup_{\tau'} E_{s,\pi}\big[e^{-r\tau'}\,(S_{\tau'} - K)^+\big]$$

where the supremum is taken τ' with respect to $\mathcal{G}_t = \sigma(S_u, \Theta_u \,|\, 0 \le u \le t)$.

$$\tau'_* = \inf\{t \ge 0 \,|\, S_t \ge a_*(\Theta_t)\}.$$

The functions $W_*(s,i)$ and the boundaries $a_*(i)$, $i = 0, 1$, solve:

$$(r-\delta_i)\, s\, W_s(s,i) + \lambda_i\, W(s, 1-i) + \frac{1}{2}\,\sigma^2\, s^2 W_{ss}(s,i) = (r+\lambda_i) W(s,i)$$

$$W(s,i)\big|_{s=a(i)-} = a(i) - K \quad (\textit{instantaneous stopping})$$

$$W_s(s,i)\big|_{s=a(i)-} = 1 \quad (\textit{smooth fit})$$

$$W(s,i) = (s-K)^+ \quad \text{for} \quad s > a(i)$$

$$W(s,i) > (s-K)^+ \quad \text{for} \quad s < a(i).$$

The case of full information

The general solution of the free-boundary problem is given by:

$$W(s,i) = C_1(i)\, s^{\beta_1} + C_2(i)\, s^{\beta_2} \quad \text{for} \quad s < a(0)$$

$$W(s,1) = C_3(1)\, s^{\gamma_1} + \frac{\lambda_1 s}{\delta_1 + \lambda_1} - \frac{\lambda_1 K}{r + \lambda_1} \quad \text{for} \quad a(0) < s < a(1)$$

where the constants $C_j(i)$ and the boundaries $a(i)$, $i = 0, 1$, $j = 1, 2$, satisfy:

$$C_1(0)\, a^{\beta_1}(0) + C_2(0)\, a^{\beta_2}(0) = a(0) - K$$

$$C_1(1)\, a^{\beta_1}(0) + C_2(1)\, a^{\beta_2}(0) = C_3(1)\, a^{\gamma_1}(0) + \frac{\lambda_1 s}{\delta_1 + \lambda_1} - \frac{\lambda_1 K}{r + \lambda_1}$$

$$C_1(0)\, \beta_1\, a^{\beta_1}(0) + C_2(0)\, \beta_2\, a^{\beta_2}(0) = a(0)$$

$$C_1(1)\, \beta_1\, a^{\beta_1}(0) + C_2(1)\, \beta_2\, a^{\beta_2}(0) = C_3(1)\, \gamma_1 a^{\gamma_1}(0) + \frac{\lambda_1}{\delta_1 + \lambda_1}$$

$$C_1(1)\, \beta_1(\beta_1 - 1)\, a^{\beta_1}(0) + C_2(1)\, \beta_2(\beta_2 - 1)\, a^{\beta_2}(0) = C_3(1)\, \gamma_1(\gamma_1 - 1)\, a^{\gamma_1}(0).$$

The case of full information

The particular solution of the system is given by:

$$W_*(s,i) = \begin{cases} W(s,i;a_*(i)), & \text{if } 0 < s < a_*(i) \\ s - K, & \text{if } s \geq a_*(i) \end{cases}$$

where

$$W(s,0;a_*(0)) = \sum_{j=1}^{2} \frac{(\beta_{3-j} - 1)a_*(0) - \beta_{3-j}K}{\beta_{3-j} - \beta_j} \left(\frac{s}{a_*(0)} \right)^{\beta_j} \quad \text{for} \quad s < a_*(0) \quad \text{and}$$

$$W(s,1;a_*(0)) = \sum_{j=1}^{2} \frac{\beta_{3-j} W(a_*(0),1;a_*(1)) - W_s(a_*(0),1;a_*(1))a_*(0)}{\beta_{3-j} - \beta_j} \left(\frac{s}{a_*(0)} \right)^{\beta_j}$$

$$W(s,1;a_*(1)) = \left(\frac{\delta_1 a_*(1)}{\delta_1 + \lambda_1} - \frac{rK}{r + \lambda_1} \right) \left(\frac{s}{a_*(1)} \right)^{\gamma_1} + \frac{\lambda_1 s}{\delta_1 + \lambda_1} - \frac{\lambda_1 K}{r + \lambda_1}$$

for $a_*(0) < s < a_*(1)$.

