Convergence results for the indifference value based on the stability of BSDEs

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Overview

Indifference valuation

Brownian setting
with variable correlation

Convergence problem?

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Indifference valuation → BSDE

Characterization

Brownian setting
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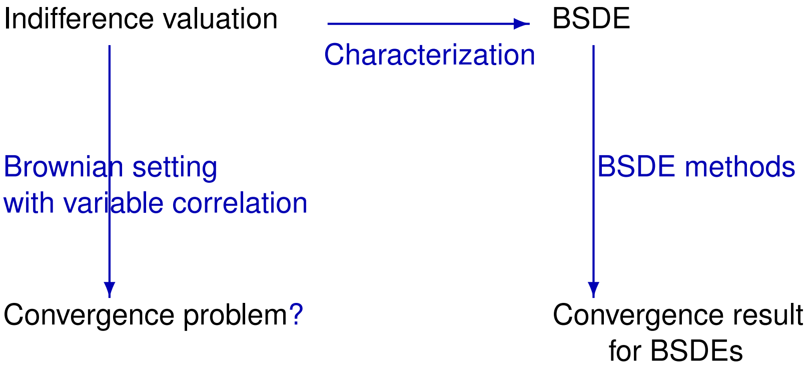
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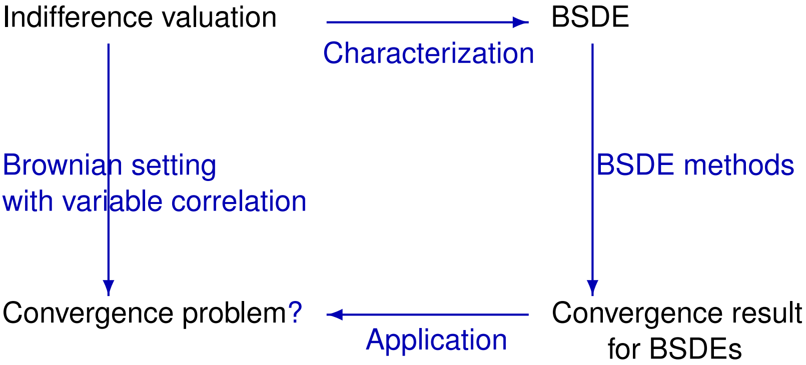
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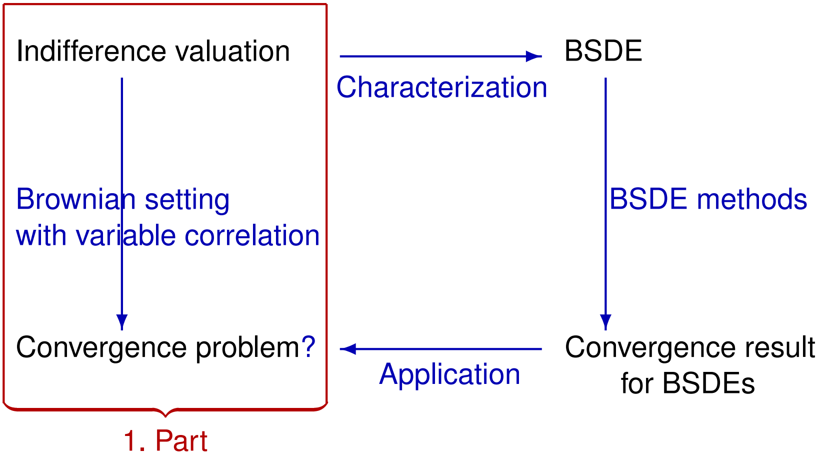
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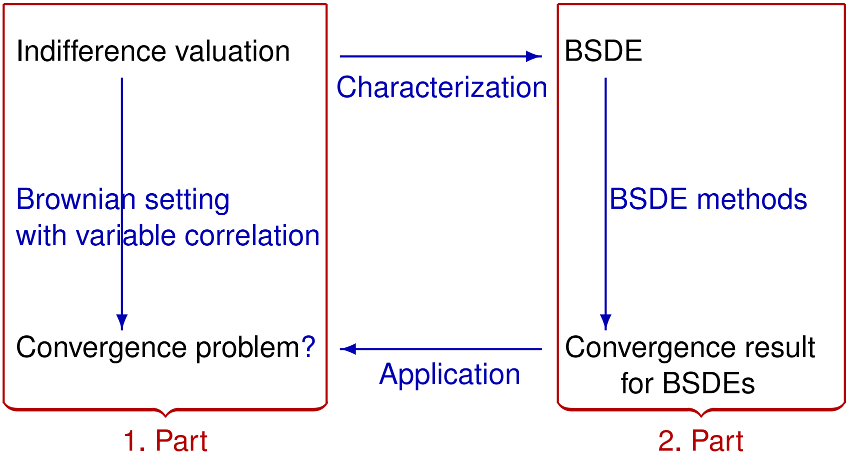
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BSDE
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BSDE methods
Convergence problem?
Application
Convergence result
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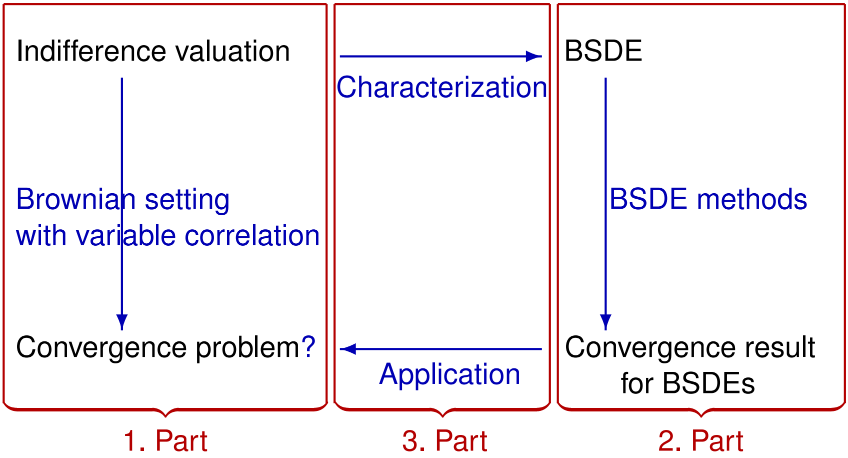
Indifference valuation
Brownian setting
with variable correlation
Convergence problem?
Characterization
Application
BSDE
BSDE methods
Convergence result
for BSDEs
1. Part
3. Part
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1. Indifference valuation

