Behavioural Portfolio Selection with Loss Control

Dr. Hanqing Jin

Mathematical Institute, University of Oxford

Oxford-Man Institute of Quantitative Finance

A joint work with Xun Yu Zhou and Song Zhang

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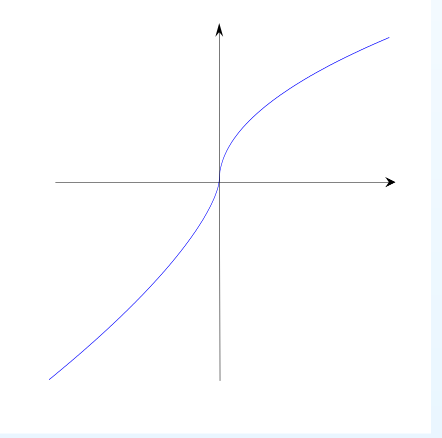

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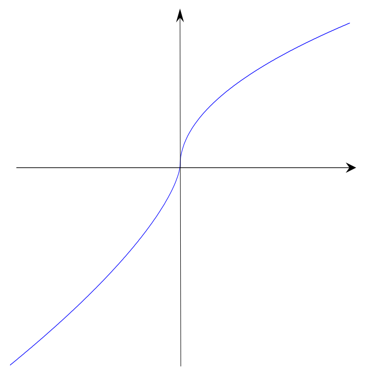
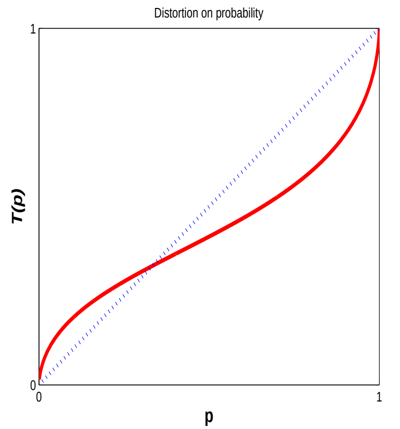
    - $u_\pm(\cdot)$ are concave, $\uparrow$
  - Probability distortions $T_\pm(\cdot) : [0,1] \mapsto [0,1]$
    - $T_\pm \uparrow$, $T_\pm(0) = 0, T_\pm(1) = 1$
    - $T_\pm(p) > p$ for small p





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- Behavioral criterion: for a r.v. Y,

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- Investor's problem

$$\begin{aligned} &\text{Maximize} \quad V(X - B) \\ &s.t. \quad \begin{cases} X \in \mathcal{A} \\ E[X\rho] = x_0 \end{cases} \end{aligned}$$

where $\mathcal{A}$ is the set of admissible terminal wealths.

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  - Optimal investment in Jin and Zhou has a deterministic loss in a bad market situation
  - But the loss can be large enough to intrigue disasters, like bankruptcy.

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- In our problem,
  - Investor are risk seeking for loss
  - Motivate the investor to borrow money for risky investor
  - Heavy loss may happen
  - Bankruptcy probability is higher when the investor is more aggressive
- To prevent disaster, a constraint on loss is necessary

Problem with bounded loss

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\begin{aligned}
&\text{Maximize} \quad V(X - B) \\
&s.t. \quad \begin{cases} X \geq B - L \\ E[X\rho] = x_0 \end{cases}
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Suppose the reference is bounded. Rewrite the problem by changing variable $\tilde{X} = X - B$,

$$\text{Maximize} \quad V_+(\tilde{X}^+) - V_-(\tilde{X}^-)$$
$$s.t. \quad \begin{cases} \tilde{X} \geq -L \\ E[\tilde{X}\rho] = \tilde{x}_0 := x_0 - E[\rho B] \end{cases}$$

where $V_\pm(Y) = \int_0^{+\infty} T_\pm(P(u_\pm(y) \geq y))dy$.