The case of full information

Here, $a_*(0)$ is determined from:

$$\sum_{j=1}^{2}(-1)^j \, \beta_j(\beta_j - 1)[\beta_{3-j} W(a_*(0), 1; a_*(1)) - W_s(a_*(0), 1; a_*(1)) a_*(0)]$$
$$= (\beta_1 - \beta_2) \, \frac{\gamma_1 r K}{r + \lambda_1}$$

and $a_*(1)$ is explicitly given by:

$$a_*(1) = \frac{\gamma_1 K}{\gamma_1 - 1} \, \frac{r}{r + \lambda_1} \, \frac{\delta_1 + \lambda_1}{\delta_1}$$

where the numbers $\beta_2 < \beta_1$ are the two largest roots of:

$$\Big(r + \lambda_0 - \beta(r - \delta_0) - \frac{1}{2}\beta(\beta - 1)\sigma^2\Big)\Big(r + \lambda_1 - \beta(r - \delta_1) - \frac{1}{2}\beta(\beta - 1)\sigma^2\Big) = \lambda_0 \lambda_1$$

and $\gamma_2 < 0 < 1 < \gamma_1$ are explicitly given by:

$$\gamma_i = \frac{1}{2} - \frac{r - \delta_1}{\sigma^2} - (-1)^i \sqrt{\left(\frac{1}{2} - \frac{r - \delta_1}{\sigma^2}\right)^2 + \frac{2(r + \lambda_1)}{\sigma^2}}.$$

Some references

[7] X. Guo (2001). Information and option pricing. *Quantitative Finance* **1** (38–44).

[8] X. Guo (2001). An explicit solution to an optimal stopping problem with regime switching. *Journal of Applied Probability* **38** (464–481).

[9] Z. Jiang and M.R. Pistorius (2008). On perpetual American put valuation and first-passage in a regime-switching model with jumps. *Finance and Stochastics* **12** (331–355).

[10] G. Peskir (2007). A change-of-variable formula with local time on surfaces. *Séminaire de Probababilité XL.* Lecture Notes in Mathematics **1899**, Springer (69–96).

[11] A.N. Shiryaev (1978). *Optimal Stopping Rules.* Springer, Berlin.

[12] G. Peskir and A.N. Shiryaev (2006). *Optimal Stopping and Free-Boundary Problems.* Birkhäuser, Basel.

The change of variables

Let us define the process $Y = (Y_t)_{t \geq 0}$ by:

$$Y_t = \frac{S_t^{-\eta} \Pi_t}{1 - \Pi_t}$$

with $\eta = (\delta_1 - \delta_0)/\sigma^2$. Then, we have:

$$dS_t = \left(r - \delta_0 - (\delta_1 - \delta_0) \frac{S_t^{\eta} Y_t}{1 + S_t^{\eta} Y_t} \right) S_t \, dt + \sigma \, S_t \, d\overline{B}_t \quad (S_0 = s)$$

$$dY_t = \left(\frac{\lambda_1 - \lambda_0 S_t^{\eta} Y_t}{1 + S_t^{\eta} Y_t} - \frac{\eta}{2} (2r - \delta_0 - \delta_1 - \sigma^2) \right) Y_t \, dt \quad \left(Y_0 = y \equiv \frac{s^{-\eta} \pi}{1 - \pi} \right)$$

for any $(s, \pi) \in (0, \infty) \times (0, 1)$. The value function is given by:

$$U_*(s, y) = \sup_{\tau} E_{s,y} \left[e^{-r\tau} (S_\tau - K)^+ \right]$$

and the optimal stopping time has the form:

$$\tau_* = \inf\{ t \geq 0 \, | \, S_t \geq g_*(Y_t) \}.$$

The free-boundary problem II

The infinitesimal operator $\mathbb{L}_{(S,Y)}$ has the structure:

$$\mathbb{L}_{(S,Y)} = \left(r - \delta_0 - (\delta_1 - \delta_0) \frac{s^\eta y}{1 + s^\eta y} \right) s \, \partial_s + \frac{1}{2} \sigma^2 s^2 \, \partial_{ss} + \left(\frac{\lambda_1 - \lambda_0 s^\eta y}{1 + s^\eta y} - \frac{\eta}{2} (2r - \delta_0 - \delta_1 - \sigma^2) \right) y \, \partial_y$$

