Resilience to contagion in financial networks

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Outline

The financial network

A network of financial counterparties can be modeled as a *weighted directed graph* whose

- n vertices (nodes) $i \in V$ represent financial market participants : banks, funds, corporate borrowers and lenders,...
- (directed) links represent counterparty exposures : e_{ij} is the exposure of i to j.
- In a market-based framework e_{ij} is understood as the fair market value of the exposure of i to j.
- Each institution i disposes of a *capital buffer* c_i which absorbs market losses. Insolvency occurs if Loss(i)$>c_i$.
- Proxy for c_i : Tier I+II capital minus required regulatory capital for non-financial assets. Other measures can be used.

Balance sheet

Assets	Liabilities
Interbank assets $A_i = \sum_j e_{ij}$	Interbank liabilities $L_i = \sum_j e_{ji}$ Deposits d_i
Other assets x_i	Net worth c_i

TABLE: Stylized balance sheet of a bank.

The capital ratio : $\gamma_i = \frac{c_i}{A_i}$

The default dynamics

The default of a market participant j affects its counterparts in the following way over a short term horizon

- Creditors lose a fraction $(1 - R)$ of their exposure.
  Loss is first absorbed by capital :
  $c_i \to \min(c_i - (1 - R)e_{ij}, 0)$.
- This leads to a writedown of $(1 - R)e_{ij}$ in the balance sheet of i, which can lead to **default** of i if

$$c_i < (1 - R)e_{ij}$$

Typically $R \simeq 0$ in the short term (liquidation takes time).

Heterogeneity in the structure of interbank networks

Example : Brazil's interbank network (data from Banco Central do Brasil 2008).

- Average number of counterparties (degree)$= 7$
- **Heterogeneity in number of debtors** : In-degree has a heavy-tailed Pareto distribution with exponent $\simeq 2$.
- **Heterogeneity in number of creditors** : Out-degree has a heavy-tailed Pareto distribution with exponent $\simeq 3$.
- Heterogeneous exposures sizes : heavy tailed distribution, a handful of bilateral exposures are > 100 times larger than most of the rest $\rightarrow$ Pareto distribution.

Financial networks under incomplete information

Financial system : weighted graph $\mathbf{e}$ with the vertex set $[1, \ldots, n]$ and the corresponding sequence of capital ratios $\gamma_{\mathbf{n}} = (\gamma_i)_{i=1}^{n}$.
The idea of this paper to do an embedding in a probability space : look at the weighted graph $\mathbf{e}$ as a realization of a random weighted graph $\mathbf{E}$, endowed with a sequence of capital ratios $\hat{\gamma}$.
We may observe partially the real network $\mathbf{e}$.
The degree sequence is prescribed as well as the exposures.

Random financial networks

Definition

The random financial network $\mathbf{E}$ is a random matrix of size n having the following properties :

- For every $1 \leq i \leq n$, the line $\mathbf{E_i}$ is a random uniform permutation of the line $\mathbf{e_i}$, with the constraint that $E_{i,i} = 0$;
- On every column $1 \leq j \leq n$, the number of non zero elements in $\mathbf{E}$ is the same as in $\mathbf{e}$.

Asymptotic study : idea

Aim : study contagion on the random financial network as its size $n \to \infty$.
We are given : the sequence $\mathbf{e}^{(\mathbf{n})}$ of financial networks.
On the probability space $(\Omega, \mathbb{P})$,
study contagion on the sequence of "rewired" networks $\mathbf{E}^{(\mathbf{n})}$.
More precisely, we introduce the final fraction of defaults

$$\alpha_n = \frac{N_{def}^n(\mathbf{E}^{(\mathbf{n})}, \hat{\gamma}^{(\mathbf{n})})}{n}$$

Question 1 : $\alpha_n \xrightarrow{p} ?$
and under which assumptions ?
Question 2 : How does the limit depend on the network topology and the individual exposures ?
Question 3 : Is the network resilient to small shocks ?