Definition of the indifference value

Financial market:

- Risk-free bank account yielding zero interest
- Risky asset with price process $S = (S_t)_{0 \le t \le T}$
- Financial product with payoff H at time T
- In mathematical terms, S is a semimartingale and H a random variable on some filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}, P\big)$.

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Problem formulation:

- Valuation of H based on the risk preferences of an investor
- Assumption: The investor has an exponential utility function $U(x) = -\exp(-\gamma x)$, $x \in \mathbb{R}$, for a fixed $\gamma > 0$
- $U(x) \mathrel{\hat{=}}$ Investor's utility if (s)he has capital $x \in \mathbb{R}$.

Definition

The **indifference value** h of H is implicitly defined by

$$\sup_{\vartheta \in \mathcal{A}} E\left[U\left(\int_0^T \vartheta_t \, \mathrm{d}S_t\right)\right] = \sup_{\vartheta \in \mathcal{A}} E\left[U\left(\int_0^T \vartheta_t \, \mathrm{d}S_t + H - h\right)\right],$$

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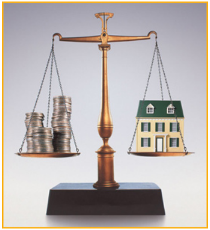

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$U(x) = -\exp(-\gamma x)$ for a fixed $\gamma > 0$

$\Downarrow$ direct calculation

The indifference value h is given by

$$h = \frac{1}{\gamma} \log \frac{V^0}{V^H},$$
$$V^H := \inf_{\vartheta \in \mathcal{A}} E\left[\exp\left(-\int_0^T \gamma \vartheta_t \, dS_t - \gamma H\right)\right].$$

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$\Downarrow$

The focus lies on V^H.

The indifference value of a nontradable asset

The underlying model:

- Two Brownian motions W and Y have constant instantaneous correlation ρ; i.e., $W = \rho Y + \sqrt{1-\rho^2}\, Y^{\perp}$ for a Brownian motion $Y^{\perp}$ independent from Y.

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- The traded stock S is given by

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t, \quad 0 \leqslant t \leqslant T, \quad S_0 > 0.$$

- Assumption: μ and σ are predictable with respect to $(\mathcal{Y}_t)_{0 \le t \le T}$, the filtration generated by Y.
- The nontradable claim H is $\mathcal{Y}_T$-measurable.