Splitting of the problem

- We use the same splitting from Jin and Zhou (2008)
- For any $c \in (\operatorname{essinf} \rho, \operatorname{esssup} \rho)$, $\tilde{x}_+ \geq \tilde{x}_0^+$, solve the following problems to get their value function $v_\pm(c, \tilde{x}_+)$

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$$\max \quad V_+(\tilde{X}_+)$$

$$s.t. \begin{cases} \tilde{X}_+ \geq 0 \\ \tilde{X} = 0 \text{ when } \rho > c \\ E[\tilde{X}_+ \rho] = \tilde{x}_+ \end{cases}$$

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\begin{aligned}
&\min \quad V_-(\tilde{X}_-) \\
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- Then find the optimal splitting c^* and $\tilde{x}_+^*$ by solving

$$\text{Maximize}_{c \in (\text{essinf}\,\rho, \text{esssup}\,\rho), \tilde{x}_+ \geq x_0^+} v_+(c, \tilde{x}_+) - v_-(c, \tilde{x}_+).$$

Recovery of optimal contingent claim

- If
  - $c^*, \tilde{x}_+^*$ is an optimal splitting
  - $\tilde{X}_+^*, \tilde{X}_-^*$ are optimal for the two subproblems respectively with parameters $c^*, \tilde{x}_+^*$,

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- If any of them fails to exist, then there is no optimal contingent claim

Positive part problem solution

The positive part problem is the same as in Jin and Zhou (2008)

Positive part problem solution

- Denote $F_\rho(\cdot)$ as the CDF of ρ. Suppose it is continuous.
- Suppose (1) $\frac{F_\rho^{-1}(\cdot)}{T_+'(\cdot)}$ is $\uparrow$ on $[0,1]$; (2) $\liminf_{x\to+\infty} \frac{-xu_+''(x)}{u_+'(x)} > 0$; (3) $E[u_+((u_+')^{-1}(\frac{\rho}{T_+'(F_\rho(\rho))}))T_+'(F_\rho(\rho))] < +\infty$.

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Theorem 1 For any $c \in (\operatorname{essinf}\rho, \operatorname{esssup}\rho]$ and $\tilde{x}_+ \geq \tilde{x}_0^+$, the optimal solution for the positive part problem is

$$\tilde{X}_+^* = (u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F(\rho))})\mathbf{1}_{\rho\leq c}.$$

The optimal value is

$$v_+(c, \tilde{x}_+) = E[u_+((u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F(\rho))}))T'_+(F(\rho))\mathbf{1}_{\rho\leq c}],$$

where λ is the unique one making $\tilde{X}_+^*$ feasible.

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- Denote $Z = F_\rho(\rho)$, $\Gamma = \{F^{-1}(\cdot) : F$ is a CDF$\}$ be the set of quantile functions. Then the problem is equivalent to

$$\min \quad \bar{v}_2(g(\cdot)) := E[u_-(g(Z))T'_-(1-Z)]$$

$$s.t. \quad \begin{cases} g(\cdot) \in \Gamma, g(\cdot) \in [0, L] \text{ on } [0,1) \\ E[g(Z)F_\rho^{-1}(Z)] = a. \end{cases}$$

Optimal quantile function

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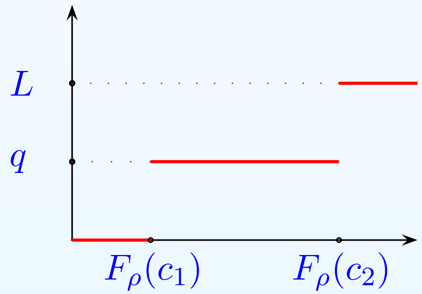
- We need to find out the boundary with the bound L

Optimal quantile

Theorem 2 If there are optimal $g(\cdot)$, then one of them is in the form $g(x; c_1, c_2) = q(c_1, c_2; a)\mathbf{1}_{x \in [F_\rho(c_1), F_\rho(c_2))} + L\mathbf{1}_{x \geq F_\rho(c_2)}$, where $q(c_1, c_2; a) = \frac{a - LE[\rho \mathbf{1}_{\rho \geq c_2}]}{E[\rho \mathbf{1}_{\rho \in [c_1, c_2)}]}$.



