Shadow DP and Equilibrium Asset Pricing in Incomplete Financial Markets

BERNARD DUMAS

INSEAD, NBER, CEPR

ANDREW LYASOFF

Boston University, School of Management

** ** **

BFS 6th World Congress

Toronto, June 22-26

http://andrew.lyasoff.com/bfs329.pdf

(1) Main motivation: *price formation*? *distribution of wealth*? *equilibrium*?

- From individual agent's point of view, *the asset prices today* depend on the pricing kernel, which depend on future consumption, which depends on today's investment, which depends on *today's asset prices*
- Therefore, each market agent must "calculate" her portfolio choice and consumption plan *simultaneously backward and forward*.
- The (heterogeneous) agents differ in their initial wealth and consumption preferences throughout time, and, consequently, use different pricing kernels. Nevertheless, in equilibrium they must agree on how the assets are priced, which ultimately determines how they trade with one another.

(2) Main motivation: *price formation*? *distribution of wealth*? *equilibrium*?

Three main contributions:

- Krusell and Smith (1998): in a vast population of individuals with independent idiosyncratic risks, incomplete-market equilibrium is close to a complete-market equilibrium.
- Heaton and Lucas (1996): equilibrium with two classes of agents, incomplete market, trading costs and borrowing constraint. They conclude that the borrowing constraint is what makes a difference.
    - Note: tâtonnement only shows how the portfolios would behave as a MC in the long run, provided that the agents somehow know how to choose optimally.

(3) Main motivation: *price formation*? *distribution of wealth*? *equilibrium*?

- However
  - Basak and Cuoco (1998) have a model with limited participation (their case of limited participation is very close to an incomplete-market case) show that, when some people are prevented from accessing the market, the market Sharpe ratio is vastly increased.
  - And Gomes and Michaelides (2006) attribute the large risk premia in their model mainly to imperfect risk sharing among stock holders rather than the limited participation.
  - Constantinides and Duffie (1996) attribute them to countercyclical variation in the cross-section of household consumption.
  - Other general discussions of this issue include Guvenen (2004, 2006) and Krueger and Lustig (2007).

(4) Our objective

- Develop a notion of equilibrium which does not rely on "stationarity," or a "fixed point argument" of any kind, i.e., applies to economies with a *finite time-horizon*.
- Nevertheless, make it possible to work with a "large" ($\geqslant 3$) number of periods and trees with "many" ($\geqslant 3$) spikes.
- Develop a method which allows one to compute incomplete-market equilibria "routinely" (when they exist) — and without the use of super-computers.
- Calculate the equilibrium as a function of the initial wealth.

(5) Our objective

- Develop a (recursion-based) “shadow DP” method which is analogous to the classical DP except that the value function is replaced by the dual variables (state prices).
- Develop a dynamic “only-backward” numerical algorithm based on the interpolation dynamic programming technique (AL, 2008).
- Explain how incompleteness *constraints the distribution of wealth* in a way that removes any degree of freedom that the incompleteness creates.

(6) The main difficulty to be overcome

- there are *exogenous state variables* driving the economy (say, initial wealth and output)
- but, in an *incomplete market*, there are also *endogenous state variables*: market prices for securities, individual endowments, individual state prices
- the system for computing these quantities is "forward-backward": to solve for tomorrow's individual state prices, one needs today's state prices, but tomorrow's wealths and security prices (as functions of tomorrow's state prices)

(7) The general approach

- We adopt the "dual approach" developed at the individual level, which is similar to He and Pearson (1991):
  - the unknowns are agent-specific state prices
- This has already been done in two ways:
  - Cuoco and He (1994, unpublished): recursive method in continuous time, but with exogenous volatility
  - Cuoco and He (2001, published): global (as opposed to recursive) method on a tree

(8) The general approach

- Will work with ***general tree structures*** (the time is discrete and all information sets are finite) and develop a recursive — as opposed to global — method.
    - however, the equilibrium that we obtain is a global equilibrium — not the recursive equilibrium discussed in Kubler and Schmedders (2002)
    - and allows the tree to be recombining — when the exogenous variables are Markovian, for example
- ***Will not be inventing the wheel***: our approach is essentially a variation of the stochastic principle of maximum with one (huge) difference: the Hamiltonian is endogenous

(9) Information structure

- $t = 0, 1, \ldots, T, \quad T < \infty$
- $\Sigma :=$ a finite set of uncertain states of the economy
  - The process of revealing the true state to the market observers is modeled by a tree-structure, defined as a finite chain of *successive partitions of the set* Σ:

$$\mathbb{F} = \{\mathbb{F}_t;\, t = 0, 1, \ldots, T\}\,,$$

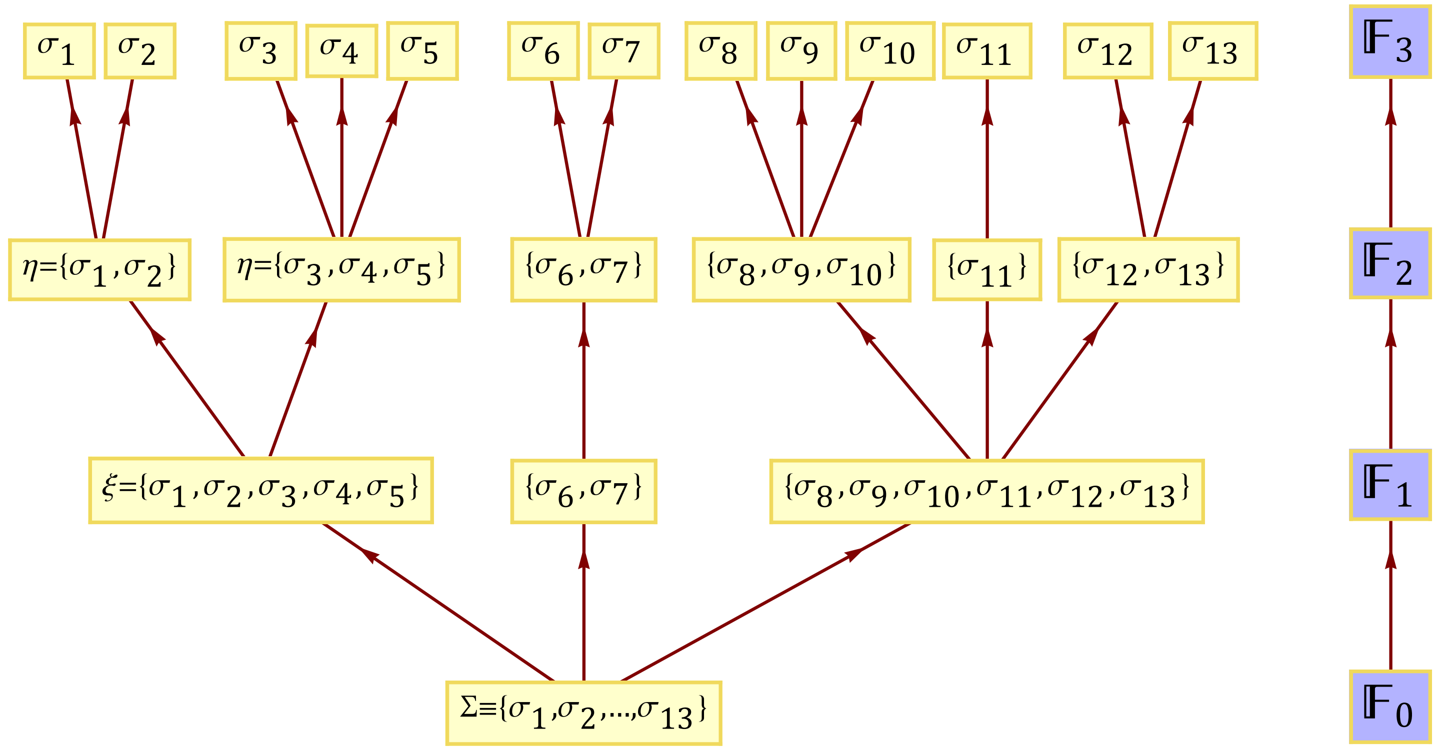
$$\mathbb{F}_0 = \{\Sigma\},\ \ \mathbb{F}_T = \{\{\sigma\};\, \sigma \in \Sigma\},$$

$$\xi \in \mathbb{F}_t \implies \exists\,!\ \xi^- \in \mathbb{F}_{t-1},\ \ \xi \subseteq \xi^-$$

$$\xi \in \mathbb{F}_t,\quad \mathbb{F}^{\xi}_{\tau} := \{\eta \in \mathbb{F}_{t+\tau};\, \eta \subseteq \xi\},\ \ 0 \leqslant \tau \leqslant T - \tau$$

$$(\xi^+) := \mathbb{F}^{\xi}_1$$

(10) Information structure



(11) Information structure

$$\ell_t\left(\xi, \mathbb{F}^{\xi}; \mathbb{R}^n\right) := \left\{(f_{t+\tau} : \xi \mapsto \mathbb{R}^n)_{\tau=0}^{T-\tau};\ f_{t+\tau} \text{ is } \mathbb{F}^{\xi}_{\tau}\text{-measurable}, \right.$$
$$\left. 0 \leqslant \tau \leqslant T - \tau \right\}$$

when $f \in \ell_0(\Sigma, \mathbb{F}; \mathbb{R}^n)$, we write $f_{t,\xi} := f_t(\xi) = f_t(\sigma), \xi \in \mathbb{F}_t, \sigma \in \xi$

- The set Σ is endowed with an objective probability measure

$$\pi(\sigma) \in]\,0,\ 1],\ \ \sigma \in \Sigma,\ \sum_{\sigma \in \Sigma} \pi(\sigma) = 1$$

(12) The economy

consists of

- a single perishable good (numeraire)
- $L+1$ agents consume the perishable good
  - individual endowment streams $\varepsilon^i \in \ell(\Sigma, \mathbb{F}; \mathbb{R}_+)$, $0 \leqslant i \leqslant L$
  - individual consumption streams $c^i \in \ell(\Sigma, \mathbb{F}; \mathbb{R}_{++})$, $0 \leqslant i \leqslant L$ (NB: agents must consume in order to survive)
  - individual (strictly concave and differentiable) consumption preferences

$$U_t^i : \mathbb{R}_{++} \mapsto \mathbb{R},\ 0 \leqslant i \leqslant L,\ 0 \leqslant t \leqslant T\,,$$

- N traded securities with associated price vector $S \in \ell_0\left(\Sigma, \mathbb{F}; \mathbb{R}_+^N\right)$
  - dividend streams $\delta^j \in \ell_0(\Sigma, \mathbb{F}; \mathbb{R}_+)$, $1 \leqslant j \leqslant N$
  - the information structure is sufficiently rich: $N \leqslant \#(\xi^+)$, $\xi \in \mathbb{F}_t$, $0 \leqslant t < T$

(13) Agents' consumption goals and constraints

upon entering state $\xi \in \mathbb{F}_t$ with wealth $W_t^i \equiv W_{t,\xi}^i$ agent i is concerned with

$$J_t^i(c) = U_t^i(c_t) + \sum_{\tau=1}^{T-t} \mathbb{E}_t\left[U_{t+\tau}^i(c_{t+\tau})\right], \; 0 \leqslant i \leqslant L,$$

given his choice of a consumption plan $c^i \in \ell_0\left(\xi, \mathbb{F}^{\xi}; \mathbb{R}_{++}\right)$ and trading strategy $\theta^i \in \ell_0\left(\xi, \mathbb{F}^{\xi}; \mathbb{R}^N\right)$ that can finance c^i (together with W_t^i) in the sense that the following *flow budget constraint* ("marketability" condition) *holds*

$$c_{t+\tau} + \theta_{t+\tau} \cdot S_{t+\tau} = \varepsilon_{t+\tau}^i + W_{t+\tau}^i, \quad \tau = 0, \ldots, T-t,$$

$$\text{where} \quad W_{t+\tau}^i = \theta_{t+\tau-1}^i \cdot (S_{t+\tau} + \delta_{t+\tau})$$

investor i's value function entering period t is

$$V_t^i(W_t) := \sup\left\{J_t^i(c)\,; \; c \text{ is fesible for the entering wealth } W_t\right\}$$

$$V_t^i\left(W_t^i\right) \equiv \left\{V_{t,\xi}^i\left(W_{t,\xi}^i\right); \; \xi \in \mathbb{F}_t\right\}$$

(14) Agents' consumption goals and constraints

Theorem 0 (The PDP holds): If $V_0^i\left(W_0^i\right) = J_0^i\left(c^i\right)$ and if $\theta^i \in \ell_0\left(\Sigma, \mathbb{F}; \mathbb{R}^N\right)$ can finance c^i, *given the initial wealth* W_0^i, then, for any $0 < t \leqslant T$, the trading strategy $\left\{\theta_t^i, \theta_{t+1}^i, \ldots, \theta_T^i\right\}$ finances the consumption plan $\left\{c_t^i, c_{t+1}^i, \ldots, c_T^i\right\}$ with entering wealth (for period t)

$$W_t^i = \theta_{t-1}^i \cdot (S_t + \delta_t)$$

and one has

$$V_t^i\left(W_t^i\right) = U_t^i\left(c_t^i\right) + \mathbb{E}_t\left[V_{t+1}^i\left(\theta_t^i \cdot (S_{t+1} + \delta_{t+1})\right)\right].$$

(15) "Primal" formulation of the agents' problem

During period t agent i observes W_t^i and S_t and decides about his consumption $c_t \in \mathbb{R}_{++}$ and portfolio plan $\theta_t^i \in \mathbb{R}^N$, so that $x^* \equiv c_t$ and $y^* \equiv \theta_t^i$ solve the optimization problem

$$\begin{aligned}
\text{Maximize } & F_t^i(x, y) := U_t^i(x) + \mathbb{E}_t\left[V_{t+1}^i(y \cdot (S_{t+1} + \delta_{t+1}))\right] \\
\text{subject to: } & x + y \cdot S_t = \varepsilon_t^i + W_t^i, \\
& x \in \mathbb{R}_{++},\ y \equiv \{y_1, \ldots, y_N\} \in \mathbb{R}^N.
\end{aligned}$$

(16) Dual formulation for the agents' problem

During period t agent i is faced with the Lagrangian

$$\mathcal{L}_t^i(x,\, y,\, \lambda) = F_t^i(x,\, y) + \lambda\left(\varepsilon_t^i + W_t - x - y \cdot S_t\right),$$
$$x \in \mathbb{R}_{++},\, y \in \mathbb{R}^N,\; \lambda \in \mathbb{R}\,.$$

and chooses his immediate consumption $c_t \in \mathbb{R}_{++}$, immediate trading strategy $\theta_t \in \mathbb{R}^N$ and local (in time and state of the economy) Arrow-Debreu shadow $\phi_t \in \mathbb{R}$ in such a way that

$$\mathcal{L}_t^i(c_t,\, \theta_t,\, \phi_t) = \inf_{\lambda \in \mathbb{R}} \sup_{x \in \mathbb{R}_{++},\, y \in \mathbb{R}^M} \mathcal{L}_t^i(x,\, y,\, \lambda)\,.$$

(17) Agents' first order conditions

$$\mathbb{E}_{t,\xi}\left[\left(\partial V^i_{t+1}\right)\left(\theta^i_t\cdot(S_{t+1}+\delta_{t+1})\right)\times\left(S^j_{t+1}+\delta^j_{t+1}\right)\right]=\phi_t\,S^j_t,$$

$$1\leqslant j\leqslant N\,,$$

$$\left(\partial U^i_t\right)(c_t)=\phi_t\,,$$

$$c^i_t+\theta^i_t\cdot S_t=\varepsilon^i_t+W_t.$$

these two properties are now crucial:

$$\left(\partial V^i_t\right)(W_t)=\phi_t(W_t)$$

$$\phi'_t(W_t)=\{c'_t(W_t),\,\theta'_t(W_t)\}^\dagger\left(\nabla^2 F^i_t\right)(c_t(W_t),\,\theta_t(W_t))\,\{c'_t(W_t),\,\theta'_t(W_t)\}$$

(18) The first order conditions suffice

Theorem 1: Given a price system $S \in \ell_0(\Sigma, \mathbb{F}; \mathbb{R}_+^N)$ and initial wealths W_0^i, $0 \leqslant i \leqslant L$, then the following constraints (in all states of the economy) on the consumption plans $c^i \in \ell_0(\Sigma, \mathbb{F}; \mathbb{R}_{++})$, the trading strategies $\theta^i \in \ell_0(\Sigma, \mathbb{F}; \mathbb{R}^N)$, and the individual state prices $\phi^i \in \ell_0(\Sigma, \mathbb{F}; \mathbb{R}_{++})$ $0 \leqslant i \leqslant L$:

$$\begin{aligned}
\mathbb{E}_t\left[\phi_{t+1}^i \left(S_{t+1}^j + \delta_{t+1}^j\right)\right] &= \phi_t^i\, S_t^j, \quad 1 \leqslant j \leqslant N,\ 0 \leqslant t < T \\
\left(\partial U_t^i\right)\left(c_t^i\right) &= \phi_t^i\,,\ 0 \leqslant i \leqslant L,\ 0 \leqslant t \leqslant T\,, \\
c_t^i + \theta_t^i \cdot S_t &= \varepsilon_t^i + W_t^i\,, \quad 0 \leqslant t \leqslant T, \\
\text{where}\quad W_t^i &:= \theta_{t-1}^i \cdot (S_t + \delta_t)\,,\ 0 < t \leqslant T,
\end{aligned} \tag{*}$$

are necessary and sufficient in order to claim that all agents achieve their goals at all times and in all states of the economy. Furthermore, the value functions $V_t^i(\cdot)$, are concave in any state and (*) can be satisfied with at most one choice for $\left(c^i, W^i, \phi^i\right)$.