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Example: Executive stock options

- Manager receives options H.
- Because of legal restrictions, (s)he can hedge H only partially by trading in a correlated stock or an index.

Proposition (An explicit formula; Tehranchi 2004)

Under boundedness assumptions, one has

$$V^H = \left(E_{\hat{P}} \left[\exp\left(-\gamma H - \frac{1}{2} \int_0^T \frac{\mu_t^2}{\sigma_t^2} \,\mathrm{d}t \right)^{1-\rho^2} \right] \right)^{\frac{1}{1-\rho^2}},$$

where the probability measure $\hat{P}$ is given by

$$\frac{\mathrm{d}\hat{P}}{\mathrm{d}P} := \exp\left(- \int_0^T \frac{\mu_t}{\sigma_t} \,\mathrm{d}W_t - \frac{1}{2} \int_0^T \frac{\mu_t^2}{\sigma_t^2} \,\mathrm{d}t \right).$$

Variable correlation:

- So far: $W_t = \rho Y_t + \sqrt{1-\rho^2}\, Y_t^{\perp}$
  $= \int_0^t \rho \, \mathrm{d}Y_s + \int_0^t \sqrt{1-\rho^2}\, \mathrm{d}Y_s^{\perp}$ with constant ρ

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Proposition (Bounds; Frei and Schweizer 2008)

For $(\mathcal{Y}_t)_{0\le t\le T}$-predictable ρ with boundedness assumptions,

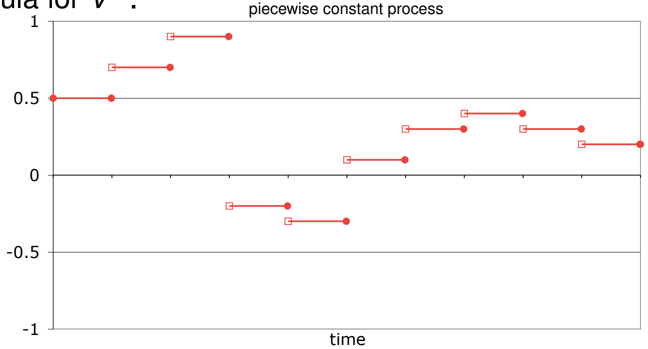
$$\left(E_{\hat{P}}\left[\exp(\hat{H})^{1/\overline{\delta}}\right]\right)^{\overline{\delta}} \le V^H \le \left(E_{\hat{P}}\left[\exp(\hat{H})^{1/\underline{\delta}}\right]\right)^{\underline{\delta}},$$

where $\hat{H} := -\gamma H - \frac{1}{2}\int_0^T \frac{\mu_t^2}{\sigma_t^2}\,\mathrm{d}t$ and

$$\overline{\delta} := \sup_{t\in[0,T]} \left\| \frac{1}{1-\rho_t^2} \right\|_{L^\infty}, \quad \underline{\delta} := \inf_{t\in[0,T]} \frac{1}{\|1-\rho_t^2\|_{L^\infty}}.$$

Ideas for an approximation of V^H:

1. If ρ is piecewise constant in time, there is an explicit formula for V^H.



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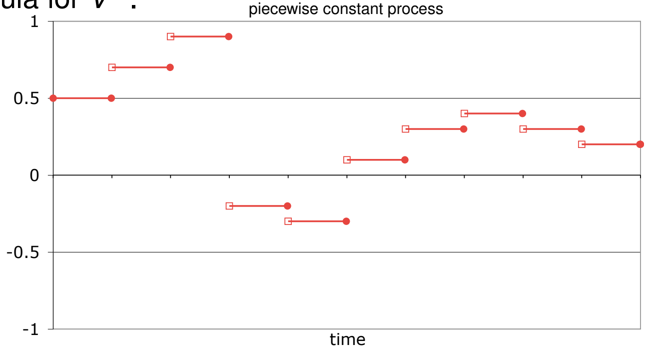