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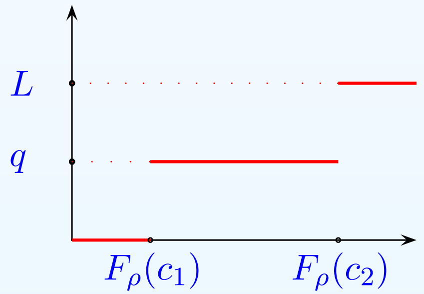

- Only need to solve the problem

$$\min \quad \bar{v}_2(g(\cdot; c_1, c_2))$$

$$s.t. \quad \operatorname{essinf}\rho \leq c_1 < c_2 \leq \operatorname{esssup}\rho$$

Optimal negative part

Theorem 3 For any $c \in [\text{essinf}\,\rho, \text{esssup}\rho)$, $\tilde{x}_+ > \tilde{x}_0^+$, the optimal value of the negative part problem is

$$v_-(c, \tilde{x}_+) = \inf_{c \le c_1 < c_2 \le \text{esssup}\rho)} v_3(c_1, c_2; c, \tilde{x}_+),$$

where

$$\begin{aligned} v_3(\cdots) &= u_-(q(c_1, c_2, \tilde{x}_+ - \tilde{x}_0))(T_-(P(\rho \ge c_2)) - T_-(P(\rho \ge c_1))) \\ &\quad + u_-(L)T_-(P(\rho \ge c_2)). \end{aligned}$$

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Furthermore, if $v_-(c, x_+)$ is obtained at (c_1^*, c_2^*), then

$$\tilde{X}_-^* = q(c_1^*, c_2^*; \tilde{x}_+^* - \tilde{x}_0)\mathbf{1}_{\rho \in [c_1^*, c_2^*)} + L\mathbf{1}_{\rho \ge c_2^*}$$

is an optimal solution for the negative part problem .

Optimal terminal wealth

The optimal splitting $c^*, \tilde{x}_+^*$ can be determined by

$$\begin{aligned} \max \quad & v_+(c, \tilde{x}_+) - v_3(c, c_2; c, \tilde{x}_+) \\ s.t. \quad & \tilde{x}_+ \geq \tilde{x}_0, \mathrm{essinf}\rho \leq c < c_2 \leq \mathrm{esssup}\rho \end{aligned}$$

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Theorem 4 Under the assumption made for positive part problem,

(i) If $(c^*, c_2^*, \tilde{x}_+^*)$ is an optimal splitting, then

$$X^* = (u_+')^{-1}\left(\lambda \frac{\rho}{T_+'(F(\rho))}\right)\mathbf{1}_{\rho \leq c^*} - q(c^*, c_2^*; \tilde{x}_+^* - \tilde{x}_0)\mathbf{1}_{\rho \in [c^*, c_2^*)} - L\mathbf{1}_{\rho \geq c_2^*} + B$$

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(ii) If there is no optimal $(c, c_2, \tilde{x}_+)$, then there is no optimal terminal wealth.

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Theorem 5 If $h(x) = T_-(f^{-1}(x)))$ is a convex function, then the optimal splitting (c^*, c_2^*, x_+^*) satisfies $c^* = c_2^*$. Hence the optimal contingent claim is

$$X^* = (u'_+)^{-1}(\lambda \frac{\rho}{T'_+(F(\rho))}) \mathbf{1}_{\rho \leq c_2^*} - L \mathbf{1}_{\rho \geq c_2^*} + B.$$

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Theorem 6 Given $h(x) = x^{\beta}$ for some $\beta > 0$. Then

- If $\beta \geq \alpha$, then $c_2^* = c^*$, and

  $X^* = (u_+')^{-1}(\lambda \frac{\rho}{T_+'(F(\rho))}) \mathbf{1}_{\rho \leq c_2^*} - L \mathbf{1}_{\rho \geq c_2^*} + B.$

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- If $\beta < \alpha$, then $c_2^* = +\infty$, and

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In any case, X^* is a two-piece function of ρ.