(19) Equilibrium

Definition: Given initial wealths W_0^i, $0 \leqslant i \leqslant L$, equilibrium in the economy is the choice of

$$S \in \ell_0\left(\Sigma, \mathbb{F}; \mathbb{R}_+^N\right),$$

$$c^i \in \ell_0(\Sigma, \mathbb{F}; \mathbb{R}_{++}),\ \theta^i \in \ell_0\left(\Sigma, \mathbb{F}; \mathbb{R}^N\right),\ \phi^i \in \ell_0(\Sigma, \mathbb{F};$$

$$\mathbb{R}_{++}),\ 0 \leqslant i \leqslant L$$

so that (*) holds and, furthermore the following *aggregate resource constraint* is satisfied at all times and in all states of the economy

$$\sum\nolimits_{i=0}^{L} c_t^i = e_t := \sum\nolimits_{i=0}^{L} \varepsilon_t^i$$

(20) Equilibrium

$$\rho_t^i := c_t^i / e_t$$

$$\phi_t^i = \left(\partial U_t^i\right)\left(c_t^i\right) = \left(\partial U_t^i\right)\left(\rho_t^i \, e_t\right) \iff \phi_t^i \equiv \phi_t^i\left(\rho_t^i\right) \equiv \phi_t^i(\rho_t), \;\; \rho_t \in \Delta_{++}^L$$

$$\Delta_{++}^L := \left\{x \equiv \{x_0, x_1, \ldots, x_L\} \in \mathbb{R}_{++}^{L+1} \,; x_0 + x_1 + \ldots + x_L = 1\right\}$$

$$\phi_t^i : \Delta_{++}^L \mapsto \mathbb{R}_{++}$$

(21) Equilibrium

To obtain an equilibrium one must solve

$$\phi_t^i(\rho_t)\, S_t = \mathbb{E}_t\left[\phi_{t+1}^i(\rho_{t+1})\,(S_{t+1} + \delta_{t+1})\right],\ \ 0 \leqslant i \leqslant L,\ \ 0 \leqslant t < T,$$

$$\rho_t^i\, e_t + \theta_t^i \cdot S_t = \varepsilon_t^i + W_t^i\,,\ \ 0 \leqslant i \leqslant L,\ \ 0 \leqslant t \leqslant T,$$

$$\rho_t^0 + \rho_t^1 + \ldots + \rho_t^L = 1\,,\ \ 0 \leqslant t \leqslant T,$$

By the beginning of period t one must be able to compute the period-t consumption levels for all agents and by the end of period t one must be able to compute the prices at which securities are to be traded in period t. However, this cannot be achieved by solving the system period by period because the consumption ratios ρ_t^i appear in the equations for period t, in which they are endogenous, and also in the equations for period $(t-1)$ in which they are exogenous.

(22) Equilibrium

To obtain an equilibrium one must solve

$$\phi^i_t(\rho_t)\,S_t = \mathbb{E}_t\Big[\phi^i_{t+1}(\rho_{t+1})\,(S_{t+1}+\delta_{t+1})\Big],\ 0\leqslant i\leqslant L,\ 0\leqslant t\leqslant T-1,$$

$$\rho^i_{t+1}\,e_{t+1} + \underbrace{\theta^i_{t+1}\cdot S_{t+1}}_{:=F^i_{t+1}} = \varepsilon^i_{t+1} + \underbrace{W^i_{t+1}}_{:=\theta^i_t\cdot(S_{t+1}+\delta_{t+1})},$$

$$0\leqslant i\leqslant L,\ 0\leqslant t\leqslant T-1,$$

$$\rho^0_{t+1}+\rho^1_{t+1}+\ldots+\rho^L_{t+1} = 1\,,\ 0\leqslant i\leqslant L,\ 0\leqslant t\leqslant T-1,$$

** ** **

$$\rho^i_0\,e_0 + \theta^i_0\cdot S_0 = \varepsilon^i_0 + W^i_0\,,\ 0\leqslant i\leqslant L\,,$$

$$\rho^0_0+\rho^1_0+\ldots+\rho^L_0 = 1\,.$$

(23) Equilibrium

To obtain an equilibrium one must solve

$$S_t = \mathbb{E}_t\Big[\frac{\phi^i_{t+1}(\rho_{t+1})}{\phi^i_t(\rho_t)}\,(S_{t+1}+\delta_{t+1})\Big],\ 0 \leqslant i \leqslant L,\ 0 \leqslant t \leqslant T-1,$$

$$\theta^i_t \cdot (S_{t+1}+\delta_{t+1}) = F^i_{t+1} + \rho^i_{t+1}\, e_{t+1} - \varepsilon^i_{t+1},\ 0 \leqslant i \leqslant L,\ 0 \leqslant t \leqslant T-1,$$

$$\rho^0_{t+1} + \rho^1_{t+1} + \ldots + \rho^L_{t+1} = 1,\ 0 \leqslant i \leqslant L,\ 0 \leqslant t \leqslant T-1,$$

$$\Updownarrow$$

$$\mathbb{E}_t\Big[\frac{\phi^0_{t+1}(\rho_{t+1})}{\phi^0_t(\rho_t)}\,(S_{t+1}+\delta_{t+1})\Big] = \mathbb{E}_t\Big[\frac{\phi^i_{t+1}(\rho_{t+1})}{\phi^i_t(\rho_t)}\,(S_{t+1}+\delta_{t+1})\Big],$$

$$1 \leqslant i \leqslant L, \qquad (**)$$

$$\theta^i_t \cdot (S_{t+1}+\delta_{t+1}) = F^i_{t+1} + \rho^i_{t+1}\, e_{t+1} - \varepsilon^i_{t+1},\ 0 \leqslant i \leqslant L,$$

$$\rho^0_{t+1} + \rho^1_{t+1} + \ldots + \rho^L_{t+1} = 1,\ 0 \leqslant i \leqslant L,$$

givens: F^i_{t+1}, $0 \leqslant i \leqslant L$, $\rho_t \in \mathbb{R}^{L+1}$ & S_{t+1}

unknowns: $\theta^i_t \in \mathbb{R}^N$, $0 \leqslant i \leqslant L$, $\rho_{t+1} \in \mathbb{R}^{L+1}$

(24) The "forward-backward" recursion

Lemma:

$$F_t^i = \mathbb{E}_t\Big[\frac{\phi_{t+1}^0(\rho_{t+1})}{\phi_t^0(\rho_t)}\big(F_{t+1}^i - \varepsilon_{t+1}^i + \rho_{t+1}^i\, e_{t+1}\big)\Big].$$

Corollary:

$$\sum_{i=0}^{L} \theta_t^i \cdot (S_{t+1} + \delta_{t+1}) = 0 \iff \sum_{i=0}^{L} F_t^i = 0\,,\ \ 0 \leqslant t \leqslant T$$

(25) The "forward-backward" recursion

Remark 1: the givens F^i_{t+1}, $0 \leqslant i \leqslant L$ & S_{t+1} are "given" only as functions of the unknowns $\rho_{t+1} \in \mathbb{R}^{L+1}$, i.e., $F^i_{t+1}(\,\cdot\,)$, $0 \leqslant i \leqslant L$, & $S_{t+1}(\,\cdot\,)$ are defined as functions on Δ^L_{++}.

Remark 2: solving (**) means writing $\rho_{t+1} = \rho_{t+1}(\rho_t)$ and $\theta^i_t = \theta^i_t(\rho_t)$.

Remark 3: ρ_{t+1} and θ^i_t depend on ρ_t only through the ratio $\frac{\phi^i_t(\rho_t)}{\phi^0_t(\rho_t)}$.

Remark 4: the functions

$$\Delta^L_{++} \ni \rho_t \longrightarrow S_t(\rho_t) = \mathbb{E}_t\Big[\frac{\phi^0_{t+1}(\rho_{t+1})}{\phi^0_t(\rho_t)}\,(S_{t+1} + \delta_{t+1})\Big]$$

$$\Delta^L_{++} \ni \rho_t \longrightarrow F^i_t(\rho_t) = \mathbb{E}_t\Big[\frac{\phi^0_{t+1}(\rho_{t+1})}{\phi^0_t(\rho_t)}\,\big(F^i_{t+1} - \varepsilon^i_{t+1} + \rho^i_{t+1}\,e_{t+1}\big)\Big]$$

can be approximated by interpolating function objects defined on some finite interpolation grid inside Δ^L_{++}.

(26) The "forward-backward" recursion

Remark 5: To close the calculations one must compute ρ_0 from the system

$$\rho_0^i\, e_0 + \theta_0^i \cdot S_0 = \varepsilon_0^i + W_0^i\,,\ \ 0 \leqslant i \leqslant L\,,$$
$$\rho_0^0 + \rho_0^1 + \ldots + \rho_0^L = 1\,.$$

Remark 6: The relation

$$S_t = \mathbb{E}_t\Big[\frac{\phi_{t+1}^i(\rho_{t+1})}{\phi_t^i(\rho_t)}\,(S_{t+1} + \delta_{t+1})\Big]$$

says that all market agents agree on the security prices. We can write

$$S_t = \frac{1}{1 + r_t^i(\rho_t)}\,\mathbb{E}_t\Big[\frac{\phi_{t+1}^i(\rho_{t+1})}{\mathbb{E}_t\big[\phi_{t+1}^i(\rho_{t+1})\big]}\,(S_{t+1} + \delta_{t+1})\Big]\,,\ \ \frac{1}{1 + r_t^i(\rho_t)} = \frac{\mathbb{E}_t\big[\phi_{t+1}^i(\rho_{t+1})\big]}{\phi_t^i(\rho_t)}$$

(27) The distribution of wealth as a dimension of incompleteness

At a given node $\xi \in \mathbb{F}_t$, with $K_\xi := \#(\xi^+)$, (**) contains $\left(K_\xi + N\right)(L+1)$ unknowns and a total of

$$N\,L + K_\xi(L+1) + K_\xi = \left(K_\xi + N\right)(L+1) + K_\xi - N$$

equations. We always suppose $N \leqslant K_\xi$ and

$$\mathrm{Rank}\left\{S_{t+1}\left(\rho_{t+1,\eta}\right) + \delta_{t+1,\eta};\, \eta \in \mathbb{F}_1^{\xi}\right\} = N\,,\ \ \forall\, \xi \in \mathbb{F}_t,\ \ 0 \leqslant t \leqslant T-1,$$

completeness $\Longleftrightarrow N = K_\xi$.

(28) The distribution of wealth as a dimension of incompleteness

When the market is complete the flow budget constraints are merely expressions for the optimal portfolios:

$$\theta_t^i = \left(F_{t+1}^i + \rho_{t+1}^i\, e_{t+1} - \varepsilon_{t+1}^i\right)\left(S_{t+1} + \delta_{t+1}\right)^{-1}$$

and can be eliminated from the system. The kernel conditions give

$$\sum\nolimits_{\eta \in \mathbb{F}_1^\xi} \left(S_{t+1}\left(\rho_{t+1,\eta}\right) + \delta_{t+1,\eta}\right) \times \left(\frac{\phi_{t+1}^i\left(\rho_{t+1,\eta}\right)}{\phi_t^i} - \frac{\phi_{t+1}^0\left(\rho_{t+1,\eta}\right)}{\phi_t^0}\right)\frac{\pi(\eta)}{\pi(\xi)} = 0$$

and are the same as

$$\frac{\phi_{t+1}^i\left(\rho_{t+1,\eta}\right)}{\phi_t^i(\rho_t)} = \frac{\phi_{t+1}^0\left(\rho_{t+1,\eta}\right)}{\phi_t^0(\rho_t)},\ \ 1 \leqslant i \leqslant L,\ \ \eta \in (\xi^+).$$

(29) The distribution of wealth as a dimension of incompleteness

We now have $L\,K_\xi \equiv L\,N$ kernel conditions and $K_\xi \equiv N$ aggregate resource constraints for a total of $K_\xi\,(L+1) \equiv N\,(L+1)$ unknowns $\rho^i_{t+1,\eta}$.

If the market is incomplete $(K_\xi > N)$ there are more constraints than the dimension of θ^i

$$\theta^i_{t,\xi} \cdot \left(S_{t+1,\eta} + \delta_{t+1,\eta}\right) = F^i_{t+1,\eta} + \rho^i_{t+1,\eta}\, e_{t+1,\eta} - \varepsilon^i_{t+1,\eta}, \;\; \eta \in (\xi^+),$$

The fact that the above system has a solution imposes $K_\xi - N$ constraints on the right sides for every fixed $0 \leqslant i \leqslant L$. This imposes $\left(K_\xi - N\right)(L+1)$ conditions on the right sides — $\left(K_\xi - N\right)$ constraints for each agent. Because of the market clearing condition, the $\left(K_\xi - N\right)$ constraints on agent 0 are redundant.

(30) The distribution of wealth as a dimension of incompleteness

Remark: The flow budget constraints entail $\left(K_{\xi} - N\right) L$ restrictions on investors wealths and consumption, which allow one to eliminate the portfolios θ_t^i. We are therefore left with

$$L\, N + \left(K_{\xi} - N\right) L + K_{\xi} = K_{\xi}\,(L+1)$$

constraints for the same number of variables $\rho_{t+1,\eta}^{i}$, $0 \leqslant i \leqslant L$, $\eta \in (\xi^{+})$.

