# A new Feynman-Kac-formula for option pricing in Lévy models

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(Joint work with E. Eberlein)

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# PIDEs for option pricing: a 'universal approach'

Types of options

- European
- Barrier
- Lookback
- American

Modeling processes

- Lévy
- Affine
- Markov (Stoch. vol.)

solve PIDE $\Longrightarrow$

pricing

hedging

calibration

# PIDE $\leftrightarrow$ options in Lévy models: Starting point

Model for stock price: $S_t = S_0 \, \mathrm{e}^{L_t}$, savings account: $r_t = \mathrm{e}^{rt}$.

European Call: payoff $\left(S_T - K\right)^+$.

Price under risk-neutral probability:

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If $V \in C^{1,2}$ Itô's formula $\Longrightarrow$

$$V(t, L_t) - V(s, L_s) = \int_s^t \left(\dot{V} + \mathcal{G}\, V\right)(u, L_u)\, \mathrm{d}u + \text{martingale}$$

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$$\underbrace{V(t, L_t) - V(s, L_s)}_{=\text{martingale}} = \int_s^t \underbrace{\Big(\dot{V} + \mathcal{G}\,V\Big)}_{=0}(u, L_u)\,\mathrm{d}u + \text{martingale}$$

$\Longrightarrow$ PIDE: $\dot{V} + \mathcal{G}\,V = 0$.

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PIDE: $\dot{V} + \mathcal{G}\, V = 0 \quad V(T) = g$

$L_t = \sigma B_t + \mu t$ Black-Scholes

$\Longrightarrow$ inf. generator $\mathcal{G} = \frac{\sigma}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu\frac{\mathrm{d}}{\mathrm{d}x}$

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$$L_t = \sigma B_t + \mu t + L_t^{\text{jump}} \quad \text{Lévy model}$$

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$$\mathcal{G}^{\text{jump}}\varphi(x) = \int \Big(\varphi(x+y) - \varphi(x) - \big(\mathrm{e}^y - 1\big) h(y)\varphi'(x)\Big) F(\mathrm{d}y)$$

## Feynman - Kac formula

Conclusion: If $V \in C^{1,2}$ $\Longrightarrow$ $V$ solves PIDE $\dot{V} + \mathcal{G}\, V = 0$

Is assumption $V \in C^{1,2}$ good? Also for barrier options?

Aim: If $V$ solves PIDE $\Longrightarrow$ $V(t, L_t) =$ option price

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Feynman-Kac representation aimed at

- Efficient numerical evaluation

  $\Longrightarrow$ wavelet Galerkin methods
- Tractability for Lévy models.

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# Hilbert space approach

Gelfand triplet $V \hookrightarrow H \hookrightarrow V^*$ and $\mathcal{A} : V \to V^*$ linear with bilinear form

$$a(u,v) := (\mathcal{A}\,u)(v) = \langle \mathcal{A}\,u, v\rangle_{V^* \times V}.$$

| | | |
|---|---|---|
| **Continuity** | $a(u,v) \le C_1 \|u\|_V \|v\|_V$ | $(u, v \in V)$, |
| **Gårding inequality** | $a(u,u) \ge C_2 \|u\|_V^2 - C_3 \|u\|_H^2$ | $(u \in V)$ |

with $C_1, C_2 > 0$ and $C_3 \ge 0$.

# Hilbert space approach

Continuity and Gårding inequality $\Longrightarrow$

**Theorem**

*For $f \in L^2\big(0, T; V^*\big)$ and $g \in H$,*

$$\begin{aligned} \partial_t u + \mathcal{A}\, u &= f \\ u(0) &= g\,, \end{aligned}$$

*has a unique solution $u \in W^1$.*

$$u \in W^1 \;:\Leftrightarrow\; u \in L^2\big(0, T; V\big) \text{ with } \dot{u} \in L^2\big(0, T; V^*\big)$$