Asymptotic study

We have $\mathbf{d^{(n)+}} = \{(d_i^{(n)+})_{i=1}^n\}$ and $\mathbf{d^{(n)-}} = \{(d_i^{(n)-})_{i=1}^n\}$ the sequences of non-negative integers representing the degrees :

$$\sum_{i=1}^n d_i^{(n)+} = \sum_{i=1}^n d_i^{(n)-}.$$

We introduce the **empirical distribution of the degrees** as

$$\mu_n(j,k) := \frac{1}{n}\#\{i : d_i^{(n)+} = j, d_i^{(n)-} = k\}.$$

Assumptions on the degree sequence

For each $n \in \mathbb{N}$, $\mathbf{d}^{(\mathbf{n})+} = \{(d_i^{(n)+})_{i=1}^n\}$ and $\mathbf{d}^{(\mathbf{n})-} = \{(d_i^{(n)-})_{i=1}^n\}$ are sequences of nonnegative integers such that $\sum_{i=1}^n d_i^{(n)+} = \sum_{i=1}^n d_i^{(n)-}$, and we assume that for some probability distribution $\mu(j,k)$ independent of n,

$$\mu_n(j,k) \to \mu(j,k) \text{ as } n \to \infty\,;$$

$$\sum_{j,k} j\mu(j,k) = \sum_{j,k} k\mu(j,k) =: \lambda \in (0,\infty)\,;$$

$$m_n/n \to \lambda \text{ as } n \to \infty\,;$$

$$\sum_{i=1}^n (d_i^{(n)+})^2 + (d_i^{(n)-})^2 = O(n).$$

Mapping continuous into discrete variables

For each node i and permutation τ of the counterparties of i, we define

$$\Theta(i,\tau) := \min\{k \geq 0, c_i < \sum_{j=1}^{k}(1-R)e^{(n)}_{i,\tau(j)}\}$$

$\Theta(i,\tau)$ is the number of counterparty defaults which will generate the default of i if defaults happen in the order prescribed by τ.

$$p_n(j,k,\theta) := \frac{\#\{(i,\tau) \mid \tau \in \underbrace{\Sigma_j}_{Perm.\ of\ 1,..,j},\ d_i^{(n)+} = j,\ d_i^{(n)-} = k,\ \Theta(i,\tau) = \theta\}}{n\mu_n(j,k)j!}.$$

Contagious links

A link is called *contagious* if it generates a default of the end node if the starting node defaults.

$n\mu_n(j,k)jp_n(j,k,1)$ is the total number of contagious links that enter a node with degree (j,k).

The value $p_n(j,k,1)$ gives the proportion of contagious links ending in nodes with degree (j,k).

Assumptions on the exposure sequence

There exists a function $p : \mathbb{N}_+^3 \to [0,1]$ such that for all $j, k, \theta \in \mathbb{N}$ $(\theta \leq j)$

$$p_n(j,k,\theta) \overset{n\to\infty}{\to} p(j,k,\theta). \tag{1}$$

as $n \to \infty$. This assumption is fulfilled for exemple in a model where exposures are exchangeable arrays.

The probability limit for the final fraction of defaults

Let us define

$$\beta(n,\pi,\theta) := \mathbb{P}(Bin(n,\pi) \geq \theta) = \sum_{j\geq\theta}^{n} \binom{n}{j} \pi^j (1-\pi)^{n-j}.$$

We introduce the out-degree and respectively in-degree biased probability measures $\hat{\mu}$ and $\tilde{\mu}$

$$\hat{\mu}(j,k) = \frac{\mu(j,k)k}{\lambda} \quad (2)$$

$$\tilde{\mu}(j,k) = \frac{\mu(j,k)j}{\lambda} \quad (3)$$

representing the probability that an edge at random begins and respectively ends in a node with in-degree j and out-degree k.

Define

$$I(\pi) := \sum_{j,k} \hat{\mu}(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi,\theta) \qquad (4)$$

Theorem

Consider the sequences of exposures and capital ratios after shock $\{(\mathbf{e_n})_{\mathbf{n}\geq 1}, (\hat{\gamma}_\mathbf{n})_{\mathbf{n}\geq 1}\}$ satisfying the Assumptions on the degree and exposure sequence. Let π^* be the smallest fixed point of I. We have

- *If $\pi^* = 1$, i.e. if $I(\pi) > \pi$ for all $\pi \in [0,1)$, then asymptotically all nodes default during the cascades*

$$\alpha_n = 1 - o_p(1).$$

- *If $\pi^* < 1$ and furthermore π^* is a stable fixed point of I, then the asymptotic fraction of defaults*

$$\alpha_n \xrightarrow{p} \sum_{j,k} \mu(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi^*,\theta).$$

The intuition :branching process approximation

We can give the for

$$I(\pi) = \sum_{j,k} \hat{\mu}(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi,\theta)$$

the following interpretation : if the counterparty of a randomly chosen node defaults with probability π defaults, $I(\pi)$ is the expected fraction of counterparty defaults after one iteration of the cascade.