for all $(s, y) \in (0, \infty)^2$. The function $U_*(s, y)$ and the boundary $g_*(y)$ solves:

$$(\mathbb{L}_{(S,Y)} U - rU)(s, y) = 0 \quad \text{for} \quad 0 < s < g(y)$$

$$U(s, y)\big|_{s=g(y)-} = g(y) - K \quad (\textit{instantaneous stopping})$$

$$U(s, y) = (s - K)^+ \quad \text{for} \quad s > g(y)$$

$$U(s, y) > (s - K)^+ \quad \text{for} \quad s < g(y)$$

$$U(s, y)\big|_{s=0+} = 0 \quad (\textit{natural boundary}), \quad U_s(s, y)\big|_{s=g(y)-} = 1 \quad (\textit{smooth fit})$$

$$U_y(s, y)\big|_{s=g(y)-} \quad \text{exists.}$$

The verification assertion

Lemma 2. *The value function takes the form:*

$$U_*(s,y) = \begin{cases} U(s,y;g_*(y)), & \text{if } 0 < s < g_*(y) \\ s-K, & \text{if } s \geq g_*(y) \end{cases}$$

and $K < \underline{g}(y) \leq g_*(y)$ *holds for the boundary* $g_*(y)$ *with:*

$$\underline{g}^{-1}(s) = (\delta_0 s - rK)s^{-\eta}/(rK - \delta_1 s).$$

for each $rK/(\delta_0 \vee \delta_1) < s < rK/(\delta_0 \wedge \delta_1)$ *with* $\eta = (\delta_0 - \delta_1)/\sigma^2$ *and* $y > 0$.

Proof. Applying the change-of-variable to the solution $e^{-rt}U(s,y)$, we obtain:

$$e^{-rt}\,U(S_t,Y_t) = U(s,y) + \int_0^t e^{-ru}\,(\mathbb{L}_{(S,Y)}U - rU)(S_u,Y_u)\,I(S_u \neq g_*(Y_u))\,du + M_t$$

with the continuous local martingale $M = (M_t)_{t \geq 0}$ defined by:

$$M_t = \int_0^t e^{-ru}\,U_s(S_u,Y_u)\,I(S_u \neq g_*(Y_u))\,\sigma\,S_u\,d\overline{B}_u.$$

The verification assertion

It follows that the inequalities:

$$(\mathbb{L}_{(S,Y)}U - rU)(s,y) \le 0 \quad \text{for} \quad (s,y) \in (0,\infty)^2$$

$$U(s,y) \ge (s-K)^+ \quad \text{or} \quad \underline{g}(y) \le g_*(y) \quad \text{for} \quad (s,y) \in (0,\infty)^2$$

hold, and thus

$$e^{-r\tau}\,(S_\tau - K)^+ \le e^{-r\tau}\,U(S_\tau, Y_\tau) \le U(s,y) + M_\tau$$

for all stopping times τ of (S,Y) started at $(s,y) \in (0,\infty)^2$.

Then, for an arbitrary localizing sequence $(\tau_n)_{n\in\mathbb{N}}$, we have:

$$E_{s,y}\big[e^{-r(\tau\wedge\tau_n)}\,(S_{\tau\wedge\tau_n} - K)^+\big] \le E_{s,y}\big[e^{-r(\tau\wedge\tau_n)}\,U(S_{\tau\wedge\tau_n}, Y_{\tau\wedge\tau_n})\big]$$
$$\le U(s,y) + E_{s,y}\big[M_{\tau\wedge\tau_n}\big] = U(s,y).$$

The verification assertion

Hence, by means of Fatou lemma, we obtain:

$$E_{s,y}\left[e^{-r\tau}\,(S_\tau - K)^+\right] \le E_{s,y}\left[e^{-r\tau}\,U(S_\tau, Y_\tau)\right] \le U(s,y)$$

for any stopping times τ and all $(s,y) \in (0,\infty)^2$.

Since $U(s,y)$ and $g_*(y)$ solves the free-boundary problem, we have:

$$e^{-r(\tau_* \wedge \tau_n)}\,(S_{\tau_* \wedge \tau_n} - K)^+ = e^{-r(\tau_* \wedge \tau_n)}\,U(S_{\tau_* \wedge \tau_n}, Y_{\tau_* \wedge \tau_n}) = U(s,y) + M_{\tau_* \wedge \tau_n}$$

for any localizing sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$.