# Variational formulation → Approximation scheme

Variational form for all $\varphi \in C_0^\infty\big(0, T\big)$ and all $v \in V$:

$$-\int_0^T \langle u(t), v\rangle_H \, \varphi'(t)\,\mathrm{d}t + \int_0^T a\big(u(t), v\big)\varphi(t)\,\mathrm{d}t = \int_0^T \langle f(t), v\rangle_{V^*\times V}\, \varphi(t)\,\mathrm{d}t$$

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Finite Element Approximation: $V^N \subset V^{N+1} \subset \ldots V$ finite dimensional with $\cup_{n\in\mathbb{N}} V^N$ dense in $V$

$\Longrightarrow$ system of linear ODEs.

## Lévy models

Model $S_t = S_0 \, \mathrm{e}^{L_t}$ with Lévy process $L$:

- model the distribution of log-returns
- high flexibility
- tractability: pricing, calibration

---

Lévy-Khintchine formula $E \, \mathrm{e}^{iuL_t} = \mathrm{e}^{t\kappa(iu)}$

$$\kappa(iu) = -\frac{\sigma}{2}u^2 + ibu + \int \left( \mathrm{e}^{iuy} - 1 - iuh(y) \right) F(\mathrm{d}y)$$

- Tractability of Lévy models: via **Fourier transform**

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Hence: Why PIDE instead of Fourier methods?

## Pricing options with Fourier methods

European options $\Longrightarrow$ convolution formula,
*Raible (2000), Carr and Madan (1999)*

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Barrier options $\Longrightarrow$ Wiener-Hopf factorization,
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pricing    hedging    calibration

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We combine PDE and Fourier method

## We study PIDEs in the light of Fourier transform

Fourier transform of $\mathcal{A}\,u$:

$\mathcal{A}$ Pseudo Diff. Op . $\mathcal{F}\big(\mathcal{A}\,u\big) = A\,\mathcal{F}(u)$

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Symbol
$$A(\xi) = \frac{\sigma}{2}\xi^2 + ib\xi - \int \big(\,\mathrm{e}^{-i\xi y} - 1 - i\xi h(y)\big)\,F(\mathrm{d}y) = -\kappa(-i\xi)$$

# Sobolev index

Bilinear form via Symbol for $u, v \in C_0^\infty$

$$a(u,v) = \int (\mathcal{A}\, u) v = \frac{1}{2\pi} \int A\, \hat{u}\, \overline{\hat{v}} \quad \text{(Parseval)}$$

Sobolev-Index of $A$ is $\alpha > 0$, if

**Continuity condition** $\left|A(\xi)\right| \leq C_1 \left(1 + |\xi|\right)^\alpha$

**Gårding condition** $\Re\left(A(\xi)\right) \geq C_2 \left(1 + |\xi|\right)^\alpha - C_3 \left(1 + |\xi|\right)^\beta$

for $\xi \in \mathbb{R}$ with $C_1, C_2 > 0$ and $C_3 \geq 0$, $0 \leq \beta < \alpha$.

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# Sobolev index and Sobolev spaces

$\Longrightarrow$ Continuity and Gårding-inequality are satisfied in a Sobolev-Slobodeckii space $H^{\alpha/2}$.

Sobolev-Slobodeckii spaces $H^s := \{u \in L^2 \mid \|u\|_{H^s} < \infty\}$ with norm

$$\|u\|_{H^s}^2 = \int \left|\hat{u}(\xi)\right|^2 \left(1 + |\xi|^2\right)^s \mathrm{d}\xi$$

# Sobolev index and parabolic PIDEs

Gelfand triplet $H^s \hookrightarrow L^2 \hookrightarrow (H^s)^* = H^{-s}$.

Let $\mathcal{A}$ PDO with Symbol $A$ with Sobolev index $\alpha$.