The function

$$\sum_{j,k} \mu(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi,\theta),$$

gives the fraction of defaulted nodes supposing that a counterparty of a randomly chosen node defaults with probability π.

Is the random network robust to small shocks ?

Corollary

If

$$\sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j,k,1) < 1 \tag{5}$$

is satisfied, then with high probability, the default of a single node cannot trigger the default of a positive fraction of the financial network.

The skeleton of contagious links

The converse also holds, if

$$\sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j,k,1) > 1,$$

then the network is not robust.

Proposition

Consider the sequence of random financial networks $(\mathbf{E}^{(\mathbf{n})}, \hat{\gamma}^{(\mathbf{n})})$ satisfying the Assumptions on the degree and exposure sequence. If

$$\sum_{j,k} k\tilde{\mu}(j,k) p(j,k,1) > 1,$$

then with high probability there exists set of nodes representing a positive fraction of the financial system, strongly interlinked such that any node belonging to this set can trigger the default of all nodes in the set.

Amplification

We suppose that the resilience condition is satisfied.
Let π^*_ϵ be the smallest fixed point of I in $[0,1]$, when a fraction ϵ of all nodes represent fundamental defaults, i.e. $p(j,k,0)=\epsilon$ for all j,k.
First order approximation of the function I :

$$\pi^*_\epsilon = \frac{\epsilon}{1-\sum_{j,k} k\tilde{\mu}(j,k)p(j,k,1)} + o(\epsilon).$$

$$\lim_{\epsilon\to 0}\frac{g(\pi^*_\epsilon)}{\epsilon} = 1 + \frac{\sum_{j,k} j\mu(j,k)p(j,k,1)}{1-\sum_{j,k} k\tilde{\mu}(j,k)p(j,k,1)}.$$

Amplification - cont.

We denote $\pi^*_\epsilon(d^+, d^-)$ the smallest fixed point of I in $[0, 1]$ in the case where $p(d^+, d^-, 0) = \epsilon$ and $p(j, k, 0) = 0$ for all $(j, k) \neq (d^+, d^-)$.
Then the good measure of how many times is the final fraction of defaults larger than the initial fraction of defaults is

$$\lim_{\epsilon \to 0} \frac{g\left(\pi^*_\epsilon(d^+, d^-)\right)}{\epsilon \mu(d^+, d^-)} = 1 + \frac{d^-}{\lambda} \frac{\sum_{j,k} \hat{\mu}(j, k) j p(j, k, 1)}{1 - \sum_{j,k} k \tilde{\mu}(j, k) p(j, k, 1)}.$$

Relevance of asymptotics

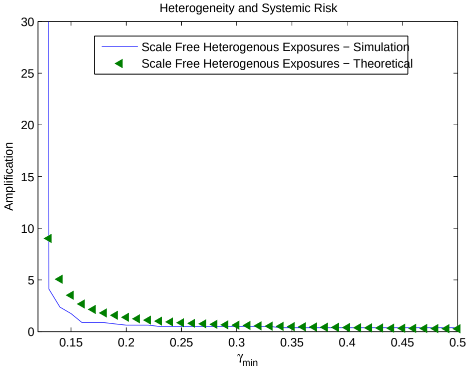

FIGURE: Amplification of the default number in a Scale-Free Network. The in and out-degree of the scale-free network are Pareto distributed with tail coefficients 2.19 and 1.98 respectively, the exposures are Pareto distributed with tail coefficient 2.61, $n = 10000$.

Amplification

We plot the simulated final fraction of defaults starting from one fundamental default in a simulated, scale free network as a function of the out-degree, versus the theoretical slope given above.