(31) Examples

Basac-Cuoco (1998) I: two agents, agent 0 holds a bond, agent 1 holds a stock and only the bond is traded. The uncertainty is represented by a binomial tree
In our setting there is only one traded security $S_t \in \mathbb{R}_+$ and the system becomes.

$$\mathbb{E}_t\Big[\frac{\phi^0_{t+1}(\rho_{t+1})}{\phi^0_t(\rho_t)}\,(S_{t+1}+\delta_{t+1})\Big] = \mathbb{E}_t\Big[\frac{\phi^1_{t+1}(\rho_{t+1})}{\phi^1_t(\rho_t)}\,(S_{t+1}+\delta_{t+1})\Big],$$

$$\theta^0_t\cdot(S_{t+1,u}+\delta_{t+1,u}) = F^0_{t+1,u}+\rho^0_{t+1,u}\,e_{t+1,u}-\varepsilon^0_{t+1,u}$$

$$\theta^1_t\cdot(S_{t+1,u}+\delta_{t+1,u}) = F^1_{t+1,u}+\rho^1_{t+1,u}\,e_{t+1,u}-\varepsilon^1_{t+1,u}$$

$$\theta^0_t\cdot(S_{t+1,d}+\delta_{t+1,d}) = F^0_{t+1,d}+\rho^0_{t+1,d}\,e_{t+1,d}-\varepsilon^0_{t+1,d}$$

$$\theta^1_t\cdot(S_{t+1,d}+\delta_{t+1,d}) = F^1_{t+1,d}+\rho^1_{t+1,d}\,e_{t+1,d}-\varepsilon^1_{t+1,d}$$

$$\rho^0_{t+1,u}+\rho^1_{t+1,u} = 1,$$

$$\rho^0_{t+1,d}+\rho^1_{t+1,d} = 1\,.$$

(32) Examples

Risk-free means that $S_{t+1,\eta} + \delta_{t+1,\eta}$ is constant across all $\eta \in (\xi^+)$. Because of the market clearing condition we only need to write the flow budget constraints for agent 0:

$$\frac{1}{2}\,\frac{\phi^0_{t+1}(\rho_{t+1,u})}{\phi^0_t(\rho_t)} + \frac{1}{2}\,\frac{\phi^0_{t+1}(\rho_{t+1,d})}{\phi^0_t(\rho_t)} = \frac{1}{2}\,\frac{\phi^1_{t+1}(\rho_{t+1,u})}{\phi^1_t(\rho_t)} + \frac{1}{2}\,\frac{\phi^1_{t+1}(\rho_{t+1,d})}{\phi^1_t(\rho_t)},$$

$$\theta^0_t(S_{t+1,u}(\rho_{t+1,u}) + \delta_{t+1,u}) = F^0_{t+1,u}(\rho_{t+1,u}) + \rho^0_{t+1,u}\,e_{t+1,u} - \varepsilon^0_{t+1,u}$$

$$\theta^0_t(S_{t+1,d}(\rho_{t+1,d}) + \delta_{t+1,d}) = F^0_{t+1,d}(\rho_{t+1,d}) + \rho^0_{t+1,d}\,e_{t+1,d} - \varepsilon^0_{t+1,d}$$

$$\rho^0_{t+1,u} + \rho^1_{t+1,u} = 1,$$

$$\rho^0_{t+1,d} + \rho^1_{t+1,d} = 1\,.$$

This is a system of 5 equations with unknowns

$$\rho^0_{t+1,u},\, \rho^1_{t+1,u},\, \rho^0_{t+1,d},\, \rho^1_{t+1,d},\, \theta^0_t$$

(33) Examples

If we complete the market (say, with the risky security) the system becomes

$$\frac{\phi^0_{t+1}(\rho_{t+1,u})}{\phi^0_t(\rho_t)} = \frac{\phi^1_{t+1}(\rho_{t+1,u})}{\phi^1_t(\rho_t)}, \quad \frac{\phi^0_{t+1}(\rho_{t+1,d})}{\phi^0_t(\rho_t)} = \frac{\phi^1_{t+1}(\rho_{t+1,d})}{\phi^1_t(\rho_t)},$$

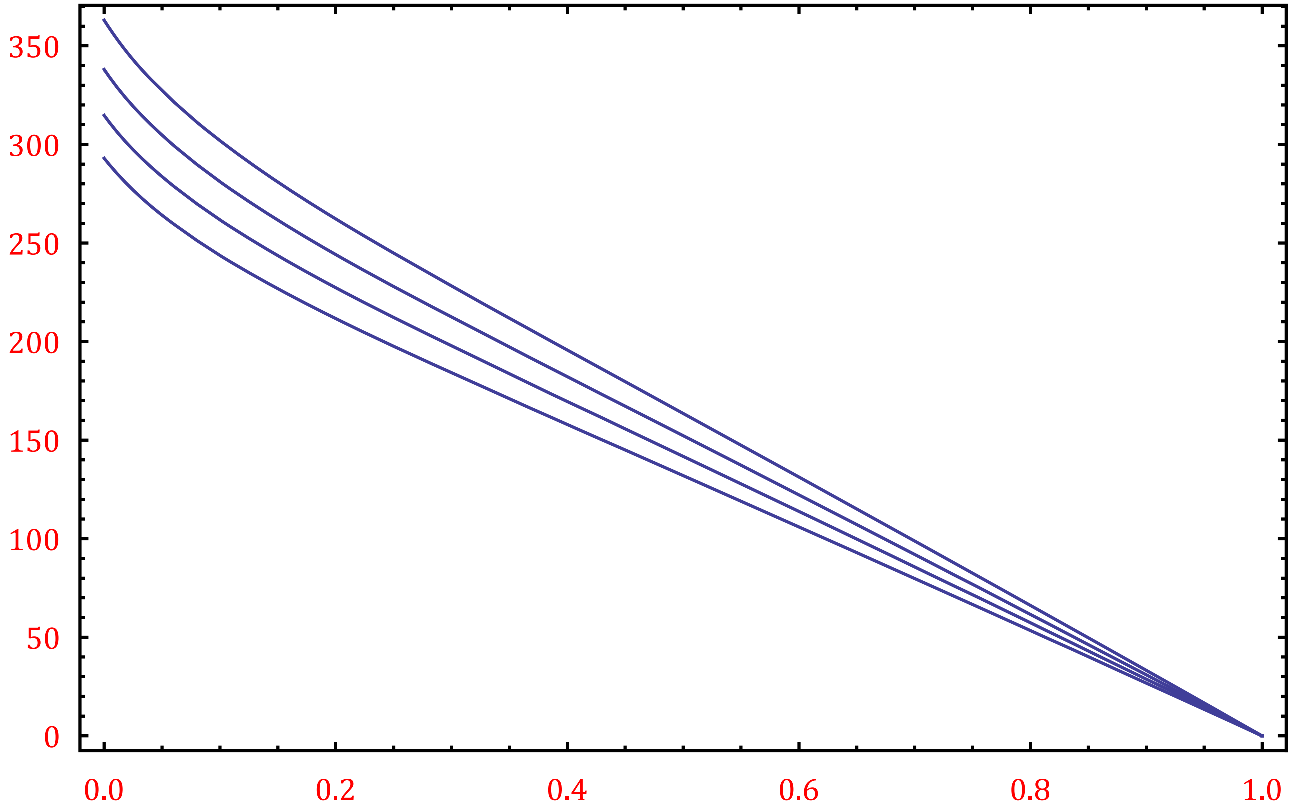
$$\rho^0_{t+1,u} + \rho^1_{t+1,u} = 1,$$

$$\rho^0_{t+1,d} + \rho^1_{t+1,d} = 1\,.$$

By using our method we extend the Basac-Cuoco (1998) economy to include agents with power utilities. In our formulation the market price of risk is

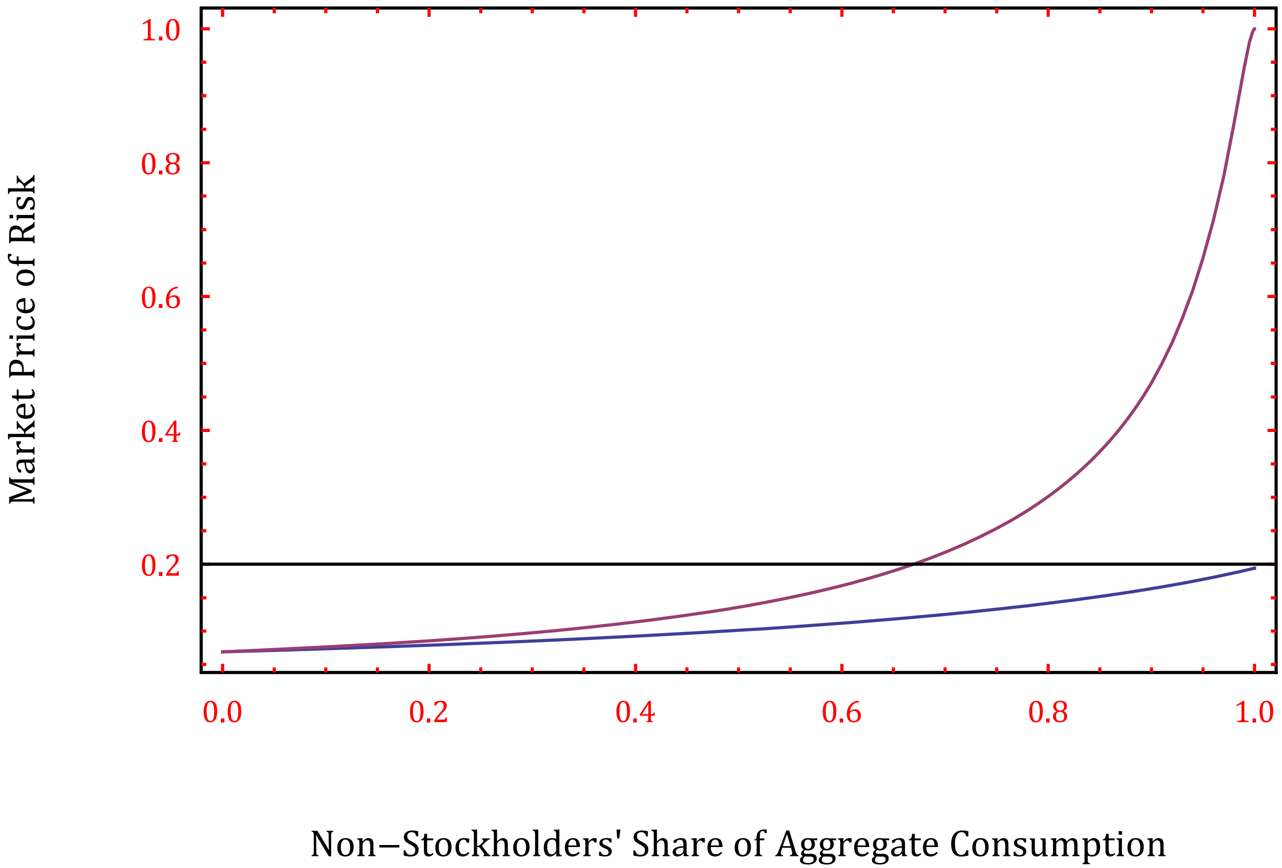
$$\frac{\left(\frac{1}{2}\,\frac{\phi^0_{t+1,u}}{\phi^0_t} + \frac{1}{2}\,\frac{\phi^0_{t+1,d}}{\phi^0_t}\right)\times\left(\frac{1}{2}\,\frac{F^0_{t+1,u}+\varepsilon^0_{t+1,u}}{F^0_t} + \frac{1}{2}\,\frac{F^0_{t+1,d}+\varepsilon^0_{t+1,d}}{F^0_t}\right) - 1}{\frac{1}{2}\,\frac{F^0_{t+1,u}+\varepsilon^0_{t+1,u}}{F^0_t} - \frac{1}{2}\,\frac{F^0_{t+1,d}+\varepsilon^0_{t+1,d}}{F^0_t}}$$

(34) Examples



The non-stock holders wealth as a function of their consumption ratio in all 4 states at $t = 3$.

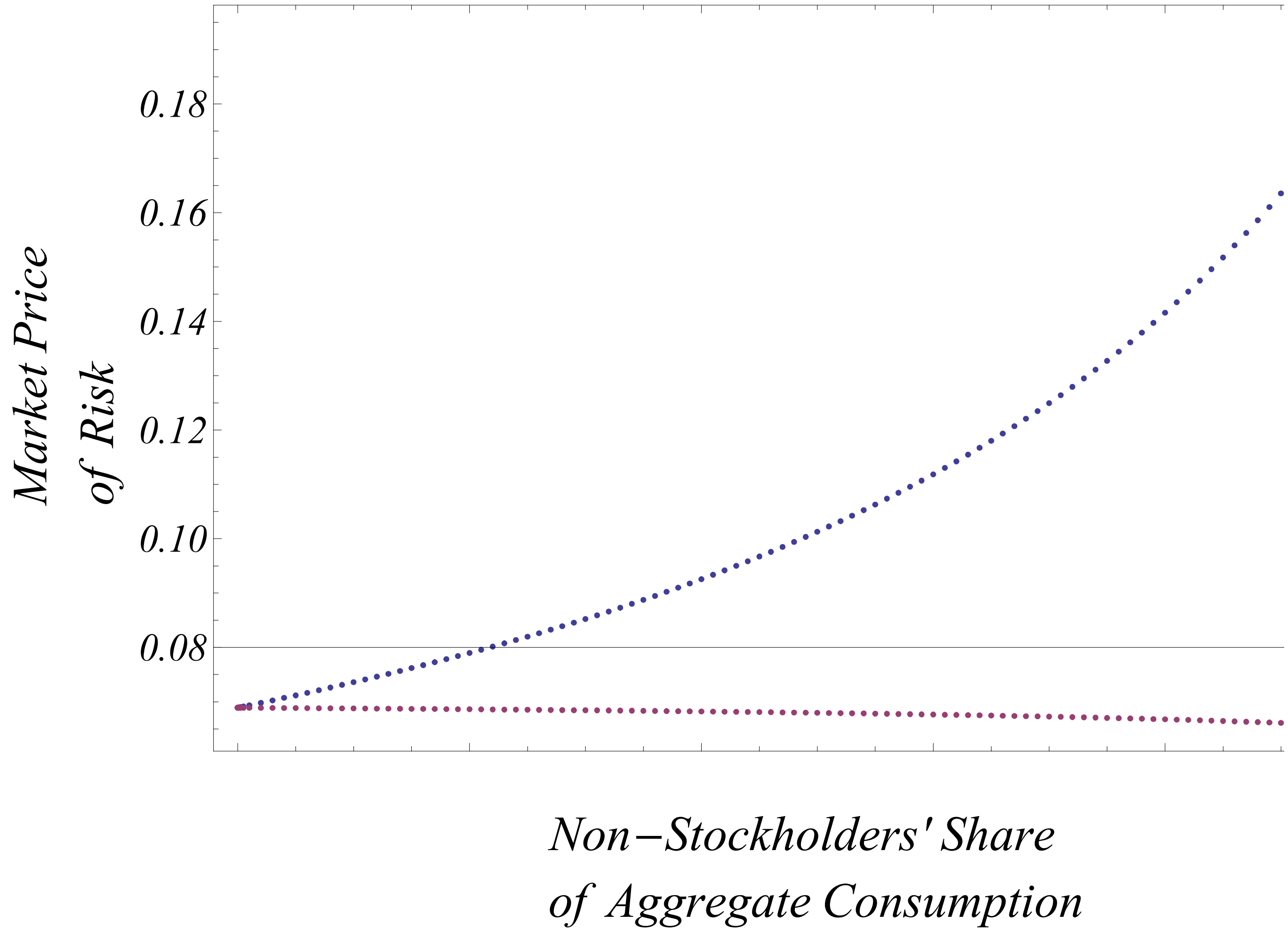
(35) Examples



The market price of risk in the case of an incomplete market with one risk-free security and the case of a complete market (under the same risk-preferences for the agents)

(36) Examples

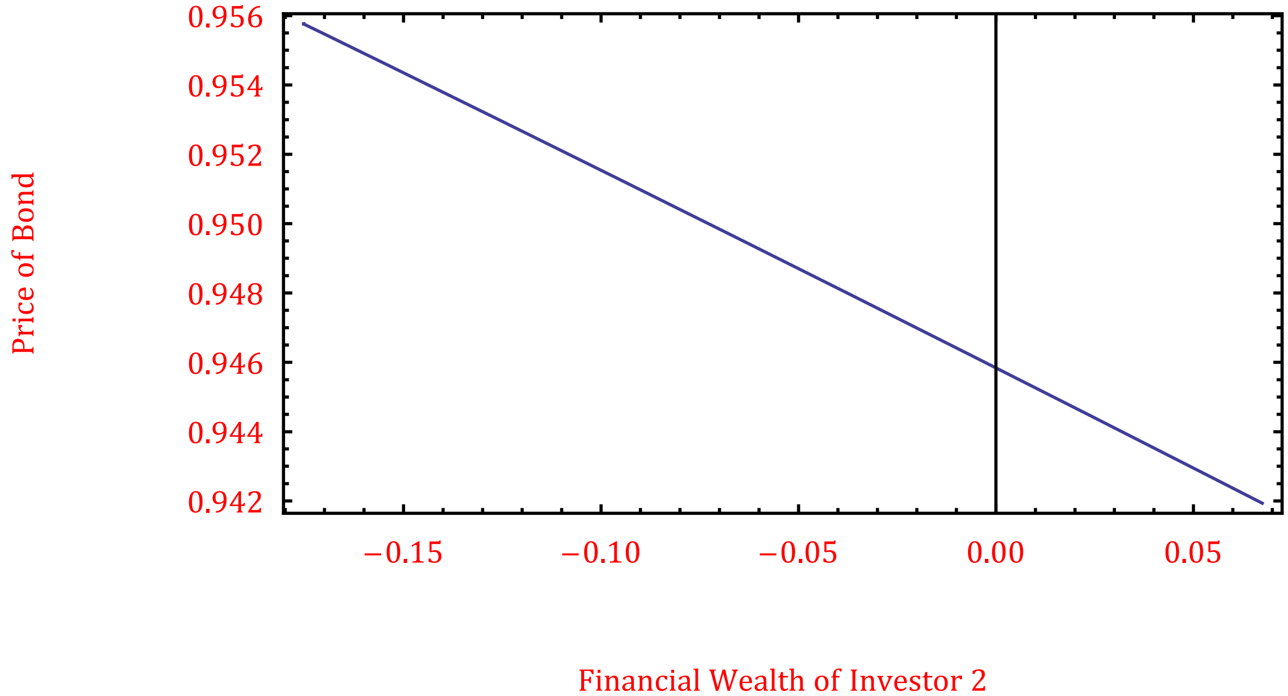
Basac - Cuoco (1998) II: only the risky security is traded



(37) Examples

- Cuoco-He (2001): Two assets on a lattice
  - it is possible to stack all the first-order conditions (**) of all the nodes into one large system and then to substitute into this system the recursions for F_t and S_t. This huge system can conceivably be solved simultaneously in one fell swoop. We call this approach the "global method," as opposed to the recursive method, for the solution of the forward-backward system.
  - In their paper of 2001, Cuoco and He write and solve a large system of that type.
  - In their numerical Example #6.2 (Page 289), they consider a two-period t=0,1,2 economy with a tree that is not binomial and is better called a lattice, and with two securities:
    - a long-term bond (maturing at time 2) and the equity claim.
  - The node of time 0 has three spokes. At time 1, one node has two spokes and the other two have three spokes.
  - The initial condition imposed is that the net financial wealth of both groups be equal to zero.