**Theorem**

*Let $s = \alpha/2$. For $f \in L^2\big((0,T); H^{-s}\big)$ and $g \in L^2$*

$$\begin{aligned} \partial_t u + \mathcal{A}\, u &= f \\ u(0) &= g\,, \end{aligned}$$

*$\exists!$ solution $u \in W^1$ i.e. $u \in L^2(0,t;H^s)$ with $\dot{u} \in L^2(0,t;H^{-s})$.*

## Stochastic representation

If $A$ is the symbol of a Lévy-Process $L$ with with Sobolev index $\alpha$

$\Longrightarrow$ the solution $u \in W^1\big(0, T; H^{\alpha/2}, L^2\big)$ of has the stochastic representation

$$u(T-t, L_t) = E\Big(g(L_T) - \int_t^T f(T-s, L_s)\,\mathrm{d}s \Big| \mathcal{F}_t\Big)$$

## Sobolev index for Lévy processes

| Distribution | Sobolev index $\alpha$ |
|---|---|
| Brownian motion (+ drift) | 2 |
| GH, NIG | 1 |
| CGMY, no drift | Y |
| CGMY, with drift, $Y \geq 1$ | Y |
| CGMY, with drift, $Y < 1$ | – |
| Variance Gamma | – |

Sobolev index $\alpha < 2 \Longrightarrow$ Blumenthal-Getoor index $\beta = \alpha$.

For option pricing: drift condition $\Longrightarrow$ Require $\alpha \geq 1$.

## PIDE for option pricing: weighted spaces

Usual options: $g \notin L^2 \Longrightarrow$ Exponential weight

$$g_\eta(x) := e^{\eta x}\, g(x) \in L^2$$

Weighted $L^2$ $\quad L^2_\eta := \{u \mid u_\eta \in L^2\}$

---

Weighting function $\quad u \to u_\eta \quad \leftrightarrow \quad \hat{u} \to \hat{u}(\cdot - i\eta)$

Weighted Sobolev-Slobodeckii spaces $H^s_\eta$ analogue to $H^s$.

# PIDE for option pricing: weighted spaces

Analytic continuation of the Lévy symbol $A$
to the complex strip $U_{-\eta} := \mathbb{R} - i\,\mathrm{sgn}(\eta)[0, |\eta|)$

(A1) Exponential moments $\int_{|x|>1} e^{-\eta x} F(dx) < \infty$.

(A2) Continuity $\big|A(z)\big| \le C_1\big(1+|z|\big)^{\alpha}$.

(A3) Gårding $\Re\big(A(z)\big) \ge C_2\big(1+|z|\big)^{\alpha} - C_3\big(1+|z|\big)^{\beta}$ .

For all $z \in U_{-\eta}$ with constants $C_1,\ C_2 > 0,\ C_3 \ge 0$
and $0 \le \beta < \alpha$

$\Rightarrow$ $\exists!$ solution of parabolic equation and stoch. representation.

# Feynman-Kac formula for boundary value pb

Boundary value problem

$$\begin{aligned} \partial_t u + \mathcal{A}\, u &= f \qquad \text{on open set } D \subset \mathbb{R}\,,\ D \neq \mathbb{R} \\ u(0) &= g \end{aligned}$$

Aim: Stoch. representation of solution $u$.

---

Precise formulation: $\widetilde{H}^{\alpha/2}_\eta(D) = \{ u \in H^{\alpha/2}_\eta(\mathbb{R}) \mid u|_{D^c} \equiv 0 \}$

Find $u \in W^1\big(0, T; \widetilde{H}^{\alpha/2}_\eta(D), \widetilde{L}^2_\eta(D)\big)$ with

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# PIDEs for barrier options

Barrier option: payoff $g(L_T)\mathbb{1}_{\{\tau_{\overline{D}}>T\}}$

Fair price $\Pi_t = \underbrace{E\big(g(L_T)\,\mathrm{e}^{-r(T-t)}\,\mathbb{1}_{\{T<\tau_{t,\overline{D}}\}}\big|\mathcal{F}_t\big)}_{u(T-t,L_t)}\,\mathbb{1}_{\{t<\tau_{\overline{D}}\}}$

