Pricing and Hedging American Options under Exponential Subordinated Levy Processes by Malliavin Calculus

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Introduction (1)

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- Two main methods under the BSM model
  1. Regression based method and its variations, e.g., Longstaff-Schwartz's least-squares method, etc.
  2. Bally *et al.*'s Malliavin calculus method.
     - The main idea of this method is to express a conditional expectation as the ratio of two unconditional expectations.

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- A subordinated Levy process (SLP) is also called subordinated Brownian motion (SBM) or time changed Brwonian motion.
- Two typical such processes are normal inverse Gaussian (NIG) process and variance gamma (VG) process.

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- Both NIG & VG processes are special cases of GH processes

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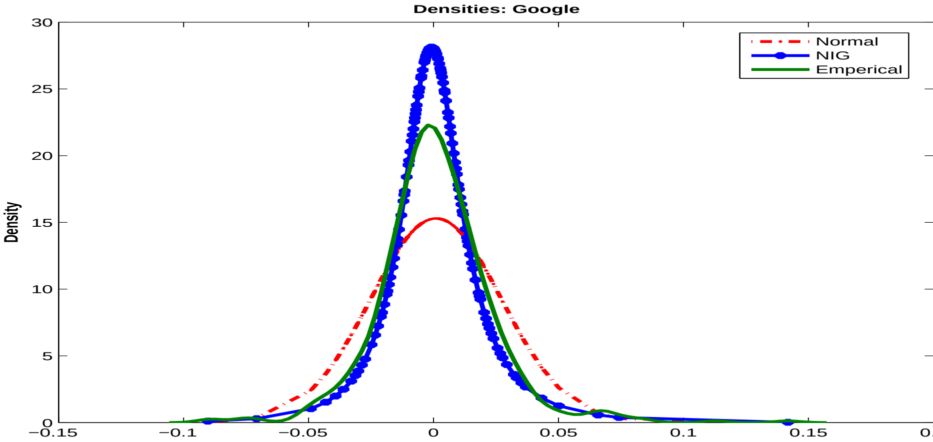

Figure: Comparisons of densities for Google data set

Introduction (5)

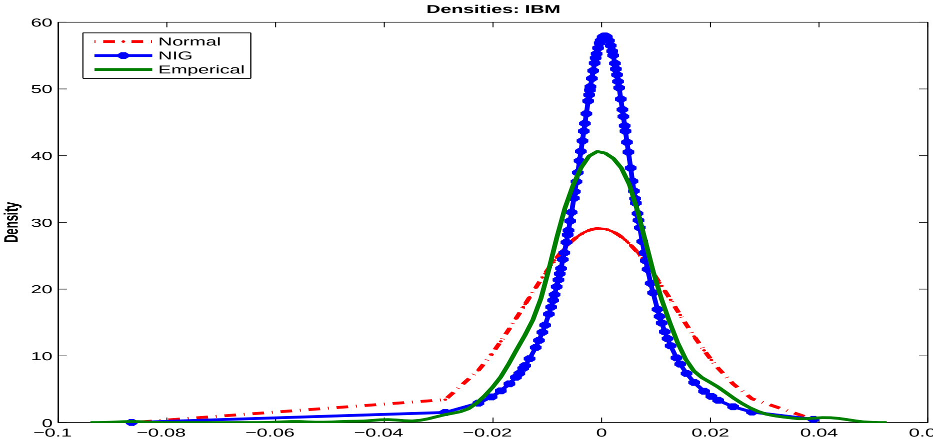

Figure: Comparisons of densities for IBM data set

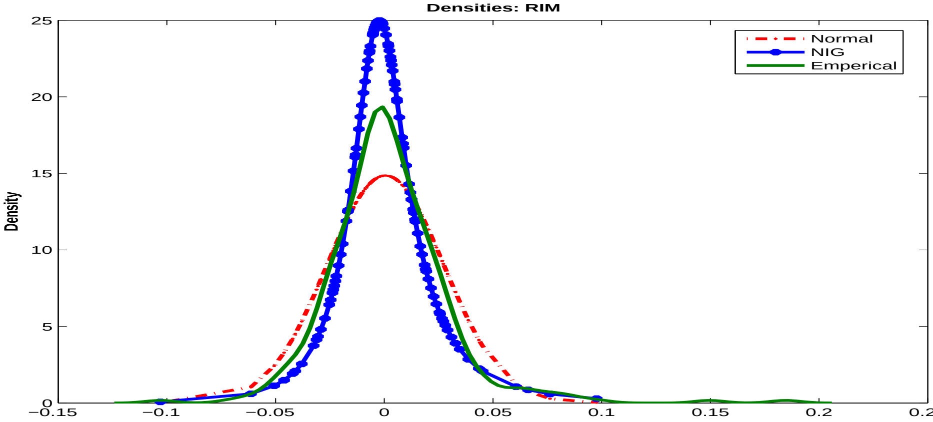

Figure: Comparisons of densities for RIM data set

Introduction (7)