Therefore, applying the Lebesgue dominated convergence, we get:

$$E_{s,y}\left[e^{-r\tau_*}\,(S_{\tau_*} - K)^+\right] = E_{s,y}\left[e^{-r\tau_*}\,U(S_{\tau_*}, Y_{\tau_*})\right] = U(s,y)$$

for all $(s,y) \in (0,\infty)^2$, that proves the desired assertion. □

The main result

Theorem. *The value function takes the form:*

$$V_*(s,\pi) = \begin{cases} U_*(s, s^{-\eta}\pi/(1-\pi)), & \text{if } 0 < s < g_*(s^{-\eta}\pi/(1-\pi)) \\ s - K, & \text{if } s \geq g_*(s^{-\eta}\pi/(1-\pi)) \end{cases}$$

and the optimal exercise boundary $b_*(\pi)$ *is the inverse to:*

$$b_*^{-1}(s) = s^\eta g_*^{-1}(s)/(1 + s^\eta g_*^{-1}(s))$$

for each $rK/(\delta_0 \vee \delta_1) < s < rK/(\delta_0 \wedge \delta_1)$ *with* $\eta = (\delta_0 - \delta_1)/\sigma^2$*.*

Moreover, both the value function $V_*(s,\pi)$ *and the boundary* $b_*(\pi)$ *are decreasing (increasing) and continuous in* $\pi \in [0,1]$*, whenever* $\delta_0 < \delta_1$ *(*$\delta_0 > \delta_1$*).*

Some estimates

Remark 1. It can be checked that the function:

$$\widehat{W}(s,\pi) = W(s,0;a_*(0))\,(1-\pi) + W(s,1;a_*(0))\,\pi$$

solves the partial differential equation above for $0 < s < \widehat{a}(\pi)$, where

$$W_*(\widehat{a}(\pi),0;a_*(0))\,(1-\pi) + W_*(\widehat{a}(\pi)),1;a_*(0))\,\pi = \widehat{a}(\pi) - K$$

for all $\pi \in [0,1]$. It follows that the function:

$$\widehat{W}(s,\pi) = \begin{cases} W(s,\pi;\widehat{a}(\pi)), & \text{if } 0 < s < \widehat{a}(\pi) \\ s - K, & \text{if } s \geq \widehat{a}(\pi) \end{cases}$$

is a lower estimate for the value function $V_*(s,\pi)$, so that

$$W_*(s,1-i) \leq \widehat{W}(s,\pi) \leq V_*(s,\pi) \leq W_*(s,i) \quad \text{whenever} \quad \delta_{1-i} > \delta_i, \quad i = 0,1.$$

Some estimates

Suppose that a function $\widehat{U}(s,y)$ and the boundary $\widehat{g}(y)$ solve:

$$(\mathbb{L}_{(S,Y)}U - rU)(s,y) = \left(\frac{\lambda_1 - \lambda_0 s^\eta y}{1 + s^\eta y} - \frac{\eta}{2}\,(2r - \delta_0 - \delta_1 - \sigma^2) \right) y\, U_y(s,y) \text{ for } s < g(y)$$

and the general solution takes the form:

$$U(s,y) = C_1(y)\, s^{\alpha_1}\, F\Big(1 + \varphi_0 + \varphi_1, 1 + \varphi_0 - \varphi_1; 1 + \varphi_0; s^\eta y\Big) + C_2(y)\, s^{\alpha_2}\, F\Big(1 - \varphi_0 + \varphi_1, 1 - \varphi_0 - \varphi_1; 1 - \varphi_0; s^\eta y\Big)$$

where

$$\alpha_i = \frac{1}{2} - \frac{r - \delta_0}{\sigma^2} - (-1)^i \varphi_0 \eta, \quad \varphi_i = \frac{1}{\eta} \sqrt{\frac{\delta_i^2}{\sigma^4} + \delta_i \left(1 - \frac{2r}{\sigma^4}\right) + \left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2}.$$

and $F(a,b;c;x)$ is a Gauss' hypergeometric function.