**Theorem**

*Let (A1)–(A3) with $\alpha \geq 1$ and $\eta \in \mathbb{R}$ and $g \in L^2_\eta(D)$. Let $g$ be bounded or $\int_{\{|x|>1\}} \mathrm{e}^{2|x\eta|}\,F(\mathrm{d}x) < \infty$. Then $u$ is the unique solution of*

$$\begin{aligned} \partial_t u + \mathcal{A}\,u + ru &= 0 \\ u(0) &= g \end{aligned}$$

*in $W^1\big(0,T;\widetilde{H}^{\alpha/2}_\eta(D);L^2_\eta(D)\big)$.*

## Digital barrier option

Payoff $\mathbb{1}_{\{L_T<B\}}\mathbb{1}_{\{\tau^B<T\}} \implies g(x) = \mathbb{1}_{(-\infty,B)}(x)$.

solution of PIDE in the weighted space $(\eta > 0)$

$$u = u^{\text{digi},B} \in W^1\big(0,T;\widetilde{H}^{\alpha/2}_{\eta}(-\infty,B);L^2_{\eta}(-\infty,B)\big).$$

# PIDE for digital barrier options: numerical result

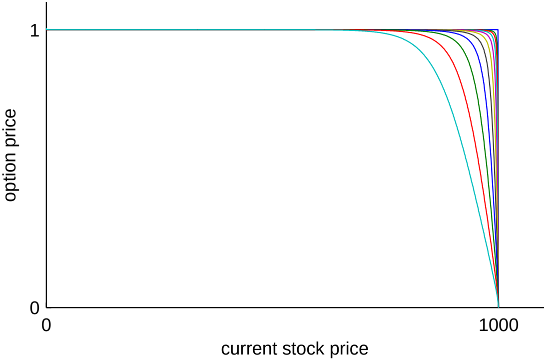


Figure: Price of digital barrier option maturity $(1/2)^k$, $k = 0, \ldots 9$ in a CGMY model.

# Distribution function of the supremum

Formula $F^{\overline{L}_T}$ of the supremum $\overline{L}_T = \sup_{0\le t\le T} L_t$ i.e.

$$F^{\overline{L}_T}(x) = P(\overline{L}_T < x) = u^{\mathrm{digi},0}(T,-x) \,.$$

The fair price $\quad V_0(S_0) = \mathrm{e}^{-rT}\, E\Big(\sup_{0\le t\le T} S_t - K\Big)^+$

$$V_0(S_0) = S_0\,\mathrm{e}^{-rT}\left(\int_{k-\log(S_0)}^{\infty} \big(1 - u^{\mathrm{digi},0}(T,-x)\big)\,\mathrm{e}^x\,\mathrm{d}x + (1 - K/S_0)^+\right).$$

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# Numerical result

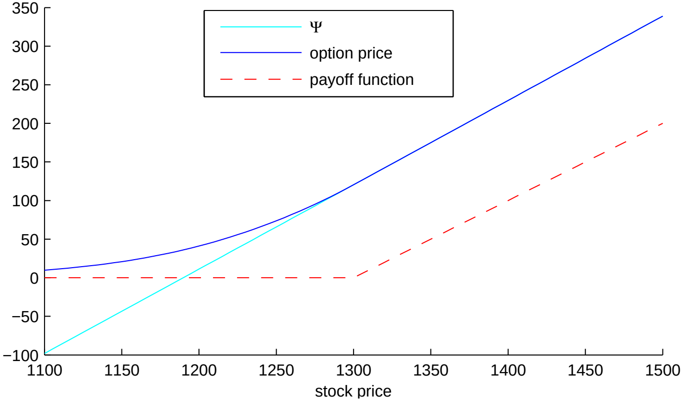


Figure: $\Psi(S_0) = E \max_{0 \le t \le T} S_t - K$, maturity 1 year, strike 1300.