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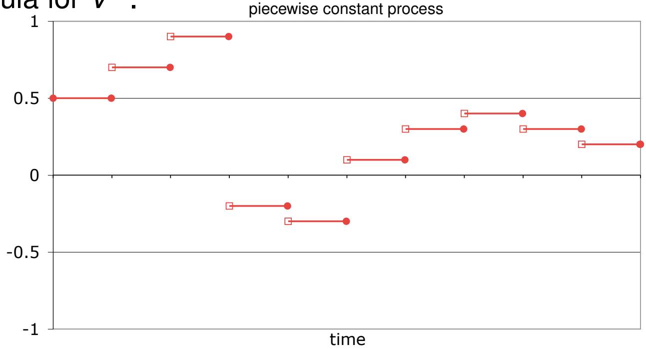

2. Approximate a general ρ by a sequence $(q_n)_{n\in\mathbb{N}}$ of piecewise constant processes.
3. Show that values corresponding to q_n converge to V^H.
Problem: It is difficult to show this directly. $\longrightarrow$ study BSDE

2. A convergence result for BSDEs

Let B be a d-dimensional Brownian motion and consider

$$\mathrm{d}\Gamma_t = f(t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}B_t, \quad 0 \le t \le T, \quad \Gamma_T = H,$$

where

- $f : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$
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The results hold not only in a Brownian setting, but more generally in a continuous filtration (i.e., a filtration where any local martingale has a continuous version).

Theorem (Convergence of BSDEs)

Fix $t \in [0, T]$ and let $(f^n, H^n)_{n=1,2,\ldots,\infty}$ be a sequence of parameters such that

- f^n satisfy some quadratic-growth and local-Lipschitz conditions in z (uniformly in $n = 1, \ldots, \infty$);
- $\lim_{n\to\infty} H^n = H^\infty$ a.s. and for almost all $(s, \omega) \in [t, T] \times \Omega$, $\lim_{n\to\infty} f^n(s, z)(\omega) = f^\infty(s, z)(\omega)$ for all $z \in \mathbb{R}^d$.

Then there exist unique solutions (Γ^n, Z^n) with parameters (f^n, H^n) for $n = 1, \ldots, \infty$, and

$$\lim_{n\to\infty} \Gamma_t^n = \Gamma_t^\infty \quad \text{a.s.}, \qquad \lim_{n\to\infty} E\left[\int_t^T |Z_s^n - Z_s^\infty|^2 \, ds\right] = 0$$

Corollary (Special form of f^n)

Suppose additionally that

- H^n converges to H^∞ in L^∞ as $n \to \infty$;
- there exist sequences $(\underline{d}^n)_{n\in\mathbb{N}}$ and $\left(\overline{d}^n\right)_{n\in\mathbb{N}}$ of deterministic functions which converge to 1 uniformly on $[t, T]$ such that $f^n = \underline{d}^n \underline{f} + \overline{d}^n \overline{f}$ for every $n = 1, \ldots, \infty$.

Then we have

$$\sup_{s\in[t,T]} |\Gamma_s^n - \Gamma_s^\infty| \to 0 \quad \text{in } L^\infty \text{ as } n \to \infty.$$

3. Applying the convergence result

A BSDE characterization of V^H

Revisiting the nontradable asset model:

- Two Brownian motions W and Y have time-dependent instantaneous correlation ρ; $\mathrm{d}W_t = \rho_t \,\mathrm{d}Y_t + \sqrt{1-\rho_t^2}\,\mathrm{d}Y_t^\perp$ for a Brownian motion $Y^\perp$ independent from Y.
- The traded stock S is given by

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$$V^H := \inf_{\vartheta\in\mathcal{A}} E\left[\exp\left(-\int_0^T \gamma\vartheta_t \,\mathrm{d}S_t - \gamma H\right)\right].$$

A BSDE characterization of V^H:

We have $V^H = \exp(-\gamma\Gamma_0)$, where Γ solves the BSDE

$$d\Gamma_t = \left(\frac{\gamma}{2}(1-\rho_t^2)Z_t^2 + \rho_t\lambda_t Z_t - \frac{\lambda_t^2}{2\gamma}\right)dt + Z_t\, dY_t, \quad \Gamma_T = H$$

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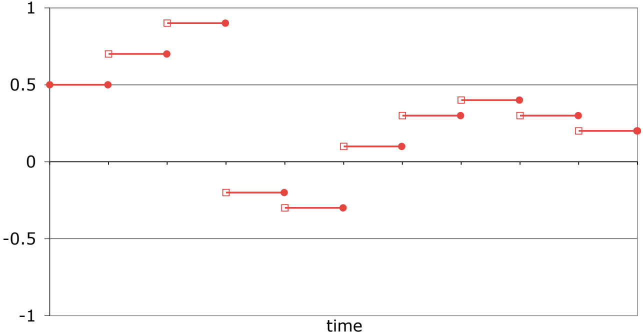
Remark:

The application can be done for $(\mathcal{Y}_t)_{0\le t\le T}$-predictable ρ, but we consider here only a deterministic, time-dependent ρ.