(38) Examples



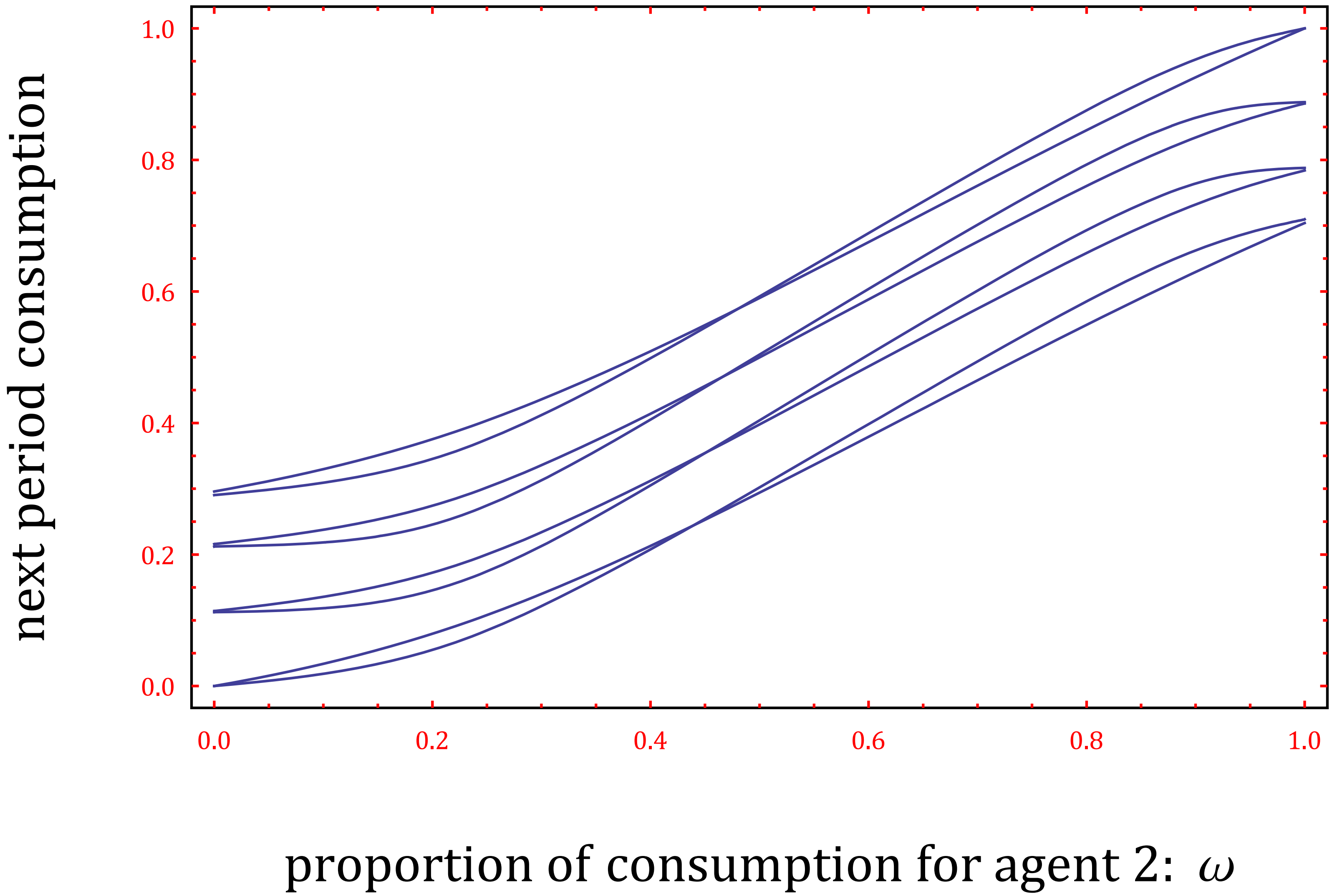
The intersection of the line of points with the x-axis gives the price of the bond corresponding to the solution of Cuoco and He (2001), Page 291.

(39) Examples

- Heaton-Lucas (1996)
  - Model calibrated to real U.S. economy, including idiosyncratic labor shocks observed on panel data
  - Two groups of households differ only in the allocation of output to individual labor income; both have CRRA=1.5
  - They have identical risk aversions and discount rates. Because of that, output is only a scale variable, which can be factored out
  - Three exogenous state variables describe the exogenous aspects of the economy at any given time:
    - the realized rate of growth of output
    - the share of output paid out as dividend, vs. labor
    - the share of labor income that is paid to Group 1, vs. Group 2
  - These follow an eight-state Markov chain, which is calibrated to U.S. data
  - One endogenous state variable defined as ρ_t above.

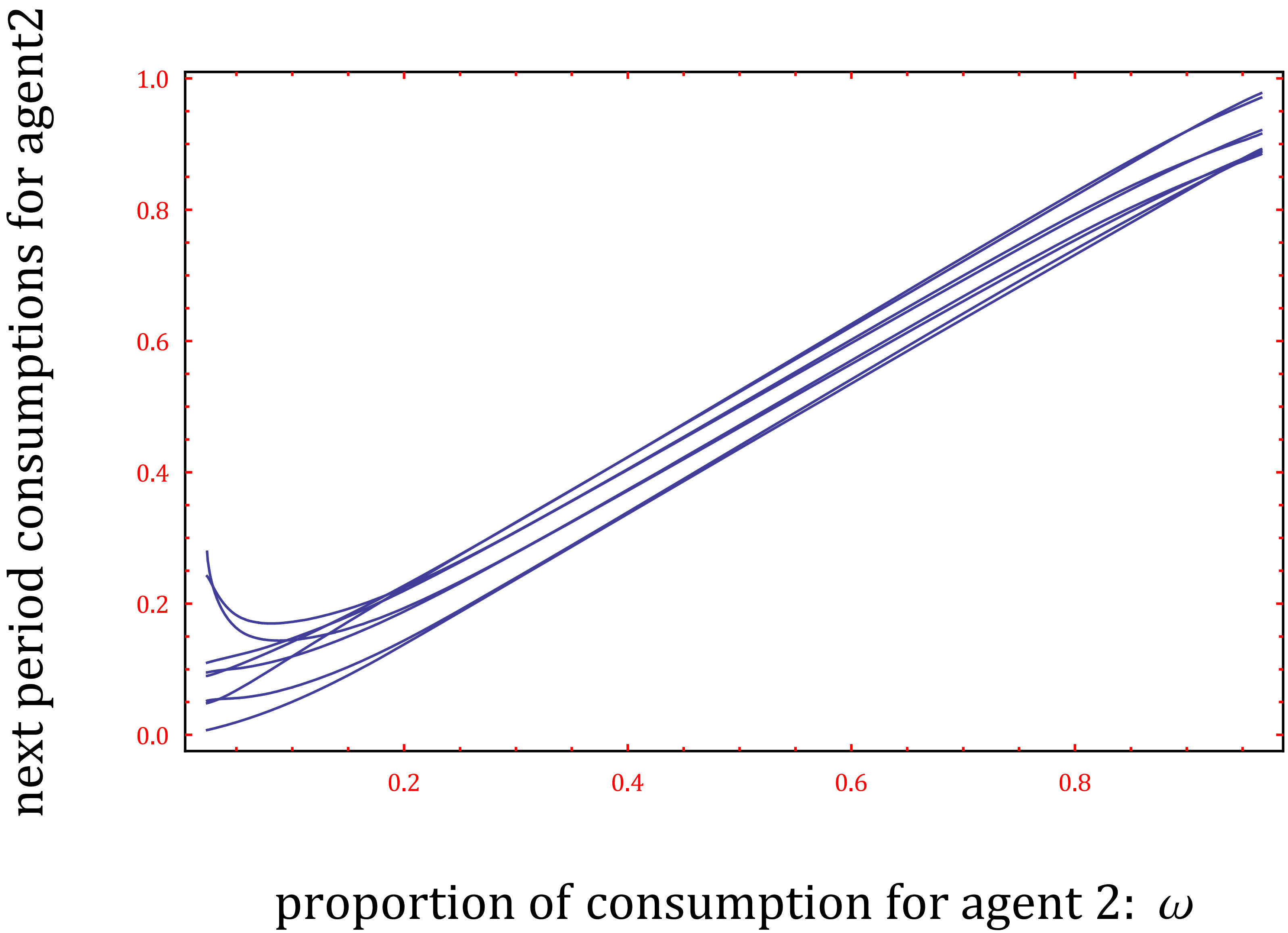
(40) Examples

$t = T - 1$ (next period consumption):



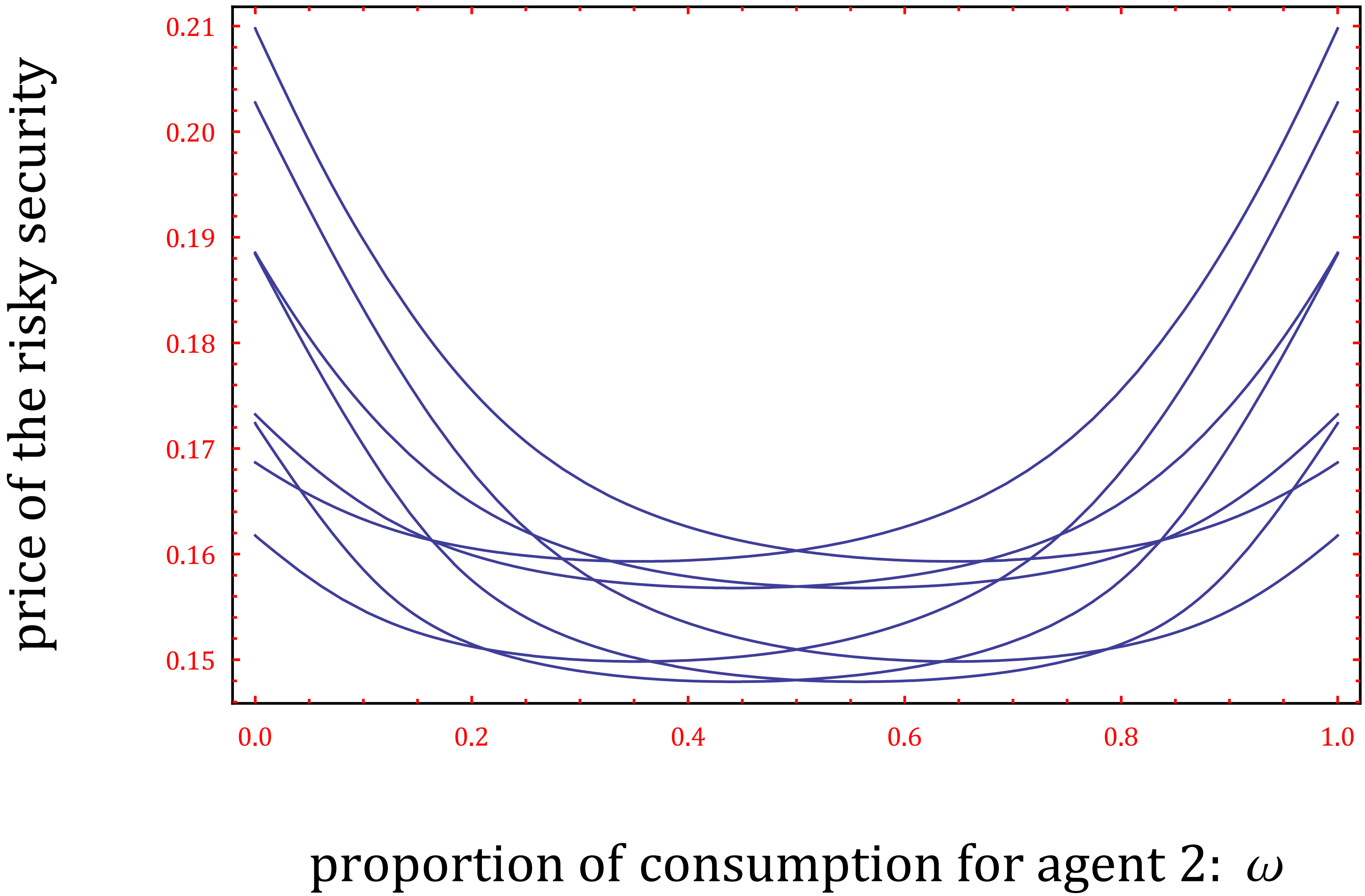
(41) Examples

$t = T - 7$ (next period consumption):