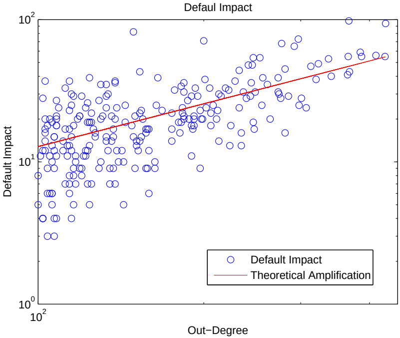

FIGURE: Number of defaulted nodes

The Impact of heterogeneity

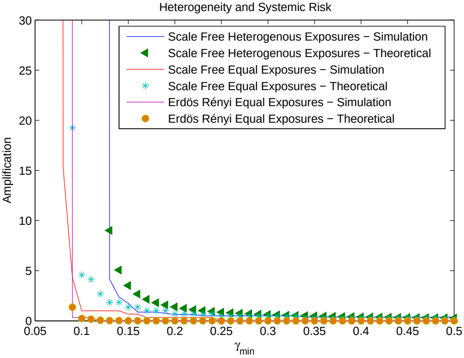

FIGURE: Amplification of the number of defaults in a Scale-Free Network (in and out-degree of the scale-free network are Pareto distributed with tail coefficients 2.19 and 1.98 respectively, the exposures are Pareto distributed with tail coefficient 2.61), the same network with equal weights and an Erdös Rényi Network with equal exposures $n = 10000$.

A simple external shock model

$$c_i = \gamma_{min} A_i.$$

We suppose that after the shock, the capital ratio becomes

$$\hat{\gamma}_i = \gamma_{min}(1 + \sigma_i(\sqrt{1-\rho}Y_i + \sqrt{\rho}Z)),$$

with $Y_i \sim \mathcal{N}(0,1)$ independent and Z is imposed.

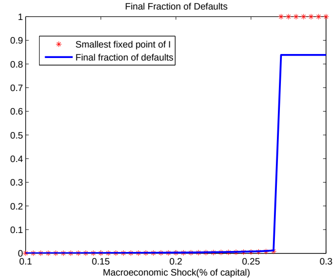

FIGURE: Final fraction of defaults

Phase transitions :Armageddon ?

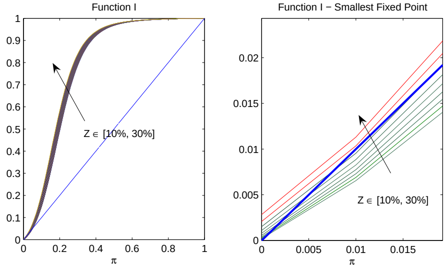

FIGURE: Function I for increasing magnitude of the macroeconomic shock. As the common factor increases, the smallest fixed point becomes 1 and the phase transition occurs.

Related problems

- Giant component
  Undirected graphs :
  - Molloy & Reed : The size of the giant component of a random graph with given degree sequence(1998)
  - Janson : A new approach to the giant component problem(2009)

  Directed graphs - Cooper&Frieze : The size of the largest strongly connected component of a random digraph with a given degree sequence(2007)
- Lukzak & Janson : A simple solution to the k-core problem(2005)
  Wormald & Cain :Encores on cores(2005)
- Balogh &Pittel : Bootstrap percolation on the random regular graph(2006)
  Amini :Bootstrap percolation and diffusion in random graphs with given vertex degrees(2010)

Configuration Model

The random graph $\mathbf{E}^{(n)}$ has the same distribution as the random multigraph G_n^* given by Configuration Model **conditional on it being simple**.

The conditions on the degree sequence insure (Janson 2009) that

$$\liminf_{n\to\infty} \mathbb{P}(G_n^* \textit{ is simple}) > 0.$$

We associate to each node two sets, W_i^+ representing its in-coming half edges and W_i^- representing its out-going half edges. We have that $|W_i^+| = d_i^+$ and $|W_i^-| = d_i^-$.

Let $W^+ = \bigcup_i W_i^+$ and $W^- = \bigcup_i W_i^-$.

We choose a random matching of W^+ with W^-, uniformly among all matchings. The in-coming half edges of node i are assigned independently from the underlying graph the exposure sequence.

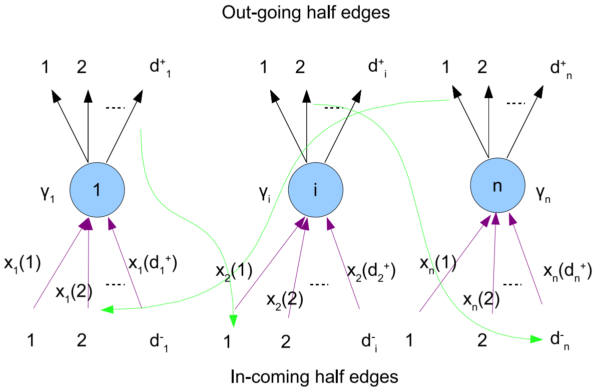

FIGURE: Configuration model. Green arrows : a random matching of out-going half edges with weighted in-coming half edges. We have re-denoted by $(x_l(i))_{l=1}^{d_i^+}$ i's set of exposures.