An approximation of V^H

1. If ρ is piecewise constant in time, there is an explicit formula for the solution of the BSDE.



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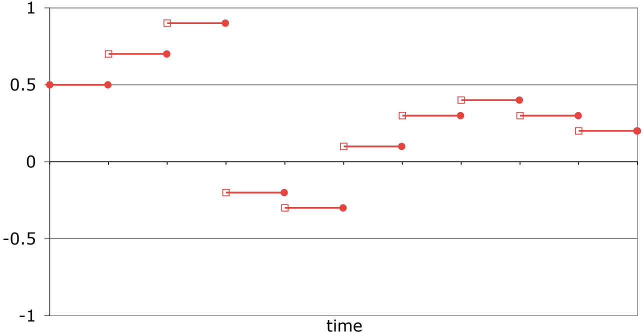

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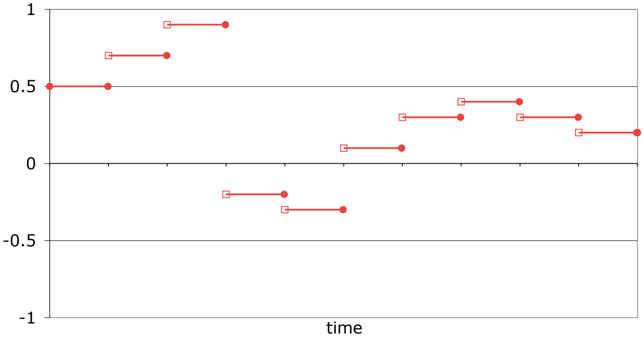

2. Approximate a general ρ by a sequence $(q_n)_{n\in\mathbb{N}}$ of piecewise constant processes.
3. Apply the convergence result to show the convergence of the solutions of the corresponding BSDEs.

1. Step: Piecewise constant processes

Let $q : [0, T] \to \,]-1, 1[$ be of the form

$$q = q^1 \mathbb{1}_{\{t_0\}} + \sum_{j=1}^{n} q^j \mathbb{1}_{]t_{j-1}, t_j]} \quad \text{for } t = t_0 \leq t_1 \leq \cdots \leq t_n = T.$$

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Then the BSDE

$$d\Gamma_t^q = \left(\frac{\gamma}{2}(1 - q_t^2)|Z_t^q|^2 + \rho_t \lambda_t Z_t^q - \frac{\lambda_t^2}{2\gamma} \right) dt + Z_t^q \, dY_t, \quad \Gamma_T = H$$

has the explicit solution Γ_0^q with $\exp\left(-\gamma \Gamma_0^q\right)$ equal to

$$E_{\hat{P}}\left[\cdots E_{\hat{P}}\left[E_{\hat{P}}\left[e^{\hat{H}(1-|q^n|^2)} \middle| \mathcal{Y}_{t_{n-1}} \right]^{\frac{1-|q^{n-1}|^2}{1-|q^n|^2}} \middle| \mathcal{Y}_{t_{n-2}} \right]^{\frac{1-|q^{n-2}|^2}{1-|q^{n-1}|^2}} \cdots \right]^{\frac{1}{1-|q^1|^2}}$$

where

$$\hat{H} := -\gamma H - \frac{1}{2} \int_0^T \lambda_t^2 \, dt, \quad \frac{d\hat{P}}{dP} := \exp\left(- \int_0^T \lambda_t \, dW_t - \frac{1}{2} \int_0^T \lambda_t^2 \, dt \right).$$

2. Step: The approximation of ρ

- Question: Which functions $\rho : [0, T] \to [-1, 1]$ can be approximated pointwise by piecewise constant functions?