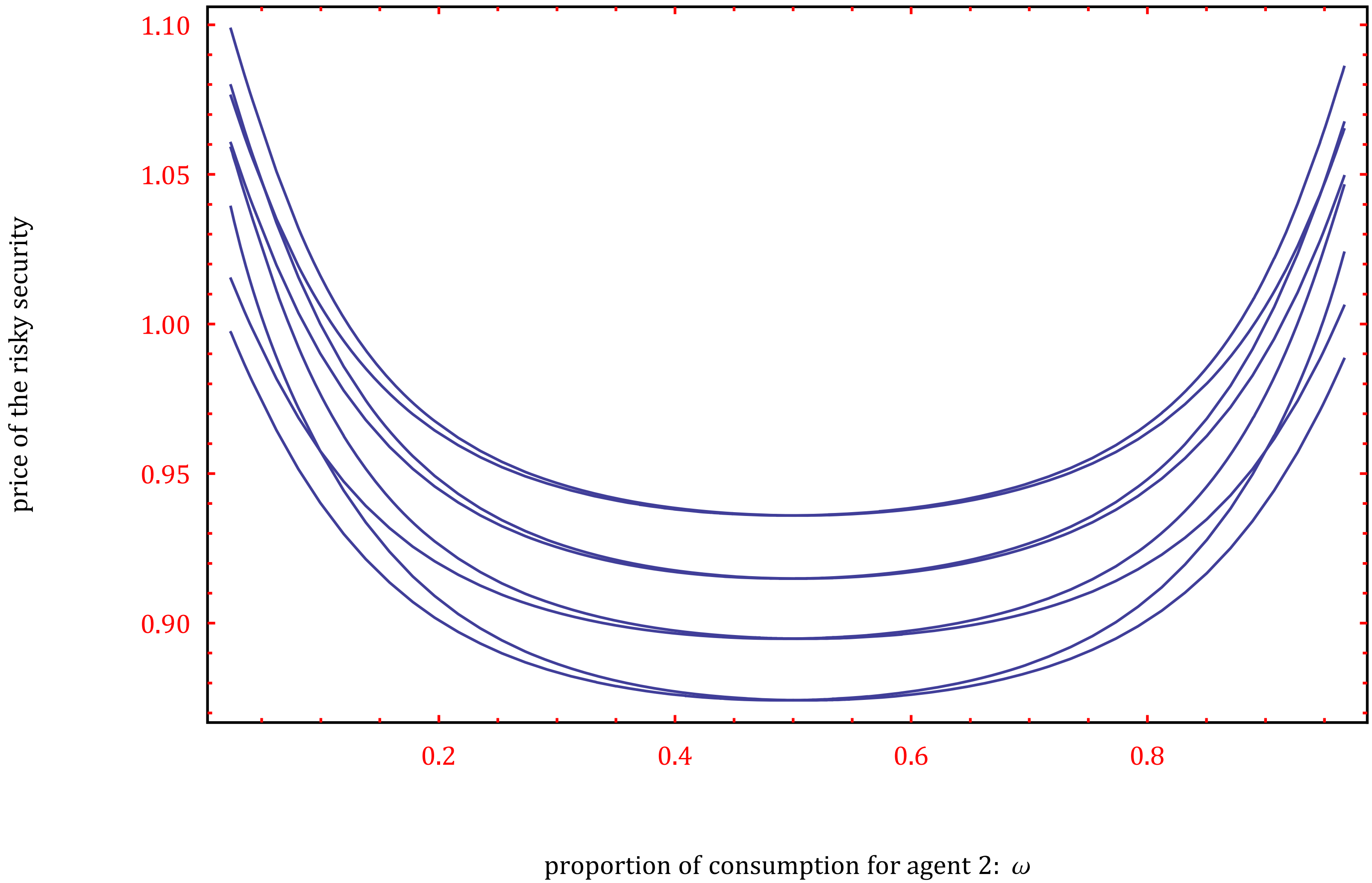
(42) Examples

$t = T - 1$ (risky security):



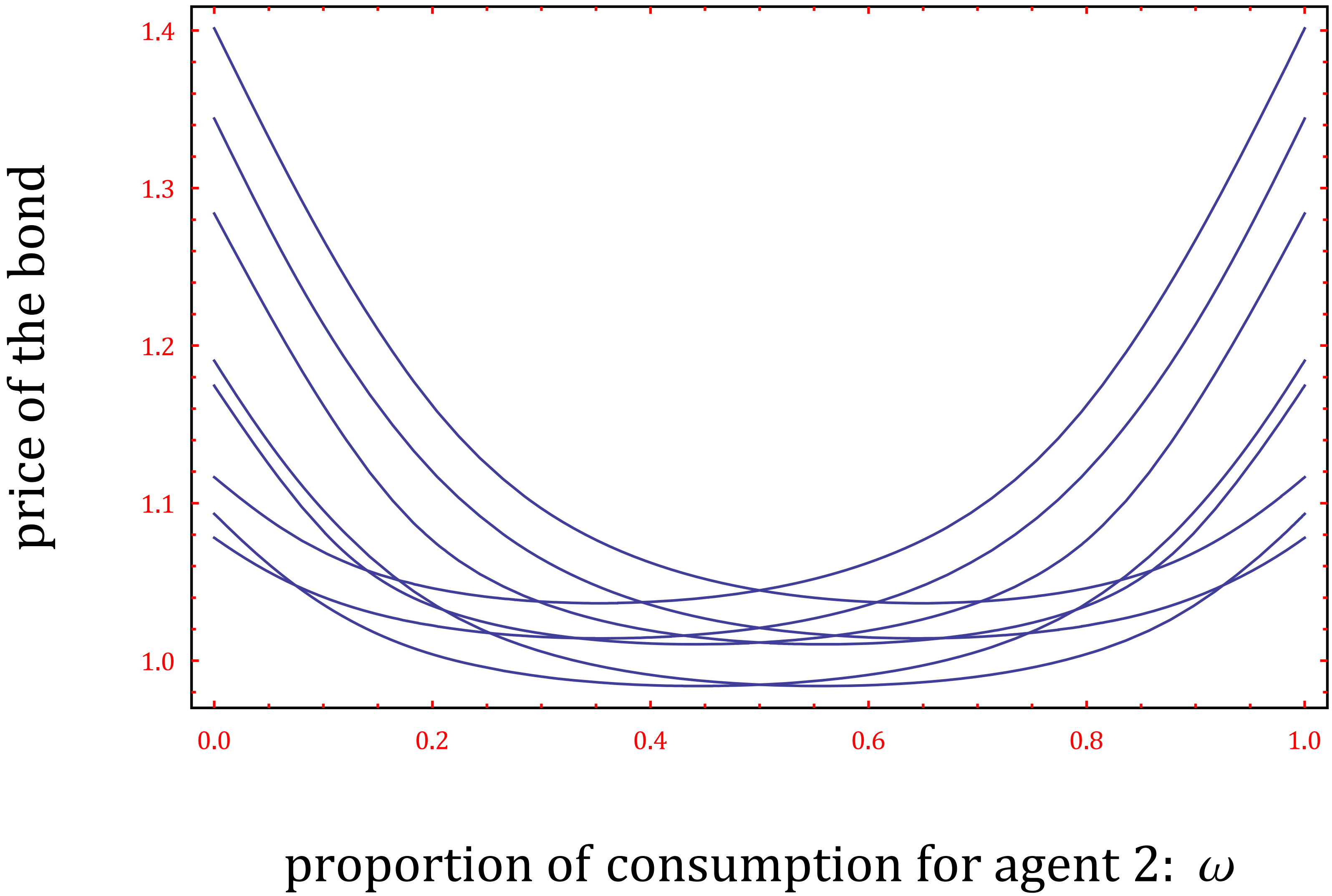
(43) Examples

$t = T - 7$ (risky security):



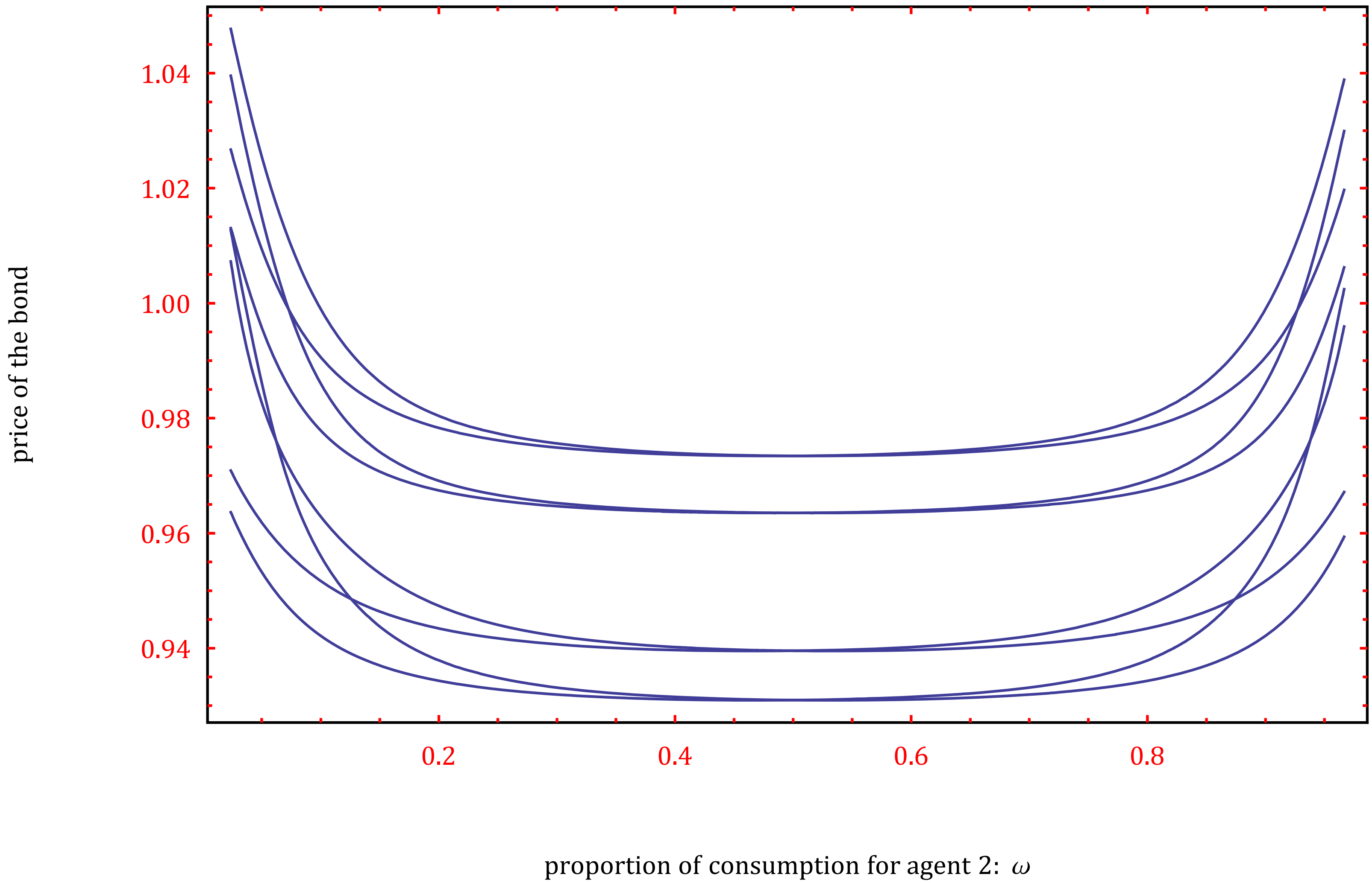
(44) Examples

$t = T - 1$ (bond):



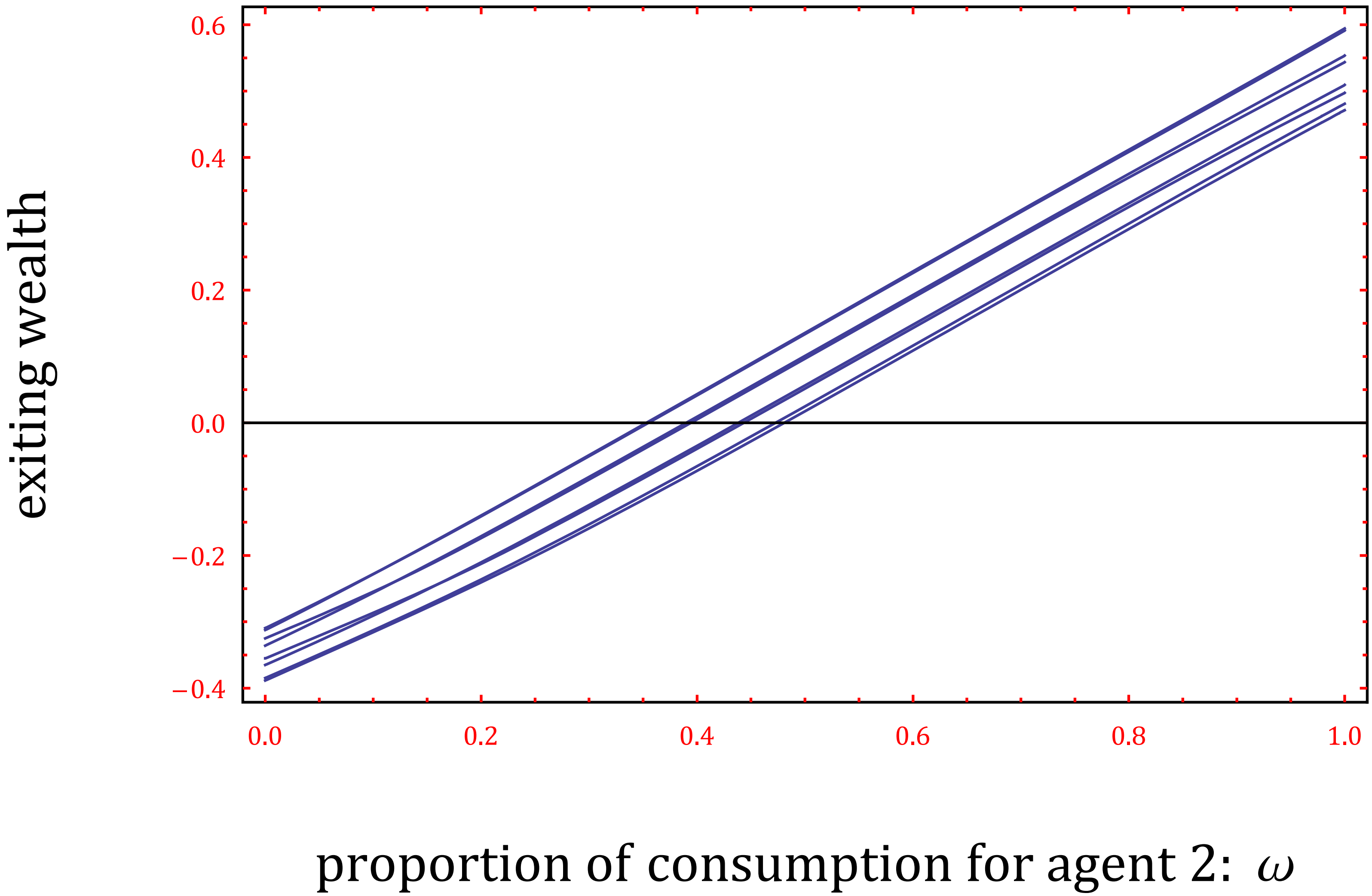
(45) Examples

$t = T - 7$ (bond):