Some estimates

Applying the boundary conditions, we get that $C_2(y) = 0$, so that:

$$C_1(y)\, g^{\alpha_1}(y)\, F(1+\varphi_0+\varphi_1, 1+\varphi_0-\varphi_1; 1+\varphi_0; g^\eta(y)y) = g(y) - K$$

$$\eta\, C_1(y)\, g^{\alpha_1+\eta}(y)y\, \frac{(1+\varphi_0)^2-\varphi_1^2}{1+\varphi_0}\, F(2+\varphi_0+\varphi_1, 2+\varphi_0-\varphi_1; 2+\varphi_0; g^\eta(y)y)$$

$$+\,\alpha_1\, C_1(y)\, g^{\alpha_1}(y)\, F(1+\varphi_0+\varphi_1, 1+\varphi_0-\varphi_1; 1+\varphi_0; g^\eta(y)y) = g(y)$$

and thus, the solution is given by:

$$U(s,y;\widehat{g}(y)) = (\widehat{g}(y) - K)\, \frac{s^{\alpha_1} F(1+\varphi_0+\varphi_1, 1+\varphi_0-\varphi_1; 1+\varphi_0; s^\eta y)}{\widehat{g}^{\alpha_1}(y) F(\rho+\varphi_0+\varphi_1, \rho+\varphi_0-\varphi_1; 1+\varphi_0; \widehat{g}^\eta(y)y)}$$

for all $0 < s < \widehat{g}(y)$ and each $y > 0$ fixed, where $\widehat{g}(y)$ is uniquely determined by:

$$\frac{(1+\varphi_0)^2-\varphi_1^2}{1+\varphi_0}\, \frac{F(2+\varphi_0+\varphi_1, 2+\varphi_0-\varphi_1; 2+\varphi_0; g^\eta(y)y)}{F(1+\varphi_0+\varphi_1, 1+\varphi_0-\varphi_1; 1+\varphi_0; g^\eta(y)y)} = \frac{\alpha_1 K + (1-\alpha_1)g(y}{(g(y)-K)\eta g^\eta(y)y}$$

Some estimates

Corollary. *Following the arguments of Lemma 2, it is shown that the function:*

$$\widehat{U}(s,y) = \begin{cases} U(s,y;\widehat{g}(y)), & \textit{if } 0 < s < \widehat{g}(y) \\ s - K, & \textit{if } s \geq \widehat{g}(y) \end{cases}$$

with $U(s,y;\widehat{g}(y))$ *defined above coincides with the value function:*

$$\widehat{U}(s,y) = \sup_{\tau} E_{s,y}\Big[e^{-r\tau}(S_\tau - K)^+ \\ \int_0^\tau e^{-rt}\left(\frac{\lambda_1 - \lambda_0 S_t^\eta Y_t}{1 + S_t^\eta Y_t} - \frac{\eta}{2}\,(2r - \delta_0 - \delta_1 - \sigma^2)\right) Y_t\,\widehat{U}_y(S_t, Y_t)\, I(S_t < \widehat{g}(Y_t))\,dt\Big]$$

and $\widehat{g}(y)$ *provides the hitting boundary for:*

$$\widehat{\tau} = \inf\{t \geq 0 \,|\, S_t \geq \widehat{g}(Y_t)\}$$

which is an optimal stopping time in the problem above.

Remark 2. Assume that $\lambda_0 = \lambda_1 = 0$ meaning that:

$$\Theta_t \equiv \theta, \quad P(\theta = 1) = \pi, \quad \text{and} \quad P(\theta = 0) = 1 - \pi, \quad \text{for} \quad \pi \in [0, 1].$$

Then, $\widehat{U}(s, y) \equiv U_*(s, y)$ and $\widehat{g}(y) \equiv g_*(y)$ holds, whenever $\delta_0 + \delta_1 = 2r - \sigma^2$.

Remark 3. Under the assumptions above, we have:

$$(\partial_y U_*)(s, y)\big|_{s=g_*(y)-} = 0$$

for all $y > 0$, and thus

$$(\partial_\pi V_*)(s, \pi)\big|_{s=b_*(\pi)-} = 0$$

with $b_*(\pi) = g_*(s^{-\eta}\pi/(1-\pi))$ for all $\pi \in (0, 1)$. At the same time, we have:

$$\widehat{W}_s(s, \pi)\big|_{s=\widehat{a}(\pi)-} < 1$$

for all $\pi \in (0, 1)$.

Thank you!