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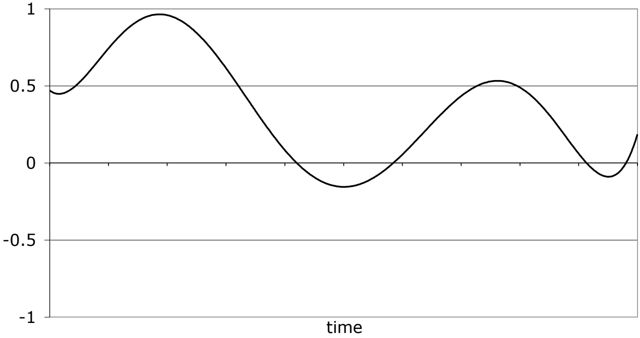

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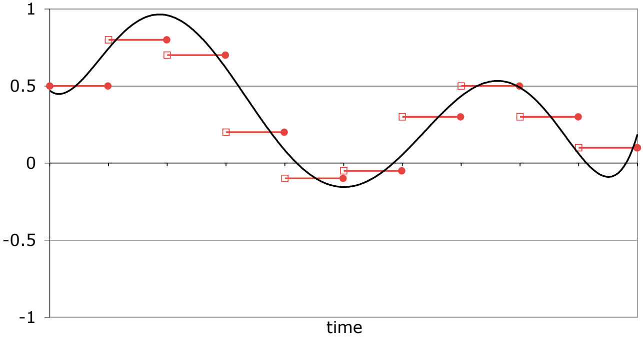

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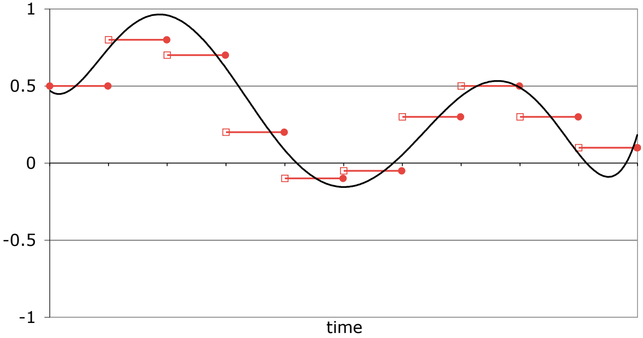


- Idea: This approximation is reminiscent of the construction of the Riemann integral.

- Recall that a bounded function $g : [0, T] \to \mathbb{R}$ is Riemann integrable if and only if it is Lebesgue-almost everywhere continuous on $[0, T]$.

- Recall that a bounded function $g : [0, T] \to \mathbb{R}$ is Riemann integrable if and only if it is Lebesgue-almost everywhere continuous on $[0, T]$.
- Assume that $\rho : [0, T] \to [-1, 1]$ is Riemann integrable. Let

$$0 = t_0^n \le t_1^n \le \cdots \le t_{\ell_n}^n = T, \quad s_j^n \in [t_{j-1}^n, t_j^n]$$

be partitions with $\lim_{n\to\infty}\left(\max_{1\le j\le \ell_n}(t_j^n - t_{j-1}^n)\right) = 0$ and set $q^n := \sum_{j=1}^{\ell_n} \rho(s_j^n)\mathbb{1}_{]t_{j-1}^n, t_j^n]}$. Then

$$\lim_{n\to\infty} q^n(x) = \rho(x) \quad \text{for almost all } x \in [0, T].$$

3. Step: The application of the convergence result

Theorem (Approximating V^H)

Assume that ρ is Riemann integrable and $]-1,1[$-valued. Let

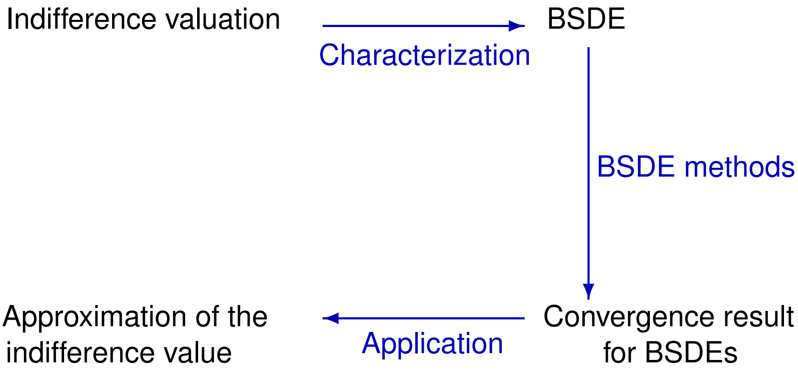
$$0 = t_0^n \leq t_1^n \leq \cdots \leq t_{\ell_n}^n = T, \quad s_j^n \in [t_{j-1}^n, t_j^n]$$