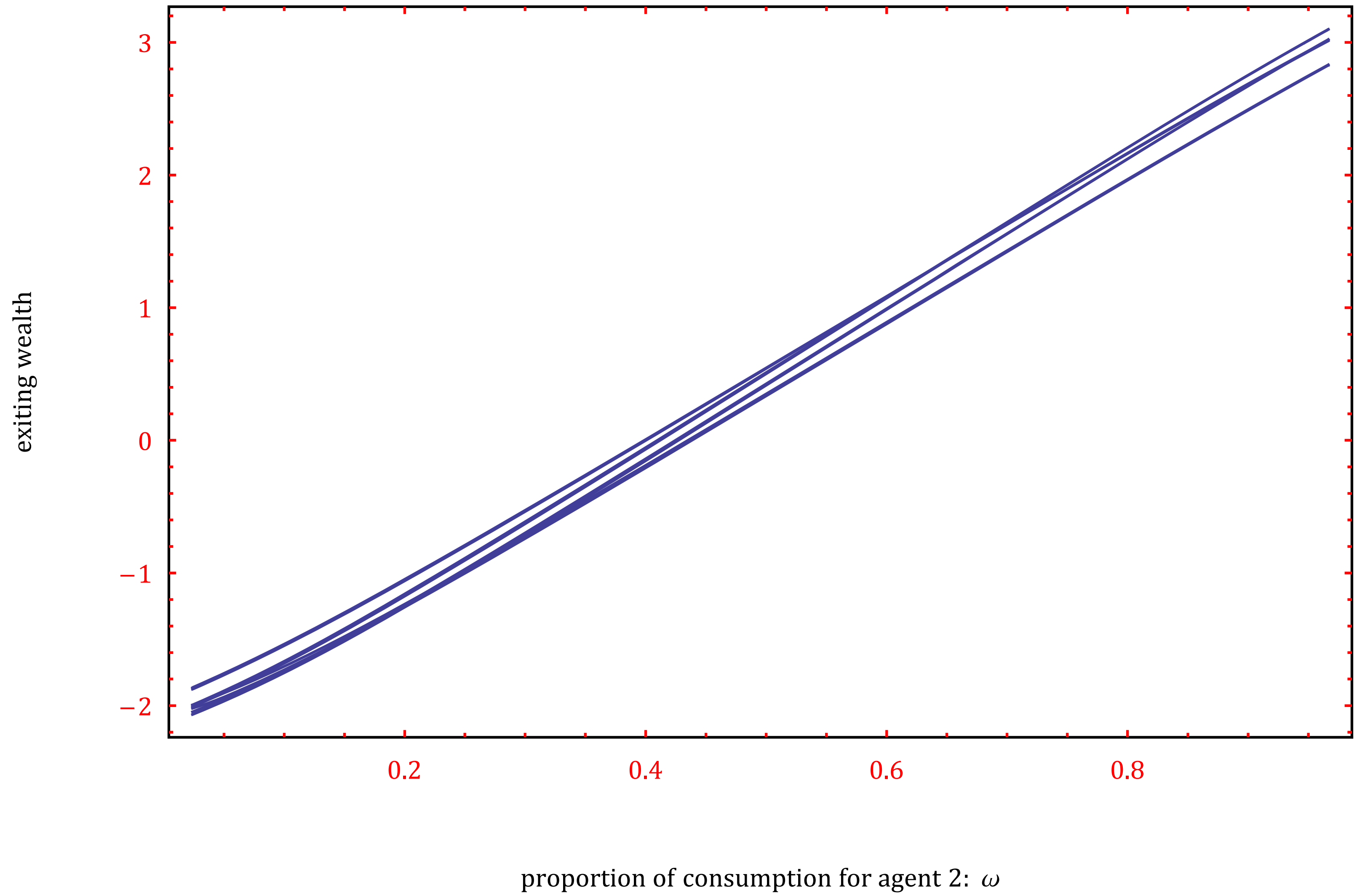
(46) Examples

$t = T - 1$ (investment):



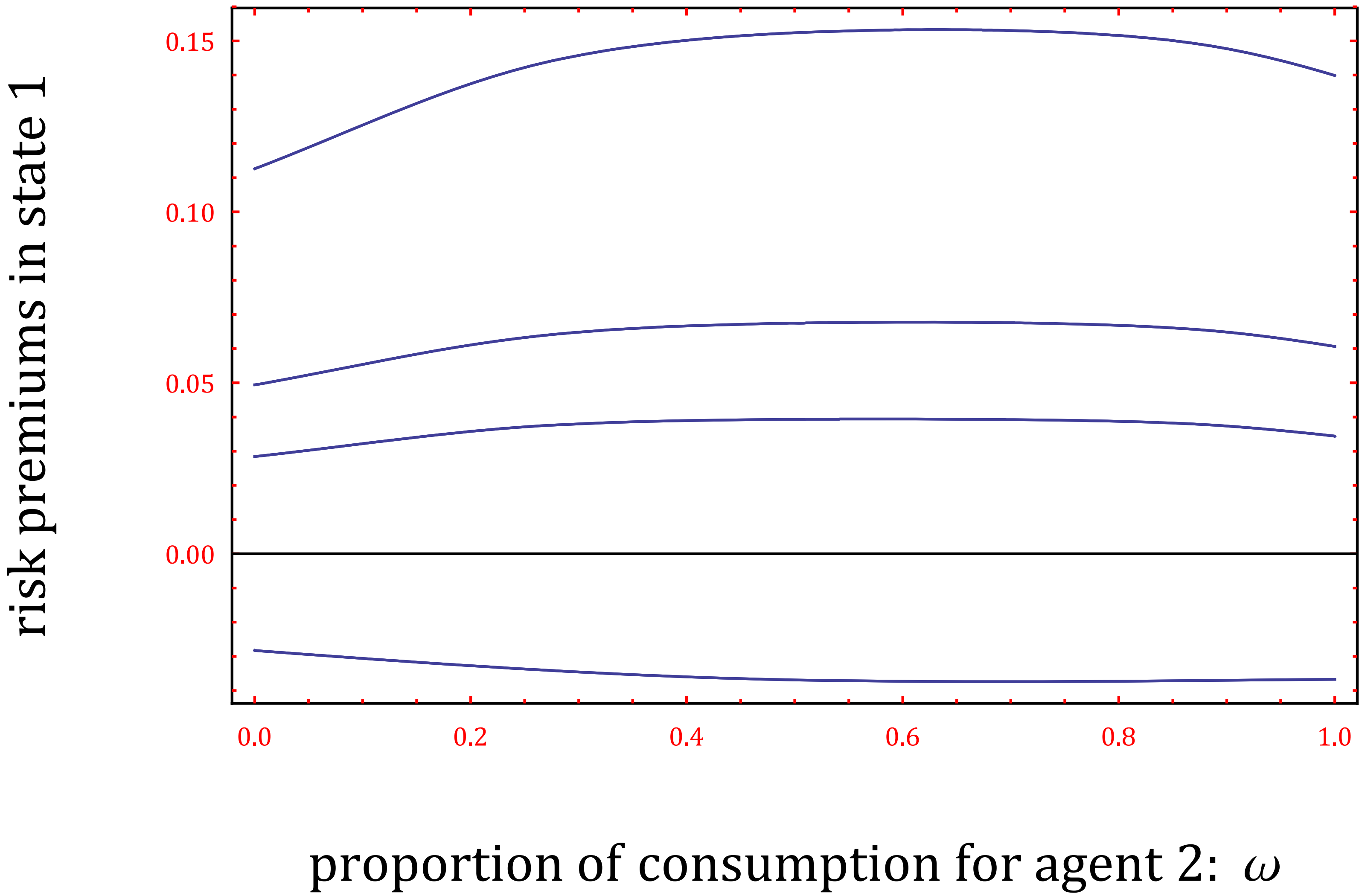
(47) Examples

$t = T - 7$ (investment):



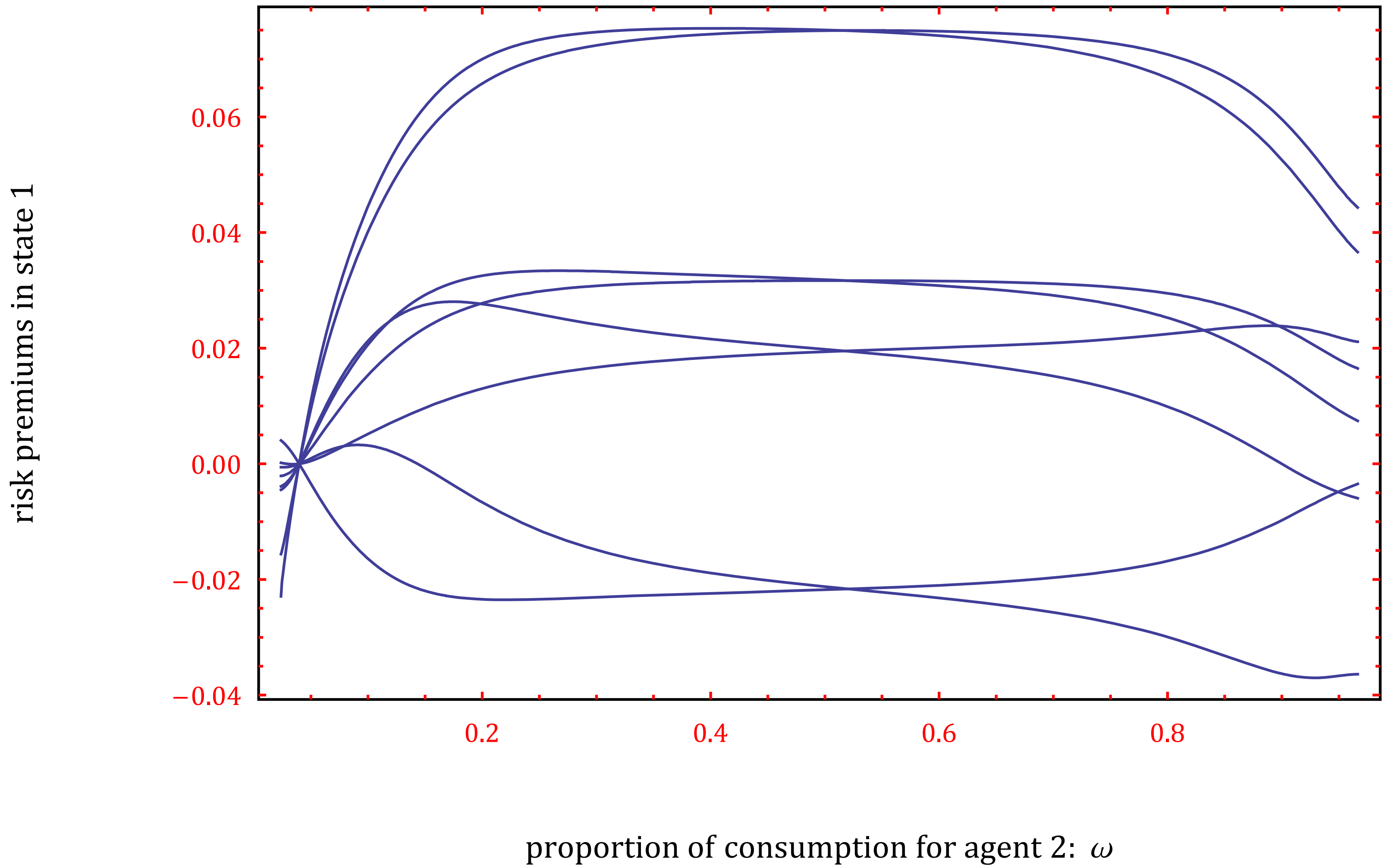
(48) Examples

$t = T - 1$ (risk premium):



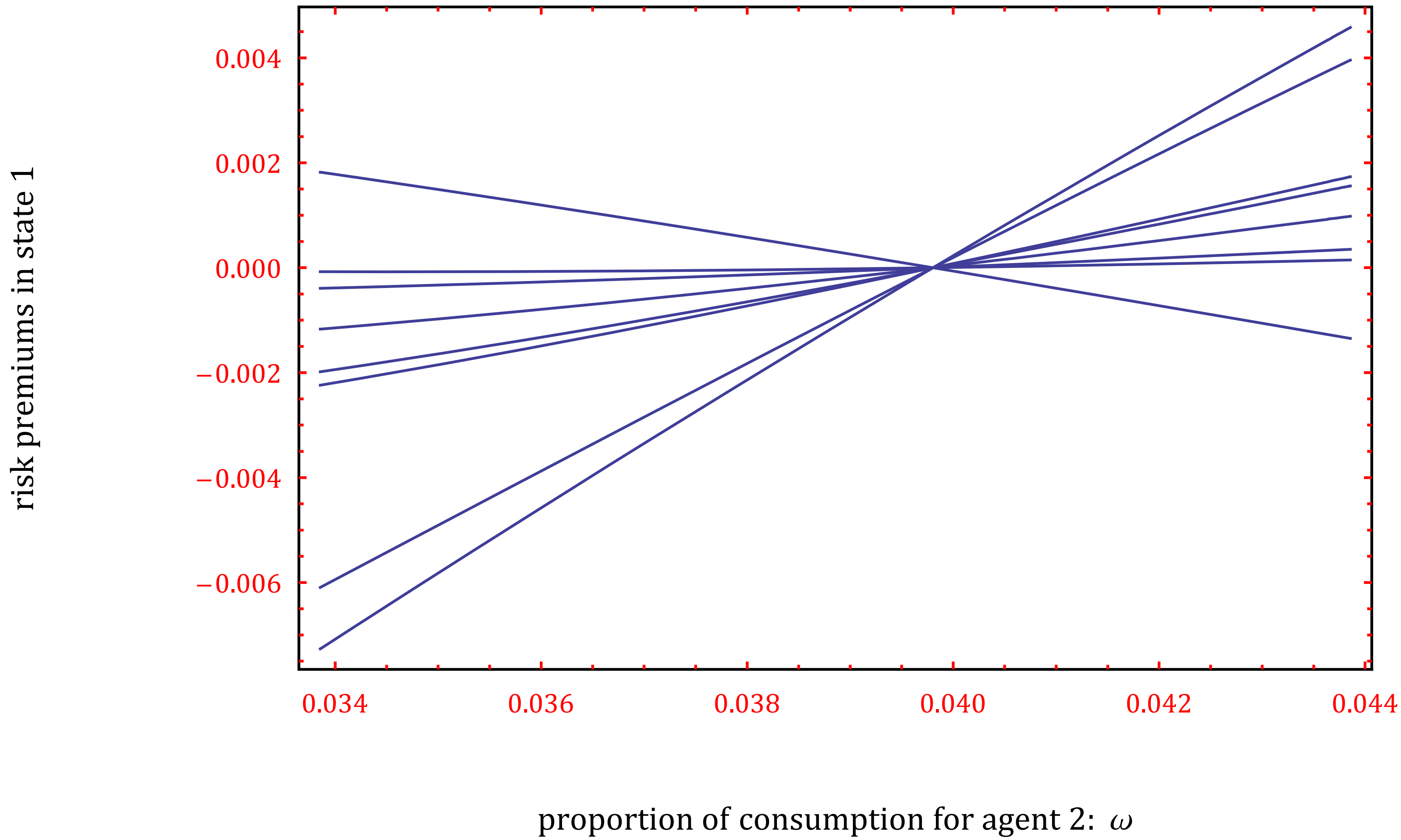
(49) Examples

$t = T - 7$ (risk-premiums):



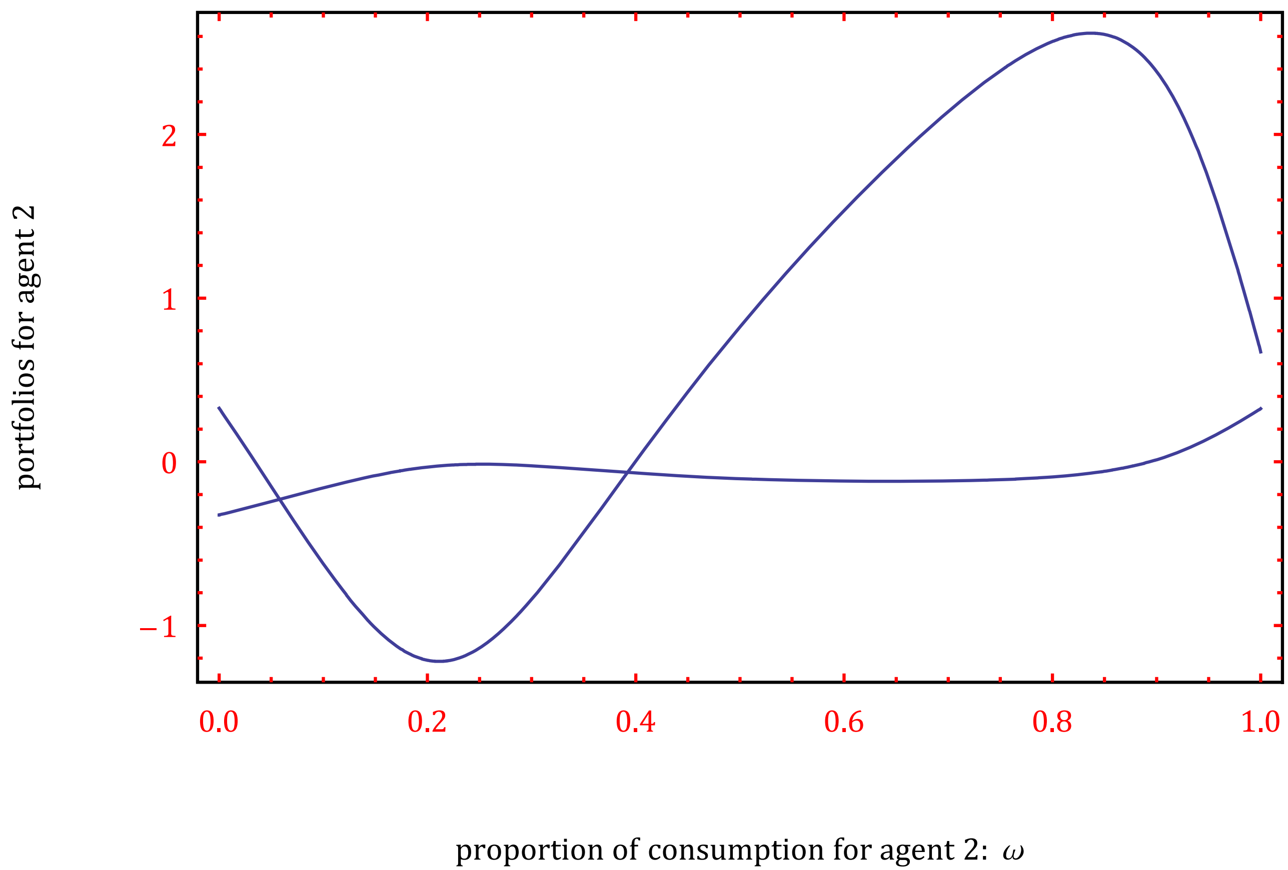
(50) Examples

$t = T - 7$ (risk-premiums):