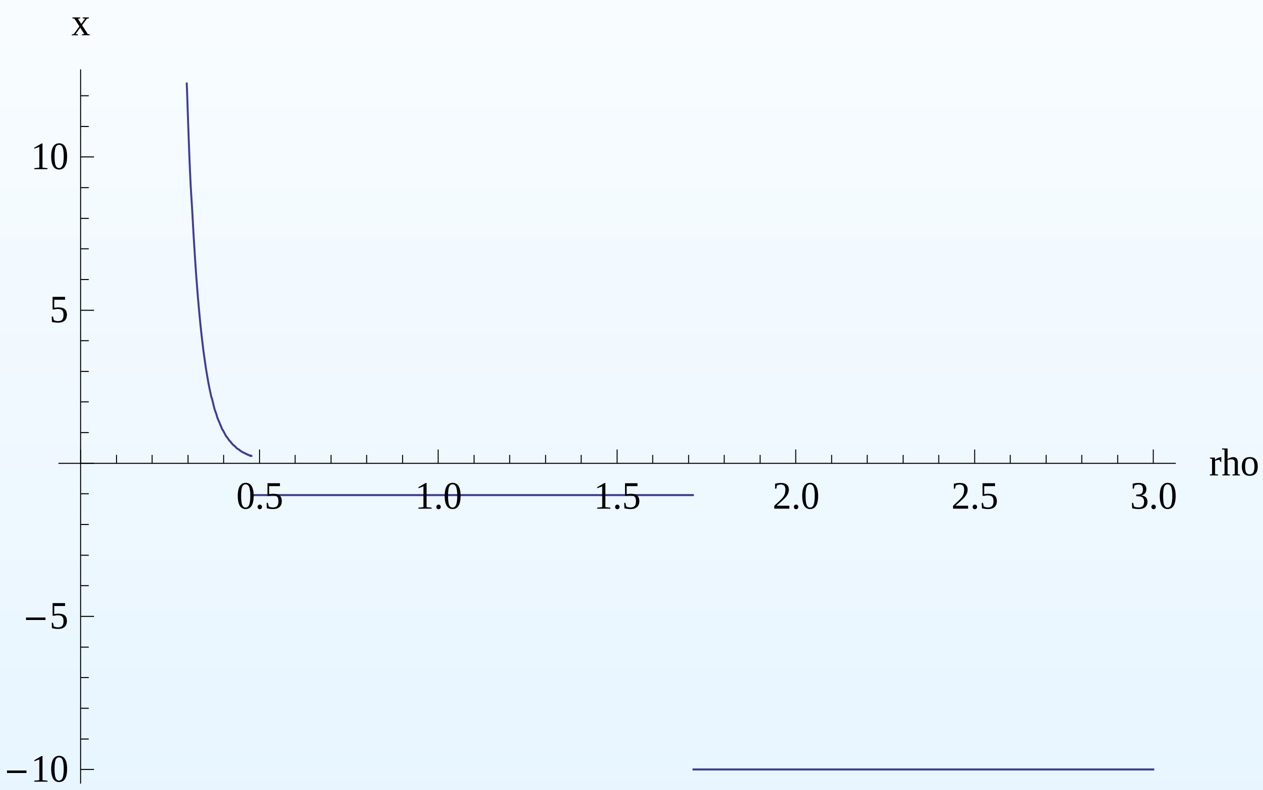
Example: power value function

- Is the optimal solution always two-piece for power value function?

Example: power value function

- Is the optimal solution always two-piece for power value function?
- A three-piece example:
  - $L = 10, \tilde{x}_0 = -1, \beta = 0.85, \alpha = 0.88, k = 2.25,$ $\rho \sim \text{Lognormal}(-0.045, 0.09)$
  - $h(x) =$
$$\begin{cases} 0.5x & x \in [0, 0.05] \\ 20 * 0.1^{\beta}(x - 0.05) + 0.025(0.1 - x) & x \in [0.05, 0.1] \\ x^{\beta} & x \in [0.1, 1] \end{cases}$$
  - The optimal solution $\tilde{X}^* = X^* - B$ is as in the next figure

Example: power value function



Thank you very much!