be partitions with $\lim_{n\to\infty}(\max_{1\leq j\leq \ell_n}(t_j^n - t_{j-1}^n)) = 0$. Then

$$V^H = \lim_{n\to\infty} E_{\hat{P}}\left[\cdots E_{\hat{P}}\left[e^{\hat{H}(1-|\rho(s_{\ell_n}^n)|^2)}\middle|\mathcal{Y}_{t_{\ell_n-1}^n}\right]^{\frac{1-|\rho(s_{\ell_n-1}^n)|^2}{1-|\rho(s_{\ell_n}^n)|^2}}\cdots\right]^{\frac{1}{1-|\rho(s_1^n)|^2}}$$

with $\hat{H} := -\gamma H - \frac{1}{2}\int_0^T \lambda_t^2\,dt$.

Overview


Indifference valuation
BSDE
Characterization
BSDE methods
Approximation of the indifference value
Convergence result for BSDEs
Application

Thank you very much for your attention!

Admissible strategies

$\mathcal{A}$ consists of all predictable $\vartheta = (\vartheta_t)_{0 \le t \le T}$ such that $\int_0^T \vartheta_t^2 \, \mathrm{d}t < \infty$ a.s. and

$$\left(\exp\left(-\gamma \int_0^t \vartheta_s \, \mathrm{d}S_s \right) \right)_{0 \le t \le T} \text{ is of class } (D).$$

The latter means that the set

$$\left\{ \exp\left(-\gamma \int_0^\tau \vartheta_s \, \mathrm{d}S_s \right) \,\middle|\, \tau \text{ is a stopping time} \right\}$$

is uniformly integrable.

Alternative measurability conditions

Assumptions:

- μ, σ are predictable w.r.t. the filtration generated by W.
- H is $\hat{\mathcal{Y}}_T$-measurable, where $\hat{\mathcal{Y}}_T$ is the sigma-field generated by $\hat{Y}_t := Y_t + \int_0^t \rho_s \frac{\mu_s}{\sigma_s}\,\mathrm{d}s$, $0 \le t \le T$.

Proposition (Bounds; Frei and Schweizer 2008)

For general ρ with boundedness assumptions, one has

$$\left(E_{\hat{P}}\left[\exp(\hat{H})^{1/\overline{\delta}}\right]\right)^{\overline{\delta}} \le V^H \le \left(E_{\hat{P}}\left[\exp(\hat{H})^{1/\underline{\delta}}\right]\right)^{\underline{\delta}},$$

where $\hat{H} := -\gamma H - \frac{1}{2} E_{\hat{P}}\left[\int_0^T \frac{\mu_t^2}{\sigma_t^2}\,\mathrm{d}t\right]$ and

$$\overline{\delta} := \sup_{t\in[0,T]} \left\| \frac{1}{1-\rho_t^2} \right\|_{L^\infty}, \quad \underline{\delta} := \inf_{t\in[0,T]} \frac{1}{\|1-\rho_t^2\|_{L^\infty}}.$$

A general BSDE characterization of V^H

Without measurability assumptions on ρ, μ, σ and H:

From Hu, Imkeller and Müller (2005), we have $V^H = \exp(-\gamma\Gamma_0)$, where Γ solves the $(\mathcal{F}_t)_{0\le t\le T}$-BSDE

$$d\Gamma_t = \left(\frac{\gamma}{2}\check{Z}_t^2 - \lambda_t\hat{Z}_t - \frac{\lambda_t^2}{2\gamma}\right)dt + \hat{Z}_t\,dW_t + \check{Z}_t\,dW_t^\perp, \quad \Gamma_T = H$$

for a Brownian motion $W^\perp$ independent of W, and $\lambda := \mu/\sigma$.

Problem: This BSDE cannot be approximated by a BSDE with an explicit solution.