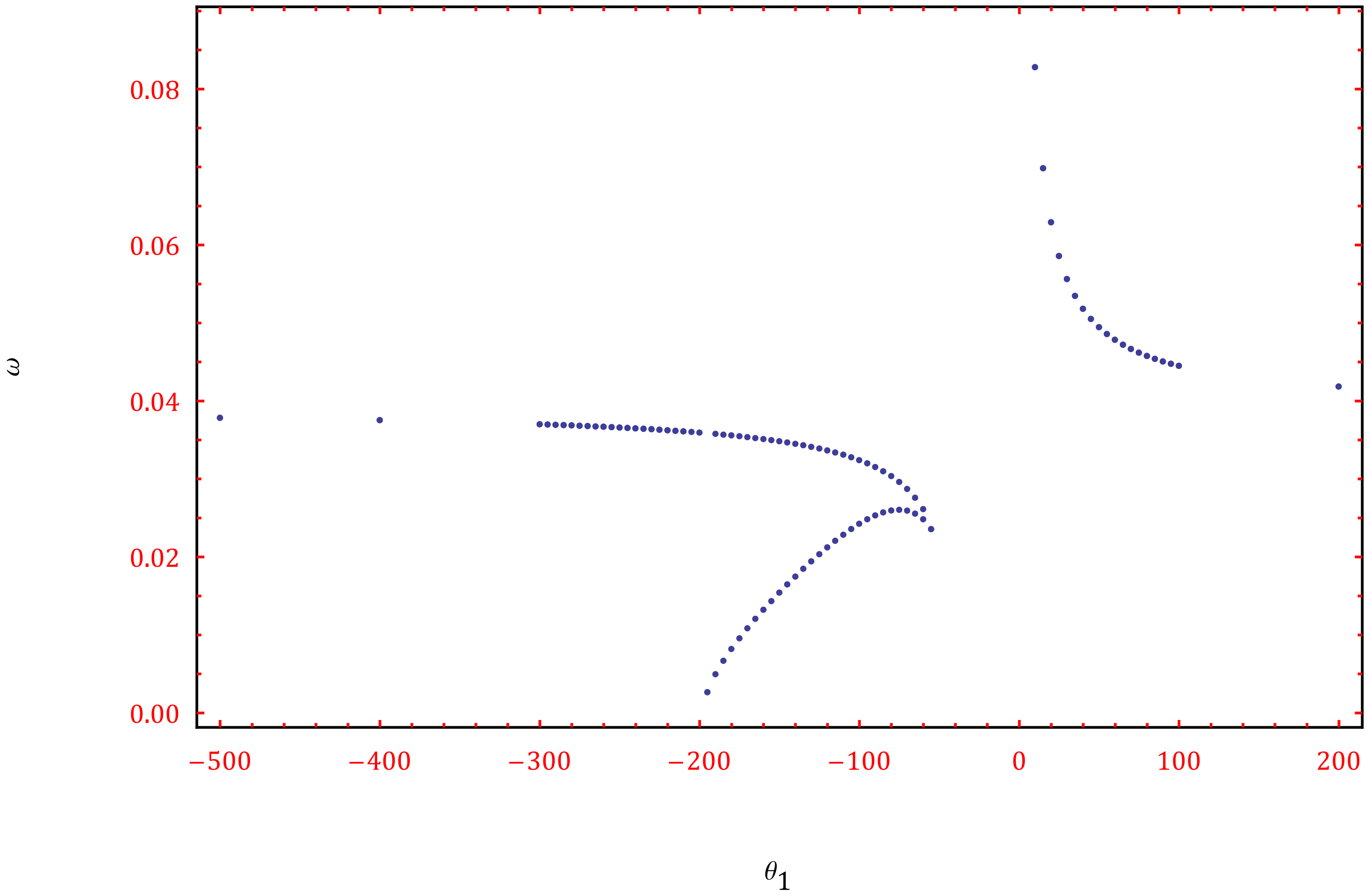
(51) Examples

$t = T - 1$ (portfolio):



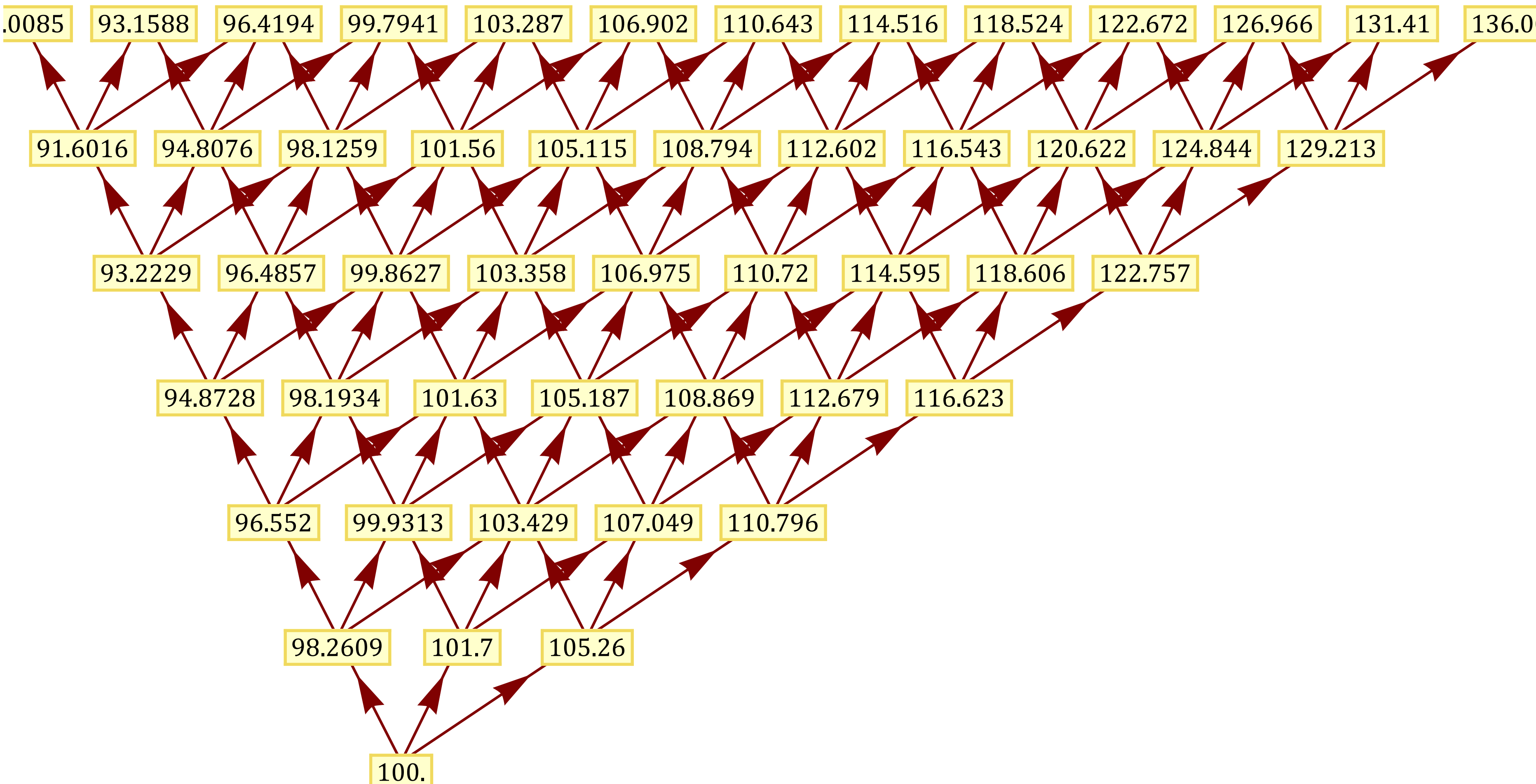
(52) Examples

$t = T - 7$ (multiple solutions):



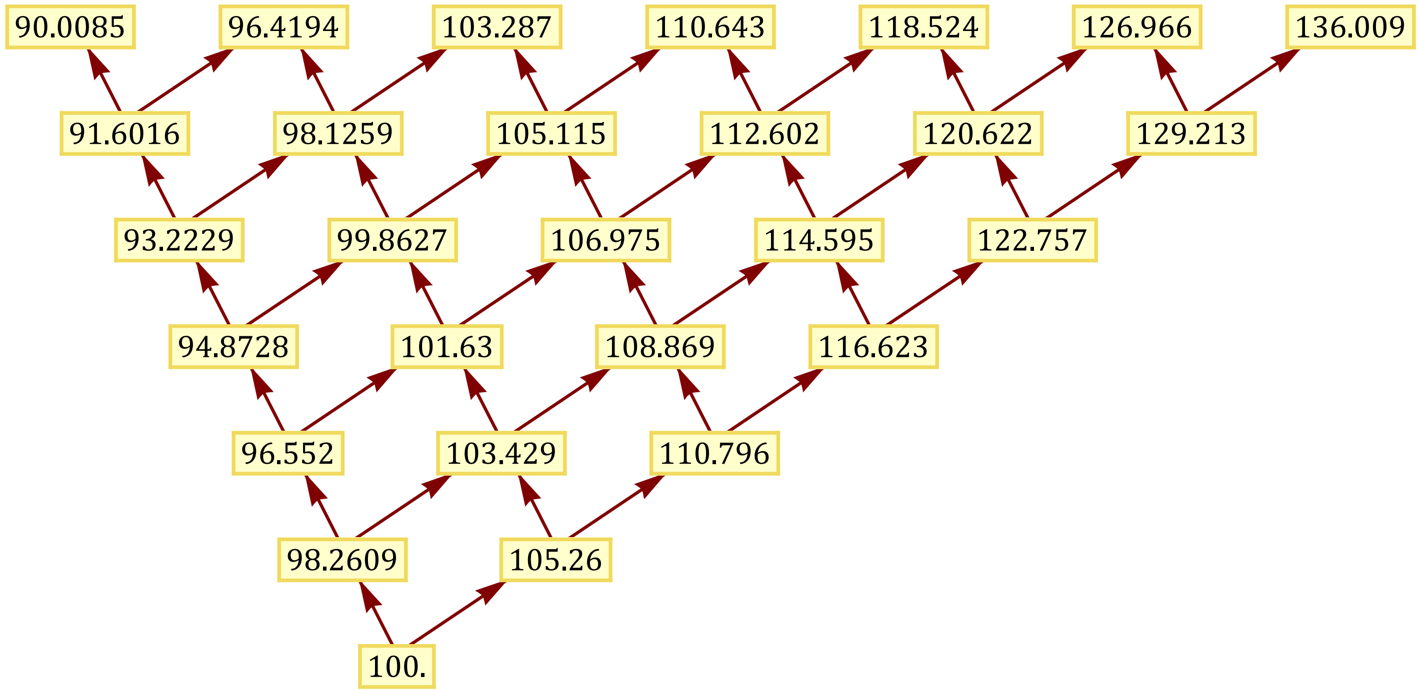
(53) Examples

One risky security and a bond on a trinomial tree (incomplete market):



(54) Examples

... or on a binomial tree (complete market):



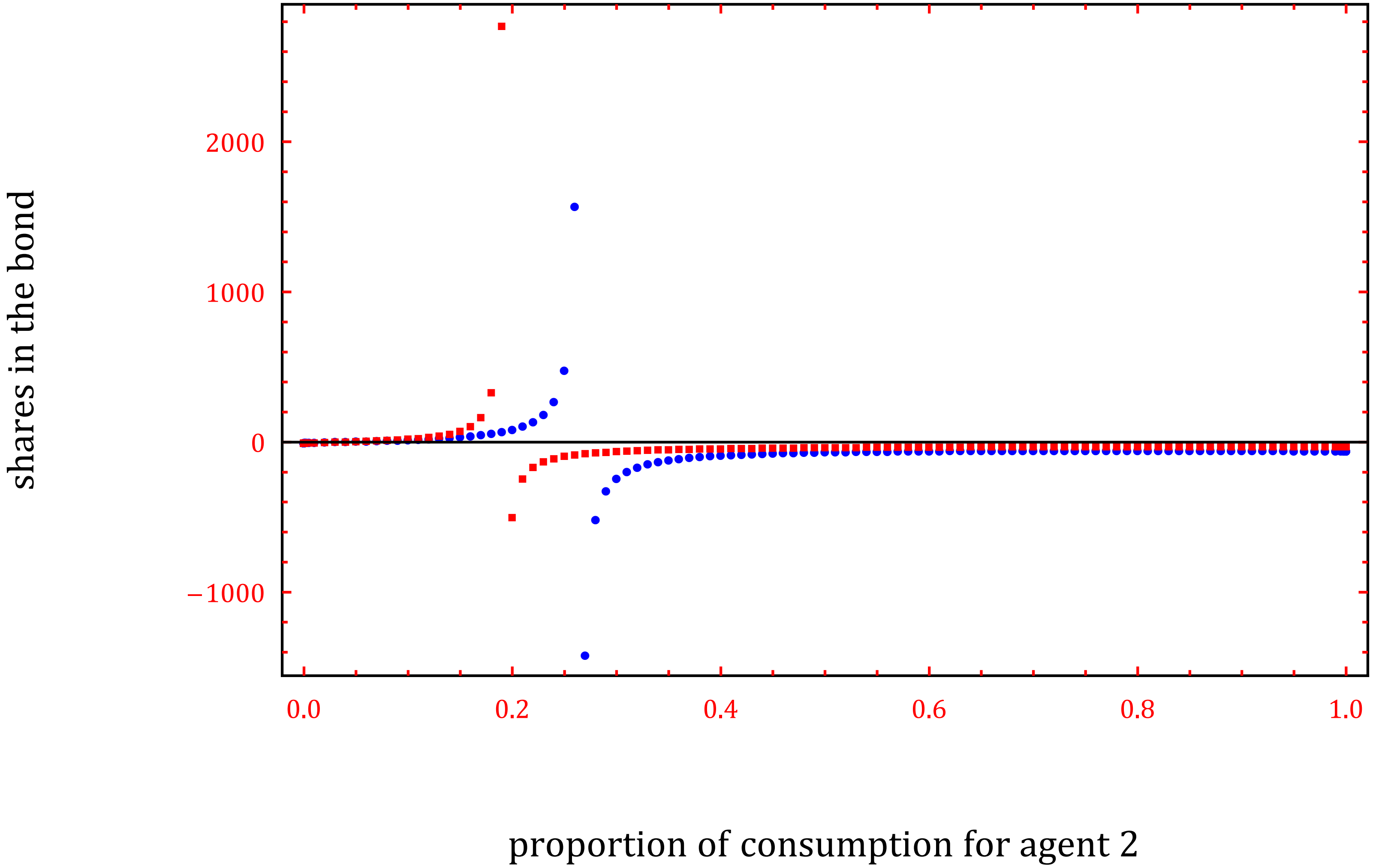
(55) Examples

One of the investors is endowed with the stream of stochastic dividends (shown on the tree) while the second, more risk averse, investor has no endowment other than his initial wealth.