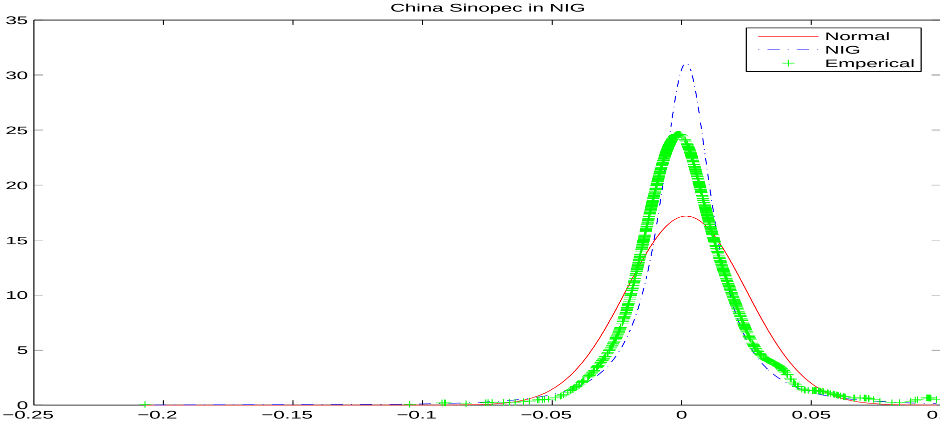

Figure: Comparisons of densities for SINOPEC data set

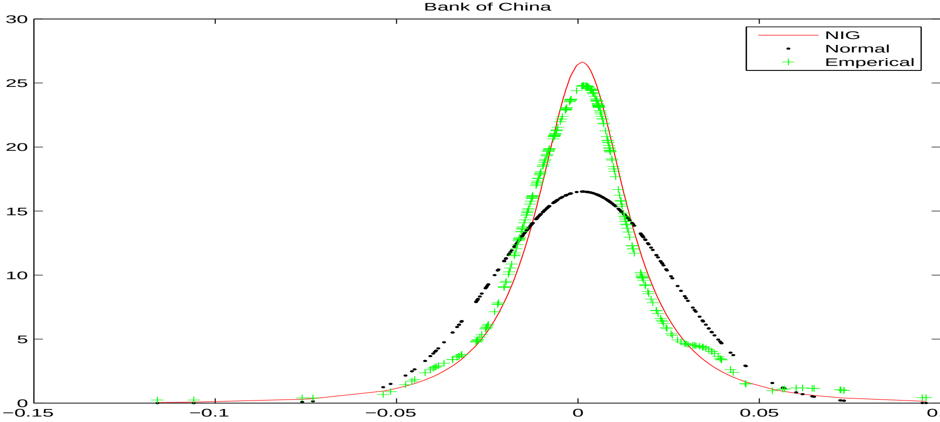

Figure: Comparisons of densities for BANKOFCHINA data set

Introduction (9)

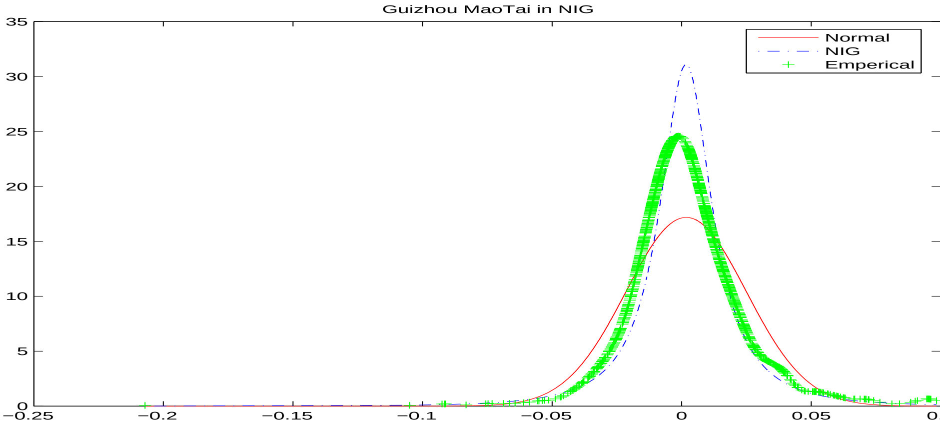

Figure: Comparisons of densities for MAOTAI data set

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- for any $\phi \in C_b^\infty(\mathbb{R})$= the set of bounded and infinitely differentiable functions.