Approximation under stochastic correlation

Theorem (Approximating V^H)

Assume that ρ is $(\mathcal{Y}_t)_{0\leq t\leq T}$-predictable, left-continuous and $]-1,1[$-valued. Let $(0=\tau_0^n\leq\cdots\leq\tau_{\ell_n}^n=T)_{n\in\mathbb{N}}$ be $(\mathcal{Y}_t)_{0\leq t\leq T}$-stopping times with $\lim_{n\to\infty}\left(\max_{1\leq j\leq \ell_n}(\tau_j^n-\tau_{j-1}^n)\right)=0$ a.s. Then we have

$$V^H=\lim_{n\to\infty}E_{\hat{P}}\left[\cdots E_{\hat{P}}\left[e^{\hat{H}(1-|\rho_{\tau_{\ell_n-1}^n}|^2)}\middle|\mathcal{Y}_{\tau_{\ell_n-1}^n}\right]^{\frac{1-|\rho_{\tau_{\ell_n-2}^n}|^2}{1-|\rho_{\tau_{\ell_n-1}^n}|^2}}\cdots\right]^{\frac{1}{1-|\rho_{\tau_0^n}|^2}}$$

with $\hat{H}:=-\gamma H-\frac{1}{2}\int_0^T\lambda_t^2\,\mathrm{d}t$.

Convergence of BSDEs in a continuous filtration

Setting:

- Assume that $\mathbb{F}$ is a general continuous filtration, i.e., all local martingales are continuous.
- Fix an $\mathbb{R}^d$-valued local martingale $M = (M_t)_{0 \leq t \leq T}$.
- Take a nondecreasing and bounded process D such that $\langle M^j \rangle \ll D$ for all $j = 1, \ldots, n$, e.g., $D = \arctan\left(\sum_{j=1}^{n} \langle M^j \rangle\right)$.

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We consider the BSDE

$$d\Gamma_t = f(t, Z_t)\,dD_t + \frac{\beta}{2} d\langle N \rangle_t + Z_t\,dM_t + dN_t, \quad 0 \le t \le T, \quad \Gamma_T = H,$$

where

- $f : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$;
- $\beta \in \mathbb{R}$;
- H is a bounded random variable.

A solution is a triple (Γ, Z, N), where

- Γ is a bounded continuous semimartingale;
- Z is a predictable process with $E\left[\int_0^T Z_t' \,\mathrm{d}\langle M\rangle_t Z_t\right] < \infty$;
- N is a square-integrable martingale null at 0 and strongly orthogonal to M.

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Theorem (Convergence of BSDEs)

Fix $t \in [0, T]$ and let $\big(f^n, \beta^n, H^n\big)_{n=1,2,\dots,\infty}$ be a sequence of parameters such that

- f^n satisfy some quadratic-growth and local-Lipschitz conditions in z (uniformly in $n = 1, \dots, \infty$);
- $\lim_{n\to\infty} \beta^n = \beta^\infty$, $\lim_{n\to\infty} H^n = H^\infty$ a.s. and for $(D \otimes P)$-almost all $(s, \omega) \in [t, T] \times \Omega$, $\lim_{n\to\infty} f^n(s, z)(\omega) = f^\infty(s, z)(\omega)$ for all $z \in \mathbb{R}^d$.

Then there exist unique solutions (Γ^n, Z^n, N^n) with parameters (f^n, β^n, H^n) for $n = 1, \dots, \infty$, and

$$\lim_{n\to\infty} \Gamma^n_t = \Gamma^\infty_t \text{ a.s.}, \quad \lim_{n\to\infty} E\big[\langle N^n - N^\infty\rangle_T - \langle N^n - N^\infty\rangle_t\big] = 0,$$

$$\lim_{n\to\infty} E\bigg[\int_t^T (Z^n_s - Z^\infty_s)' \, \mathrm{d}\langle M\rangle_s (Z^n_s - Z^\infty_s)\bigg] = 0.$$

Precise assumptions of the convergence result

- There exist a nonnegative predictable κ^1 with $\left\| \int_0^T \kappa_s^1 \, \mathrm{d}s \right\|_{L^\infty} < \infty$ and a constant c^1 such that
$$\left| f^n(s,z) \right| \leq \kappa_s^1 + c^1 |z|^2$$
for all $s \in [0,T]$, $z \in \mathbb{R}^d$ and $n = 1, \ldots, \infty$.

- There exist a nonnegative predictable κ^2 with $\left\| \int_0^T |\kappa_s^2|^2 \, \mathrm{d}s \right\|_{L^\infty} < \infty$ and a constant c^2 such that
$$\left| f^n(s,z^1) - f^n(s,z^2) \right| \leq c^2 \left(\kappa_s^2 + |z^1| + |z^2| \right) |z^1 - z^2|$$
for all $s \in [0,T]$, $z^1, z^2 \in \mathbb{R}^d$ and $n = 1, \ldots, \infty$.

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