Contagion study

Ideas :

- Construct the random graph as contagion spreads.
- Construct the matching of half edges in CM in two steps : choose independently for each node i a random permutation τ_i of its in-coming half edges giving their relative order of matching to the out-going half edges and then do the global matching.
- Replace the information given by the capital ratios and the exposures by default thresholds :

$$\Theta_i = \min\{k \geq 0, c_i < \sum_{j=1}^{k} (1-R) e^{(n)}_{i,\tau_i(j)}\}$$

All nodes of degree (j, k) and threshold equal to θ become now indistinguishible.

Contagion explained by a simple Markov chain

We describe our Markovian system in terms of

- $D^{j,k,\theta}(t)$: the number of defaulted banks with in-degree j, out-degree k at time t and default threshold θ,
- $S_l^{j,k,\theta}(t)$, $l < \theta \leq j$, the number of solvent banks with in-degree j, out-degree k, default threshold θ and l defaulted debtors before time t,
- $D(t)$: the number of defaulted banks at time t,
- $D_{in}(t)$: the number of in-coming edges belonging to defaulted banks,
- D_{out} : the number of out-going edges belonging to defaulted banks,
- $S_{in}(t)$: the number of in-coming edges belonging to solvent banks at time t.

$$
\begin{aligned}
S_{in}(t) &= \sum_{j,k} \sum_{0 \le l < \theta} (j-l) S_l^{j,k,\theta}(t), \\
D_{out}(t) &= \sum_{j,k,0 \le \theta \le j} k D^{j,k,\theta}(t) - t, \\
D(t) &= \sum_{j,k,0 \le \theta \le j} D^{j,k,\theta}(t).
\end{aligned}
$$

The process will finish at the stopping time T_f which is the first time $t \in \mathbb{N}$ where $D_{out}(t) = 0$. The final number of defaulted banks will be $D(T_f)$.

Transition probabilities

$\mathbf{Y}(t) = \left(D^{j,k,\theta}(t), S_l^{j,k,\theta}(t)\right)_{j,k,0\leq l<\theta\leq j}$ represents a Markov chain. Choose an available out-going half edge belonging to a defaulted node A. Let B be its counterparty.

- B is defaulted, the next state is $\mathbf{Y}(t+1) = \mathbf{Y}(t)$.
- B is solvent of in-degree j, out-degree k, default threshold θ and this is the $(l+1)$-th deleted in-coming edge and $l+1<\theta$. The probability of this event is $\frac{(j-l)S_l^{j,k,\theta}(t)}{m_n-t}$. The changes for next state will be

$$S_l^{j,k,\theta}(t+1) = S_l^{j,k,\theta}(t) - 1,$$
$$S_{l+1}^{j,k,\theta}(t+1) = S_{l+1}^{j,k,\theta}(t) + 1.$$

- B is solvent of in-degree j, out-degree k, default threshold θ and this is the θ-th deleted in-coming edge. Then with probability $\frac{(j-\theta+1)S^{j,k,\theta}_{\theta-1}(t)}{m_n-t}$ we have

$$D^{j,k,\theta}(t+1) = D^{j,k,\theta}(t) + 1,$$
$$S^{j,k,\theta}_{\theta-1}(t+1) = S^{j,k,\theta}_{\theta-1}(t) - 1.$$

When $n \to \infty$ we have that $\frac{Y(n\tau)}{n} \xrightarrow{p} y(\tau)$ (Wormald 1995), with $y(\tau)$ the solutions the associated differential equations.

The differential equations can be solved in closed form.

Conclusions

- We can find in closed form the asymptotic limit in probability for the final fraction of defaults in the probabaility space of “rewired” networks.
- The formula suggests that the largest threat to the system is posed by banks that have both high out-degree and are high fraction of contagious links.
- Systemic nodes both highly interconnected and over-exposed.
- Certain networks may be intrinsicly fragile, and an external shock may trigger a phase transition.