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- with the convention that $E(G \mid F = \alpha) = 0$ if $E\left[H(F-\alpha)\pi_F(1)\right] = 0$;
- $H(x) = 1_{\{x \geq 0\}}(x)$, $x \in \mathbb{R}$ - the Heaviside function.

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  for $f, g \in C^1$.

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- where $\{Y_t\}$ is a subordinator process.
- Denote $\mathcal{F}_t = \sigma(Y_r, r \in [0, t])$, the $\sigma-$ field generated by $\{Y_r, r \in [0, t]\}$.

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- Proposition 1: Assume that S_t is given as before. Let $0 < s < t$, $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function with polynomial growth.

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- with $\Delta W_{s,t} = (Y_t - Y_s)W_{Y_s} - Y_s(W_{Y_t} - W_{Y_s}) + \sigma Y_s(Y_t - Y_s) = Y_t W_{Y_s} - Y_s W_{Y_t} + \sigma Y_s(Y_t - Y_s)$.

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- (2) For any $0 < s < t$, $\alpha > 0$ and Φ,

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- where $\mathbb{T}^{\psi}_{s,t}[\Phi](\alpha)$ and $\mathbb{R}^{\psi}_{s,t}[\Phi](\alpha)$ are given in the above.

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- Let $0 < s < t$, $\alpha > 0$ and $\Phi \in \varepsilon_b(\mathbb{R}^d)$ - the set of measurable functions with polynomial growth.
- To express the conditional expectation $E\left[\Phi(S_t)|S_s = \alpha\right]$, we try to use the results in one-dimensional case.

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- To this purpose, we consider an auxiliary process $\widetilde{S}_t$ with independent coordinates conditional on $\mathcal{F}_t$.
- Let $p_t = (p_{1;t}, \cdots, p_{d;t})$ be a fixed C^1 function (to be determined later) and let

$$\widetilde{S}_{i;t} = S_{i;0} \exp\left(\mu_i Y_t + p_{it} + c_{ii} W_{i;Y_t}\right), \ i = 1, \cdots, d,$$

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- $\widetilde{S}_{i;t} = G_{i;t}(S_t) = S_{i;0} e^{p_{i;t}} \prod_{l=1}^{d} \left(\frac{S_{l;t}}{S_{l;0}} e^{-\mu_l Y_t} \right)^{\widehat{c}_{il}}, i = 1, \cdots, d.$

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- and $\Delta W_{s,t;l} = Y_t W_{l;Y_s} - Y_s W_{l;Y_t} + c_{ll} Y_s(Y_t - Y_s)$.

Multi-asset case (5)

- (2) For any $0 < s < t$, $\alpha \in \mathbb{R}^d_+$, $\Phi \in \varepsilon_b(\mathbb{R}^d)$, and $i = 1, \cdots, d$, we have

$$\partial_{\alpha_i} E\left[\Phi(S_t)|S_s = \alpha\right] = \sum_{l=1}^{d} \widehat{c}_{il} \frac{\widetilde{\alpha}_l}{\alpha_i} \frac{\mathbb{R}_{s,t;l}[\Phi](\alpha)\mathbb{T}_{s,t}[1](\alpha) - \mathbb{R}_{s,t;l}[1](\alpha)\mathbb{T}_{s,t}[\Phi](\alpha)}{\left(\mathbb{T}_{s,t}[1](\alpha)\right)^2},$$

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$$\mathbb{R}_{s,t;l}[f](\alpha) = -E\left\{ f(S_t) \frac{H(\widetilde{S}_{l;s} - \widetilde{\alpha}_l)}{c_{ll} Y_s (Y_t - Y_s) \widetilde{S}_{l;s}} \left[\frac{(\Delta W_{s,t;l})^2}{c_{ll} Y_s (Y_t - Y_s)} + \Delta W_{s,t;l} - \frac{Y_t}{c_{ll}} \right] \prod_{j=1, j \neq l}^{d} \frac{H(\widetilde{S}_{j;s} - \widetilde{\alpha}_j)}{c_{jj} Y_s (Y_t - Y_s) \widetilde{S}_{j;s}} \Delta W_{s,t;j} \right\}.$$

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- Lemma 5 (Localization)

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- Let $\psi(x) = \prod_{i=1}^{d} \psi_i(x_i)$, where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, each $\psi_i : \mathbb{R} \to \mathbb{R}_+ = [0,\infty)$ is a PDF with its corresponding CDF Ψ_i.

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$$\mathbb{T}_{s,t}[f](\alpha) = \mathbb{T}^{\psi}_{s,t}[f](\alpha)$$
$$= E\left[f(S_t) \prod_{i=1}^{d} \left(\psi_i(S_{i;s} - \alpha_i) + \frac{H(\widetilde{S}_{is} - \widetilde{\alpha}_i) - \Psi_i(\widetilde{S}_{is} - \widetilde{\alpha}_i)}{c_{ii} Y_s (Y_t - Y_s) \widetilde{S}_{is}} \Delta W_{s,t;i} \right) \right]$$

Multi-asset case (7)

- and

$$\mathbb{R}_{s,t;l}[f](\alpha) = \mathbb{R}^{\psi}_{s,t;l}[f](\alpha) = -E\left\{ f(S_t)\left[\psi_l(\widetilde{S}_{l;s} - \widetilde{\alpha}_l)\frac{\Delta W_{s,t;l}}{c_{ll} Y_s (Y_t - Y_s)\widetilde{S}_{l;s}} \right.\right.$$

$$\left. + \frac{H(\widetilde{S}_{l;s} - \widetilde{\alpha}_l) - \Psi_l(\widetilde{S}_{l;s} - \widetilde{\alpha}_l)}{c_{ll} Y_s (Y_t - Y_s)(\widetilde{S}_{l;s})^2} \left(\frac{\Delta W^2_{s,t;l}}{c_{ll} Y_s (Y_t - Y_s)} + \Delta W_{s,t;l} - \frac{Y_t}{c_{ll}} \right) \right]$$

$$\left. \times \prod_{q=1, q\neq l}^{d} \left(\psi_q(\widetilde{S}_{q;s} - \widetilde{\alpha}_q) + \frac{H(\widetilde{S}_{q;s} - \widetilde{\alpha}_q) - \Psi_q(\widetilde{S}_{q;s} - \widetilde{\alpha}_q)}{c_{ss} Y_s (Y_t - Y_s)\widetilde{S}_{q;s}} \Delta W_{s,t;q} \right) \right\},$$

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- To find $V(0, S_0)$ and $\Delta(0, S_0)$, we can use the formulas for the conditional expectations discussed in the previous sections.
- To this end, we equally subdivide the interval $[0, T]$ into $m(> 1)$ subintervals: $0 = t_0 < t_1 < \cdots < t_m = T$, $t_j = jh$ with step size $h = T/m$.

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- 5^0: Computation of $\{\Delta W_{i,j,k}\}$:

$$\Delta W_{i,j,k} = \Delta W^k_{t_j,t_{j+1};i} = Y^k_{t_{j+1}} W_{i;Y^k_{t_j}} - Y^k_{t_j} W_{iY^k_{t_{j+1}}} + c_{ii} Y^k_{t_j}(Y^k_{t_{j+1}} - Y^k_{t_j}).$$

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- $$= \frac{E\left[V_{j+1}(S_{t_{j+1}})\prod_{l=1}^{d}\frac{H(\widetilde{S}_{l;t_j}-\widetilde{\alpha}_l)}{\sigma_{ll}Y_{t_j}(Y_{t_{j+1}}-Y_{t_j})\widetilde{S}_{l;t_j}}\Delta W_{Y_{t_j},Y_{t_{j+1}};l}\right]}{E\left[\prod_{l=1}^{d}\frac{H(\widetilde{S}_{l;t_j}-\widetilde{\alpha}_l)}{\sigma_{ll}Y_{t_j}(Y_{t_{j+1}}-Y_{t_j})\widetilde{S}_{l;t_j}}\Delta W_{Y_{t_j},Y_{t_{j+1}};l}\right]}$$

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- $$\approx \widehat{V}_j(S^k_{t_j}) = \frac{\sum_{q=1}^{N} V_{j+1}(S^q_{t_{j+1}})\prod_{l=1}^{d}\frac{H(\widetilde{S}^q_{l;t_j}-\widetilde{S}^k_{l;t_j})}{\sigma_{ll}Y^q_{t_j}(Y^q_{t_{j+1}}-Y^q_{t_j})\widetilde{S}^q_{l;t_j}}\Delta W_{l,j;q}}{\sum_{q=1}^{N}\prod_{l=1}^{d}\frac{H(\widetilde{S}^q_{l;t_j}-\widetilde{S}^k_{l;t_j})}{\sigma_{ll}Y^q_{t_j}(Y^q_{t_{j+1}}-Y^q_{t_j})\widetilde{S}^q_{l;t_j}}\Delta W_{l,j;q}}$$

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- 7^0: Computation of option price $V(0, S_0)$:

$$V(0, S_0) \approx V_0(S_0) = \max\left(\Phi(S_0), \frac{1}{N}\sum_{k=1}^{N} V_1(S_{t_1}^k)\right),$$

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- $V_m(S_{t_m}^k) = \Phi(S_{t_m}^k)$, $k = 1, \cdots, N$, and
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- $$\Delta_i(S_0) \approx \overline{\Delta}_i(S_0) = \frac{1}{N}\sum_{k=1}^{N} D_{ik}(S_{t_1}^k),$$
- $$D_{ik}(S_{t_1}^k) = \begin{cases} \partial_{\alpha_i}\Phi(\alpha)|_{\alpha=S_{t_1}^k}, & \text{if } U_1(S_{t_1}^k) < \Phi(S_{t_1}^k) \\ e^{-hr}\partial_{\alpha_i} E\left[V_2(S_{t_2})|S_{t_1} = \alpha\right]|_{\alpha=S_{t_1}^k}, & \text{if } U_1(S_{t_1}^k) > \Phi(S_{t_1}^k) \end{cases},$$

Algorithms (9)

- and

$$\partial_{\alpha_i} E\left[V_2(S_{t_2})|S_{t_1}=\alpha\right]|_{\alpha=S^k_{t_1}} = \sum_{l=1}^{d} \widehat{\sigma}_{il} \frac{\widetilde{S}^k_{l;t_1}}{S^k_{i;t_1}} \times$$

$$\frac{\mathbb{R}_{t_1,t_2;l}[\Phi](S^k_{t_1})\mathbb{T}_{t_1,t_2}[1](S^k_{t_1}) - \mathbb{R}_{t_1,t_2;l}[1](S^k_{t_1})\mathbb{T}_{t_1,t_2}[\Phi](S^k_{t_1})}{\left(\mathbb{T}_{t_1,t_2}[1](S^k_{t_1})\right)^2}$$

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$$\partial_{\alpha_i} E\left[V_2(S_{t_2})|S_{t_1} = \alpha\right]|_{\alpha=S_{t_1}^k} = \sum_{l=1}^{d} \widehat{\sigma}_{il} \frac{\widetilde{S}_{l;t_1}^k}{S_{i;t_1}^k} \times$$
$$\frac{\mathbb{R}_{t_1,t_2;l}[\Phi](S_{t_1}^k)\mathbb{T}_{t_1,t_2}[1](S_{t_1}^k) - \mathbb{R}_{t_1,t_2;l}[1](S_{t_1}^k)\mathbb{T}_{t_1,t_2}[\Phi](S_{t_1}^k)}{\left(\mathbb{T}_{t_1,t_2}[1](S_{t_1}^k)\right)^2}$$

-

$$\mathbb{T}_{t_1,t_2}[f](S_{t_1}^k) \approx \frac{1}{N}\sum_{q=1}^{N} f(S_{t_2}^q)\prod_{l=1}^{d}\frac{H(S_{l;t_1}^q - \widetilde{S}_{l;t_1}^k)}{\sigma_{ll} Y_{t_1}^q (Y_{t_2}^q - Y_{t_1}^q)\widetilde{S}_{l;t_1}^q}\Delta W_{l,1;q},$$

Algorithms (10)

- $$\mathbb{R}_{t_1,t_2;l}[f](S_{t_1}^k) \approx -\frac{1}{N}\sum_{q=1}^{N} f(S_{t_2}^q)\frac{H(\widetilde{S}_{l;t_1}^q - \widetilde{S}_{l;t_1}^k)}{\sigma_{ll} Y_{t_1}^q (Y_{t_2}^q - Y_{t_1}^q)\widetilde{S}_{l;t_1}^q} \times \left[\frac{(\Delta W_{l,1;q})^2}{\sigma_{ll} Y_{t_1}(Y_{t_2} - Y_{t_1})} + \Delta W_{l,1;q} - \frac{Y_{t_2}}{\sigma_{ll}}\right] \times \prod_{n=1,n\neq l}^{d} \frac{H(\widetilde{S}_{n;t_1}^q) - \widetilde{S}_{n;t_1}^k)}{\sigma_{nn} Y_{t_1}(Y_{t_2} - Y_{t_1})\widetilde{S}_{n;t_1}^q}\Delta W_{n,1;q}.$$

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- What can be used when estimating the ratio of two expectations $\frac{E(X)}{E(Y)}$?
- Other type of Levy processes?