$T = 6$, $N = 2$, $K_t^{\xi} = 3$, $p_1 = p_2 = p_3 = \frac{1}{3}$

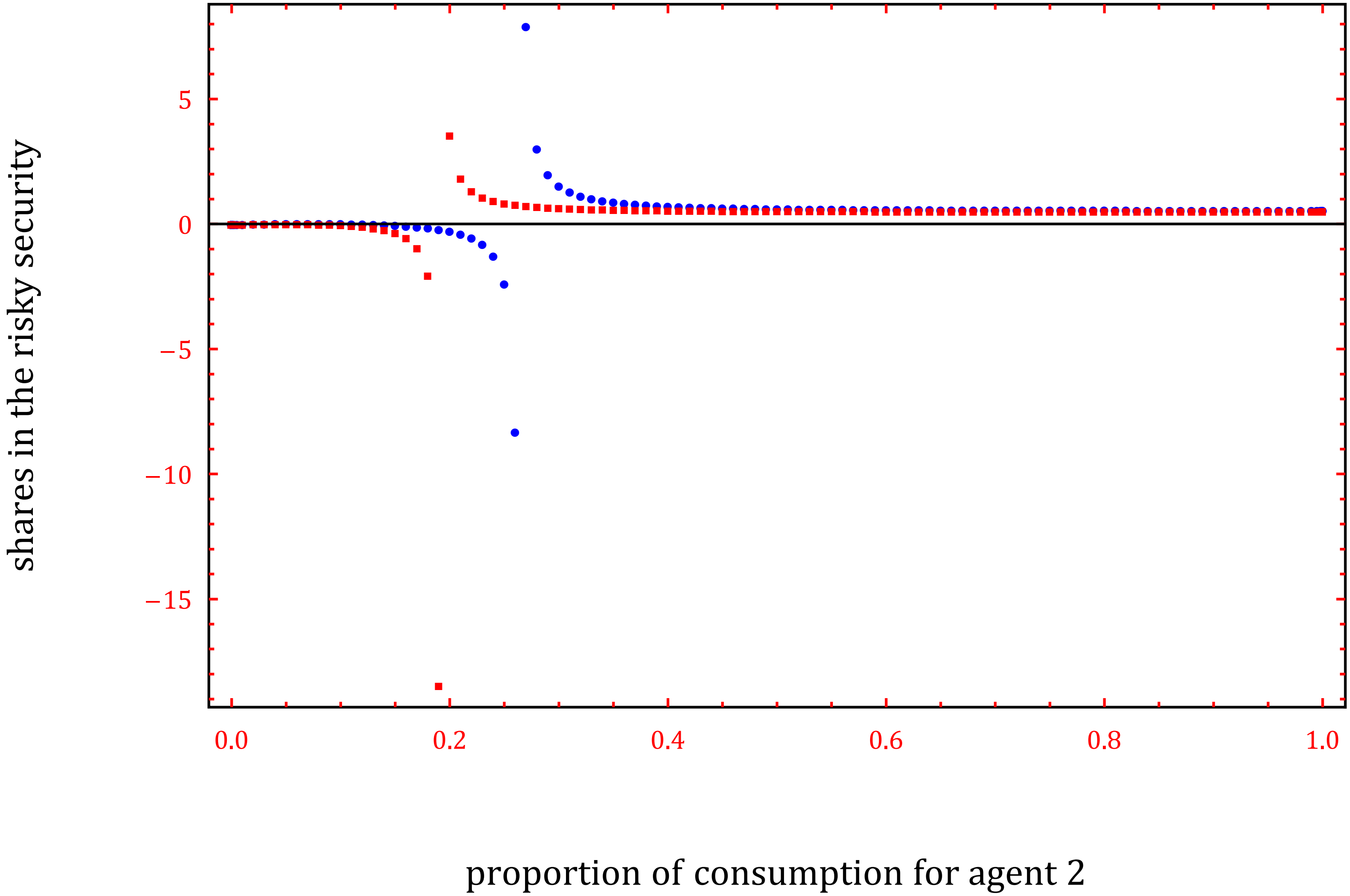
$$U_t^i(c) = \rho^{T-t}\,\frac{c^{1-\gamma_i}}{1-\gamma_i}\,,\ i = 1, 2,\ \gamma_1 = 1,\ \gamma_2 = 5\,.$$

(56) Examples



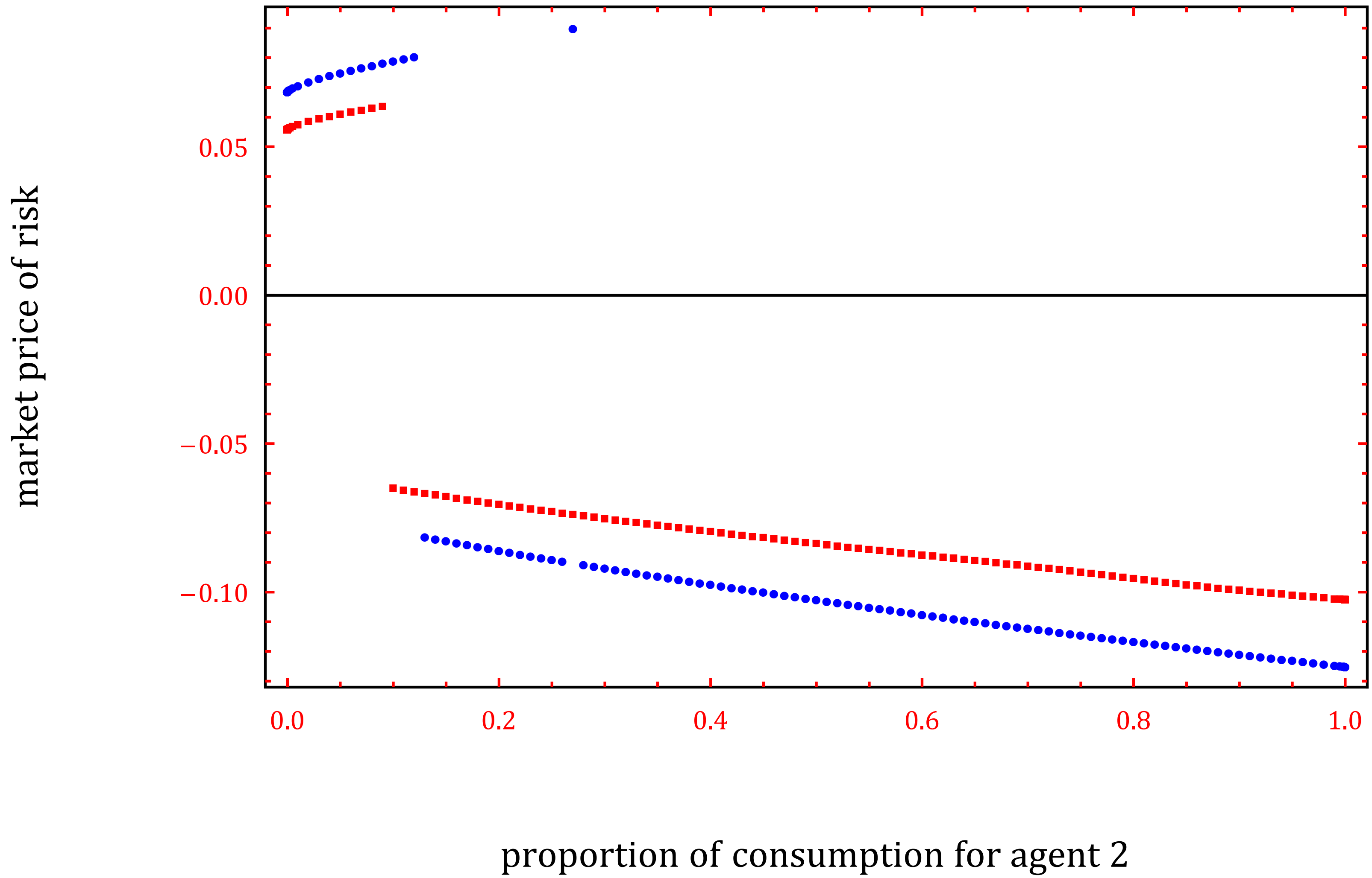
2000
1000
0
−1000
0.0
0.2
0.4
0.6
0.8
1.0
shares in the bond
proportion of consumption for agent 2

(57) Examples



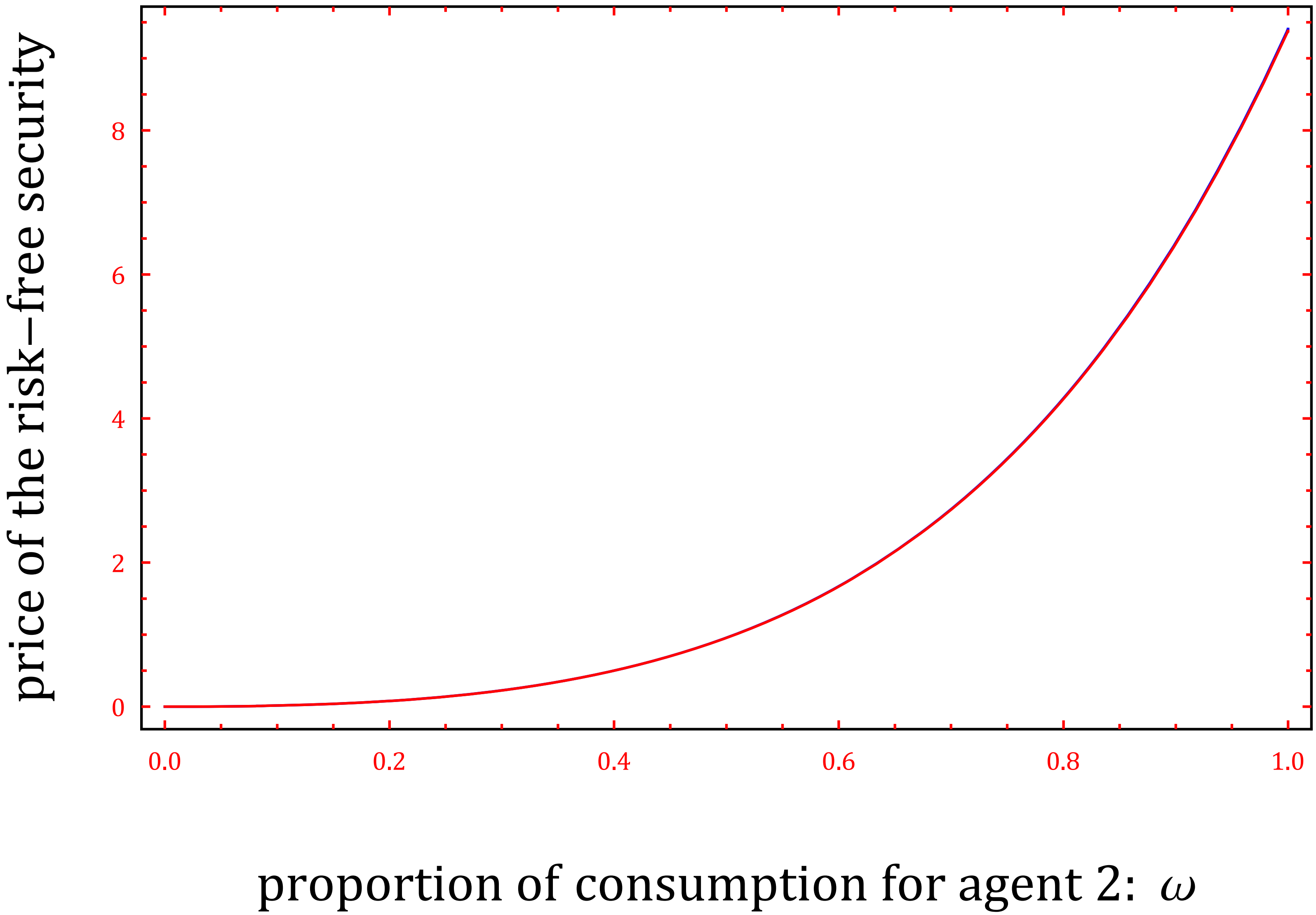
shares in the risky security
5
0
−5
−10
−15
0.0
0.2
0.4
0.6
0.8
1.0
proportion of consumption for agent 2

(58) Examples



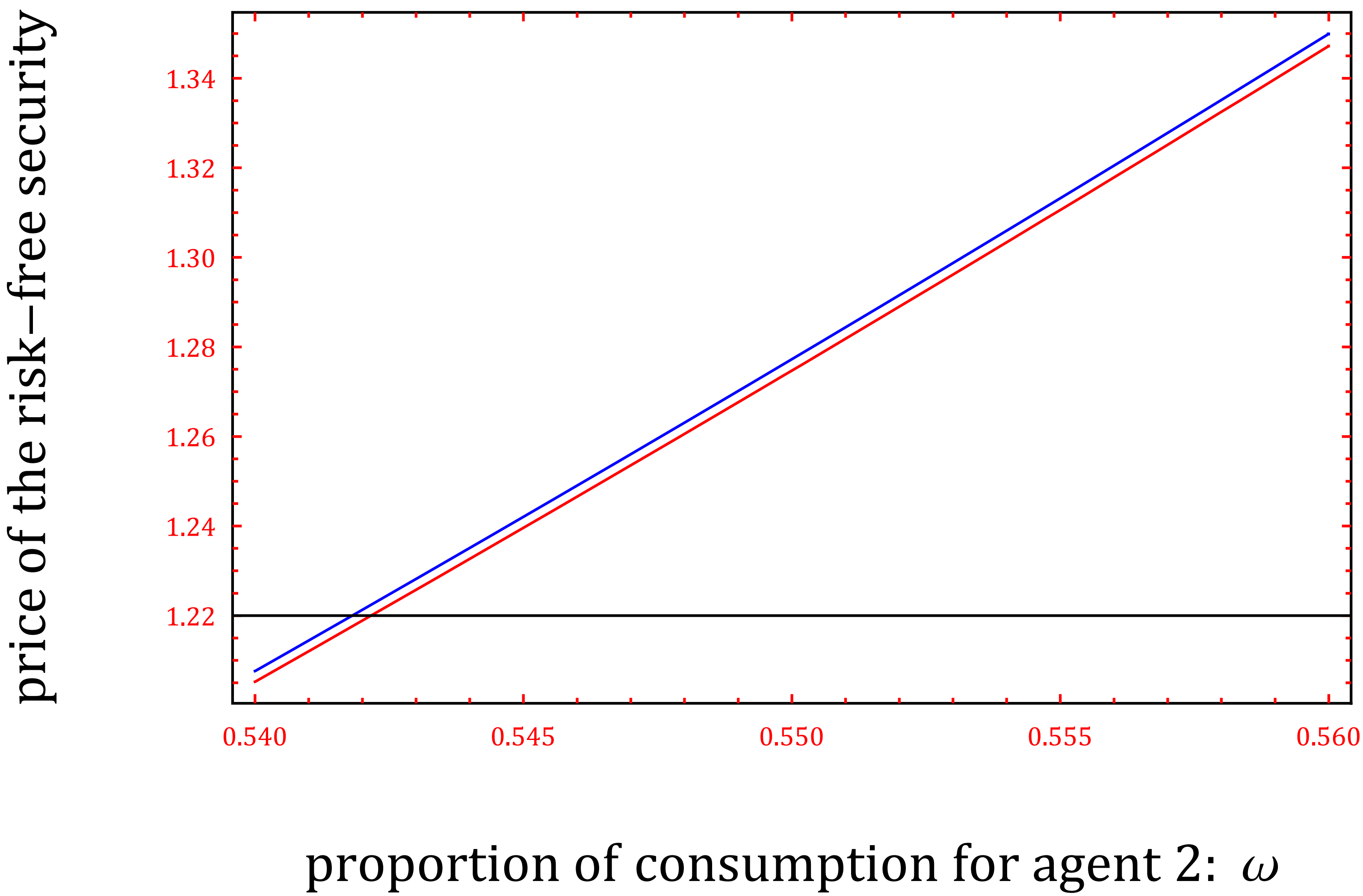
market price of risk
0.05
0.00
−0.05
−0.10
0.0
0.2
0.4
0.6
0.8
1.0
proportion of consumption for agent 2

(59) Examples



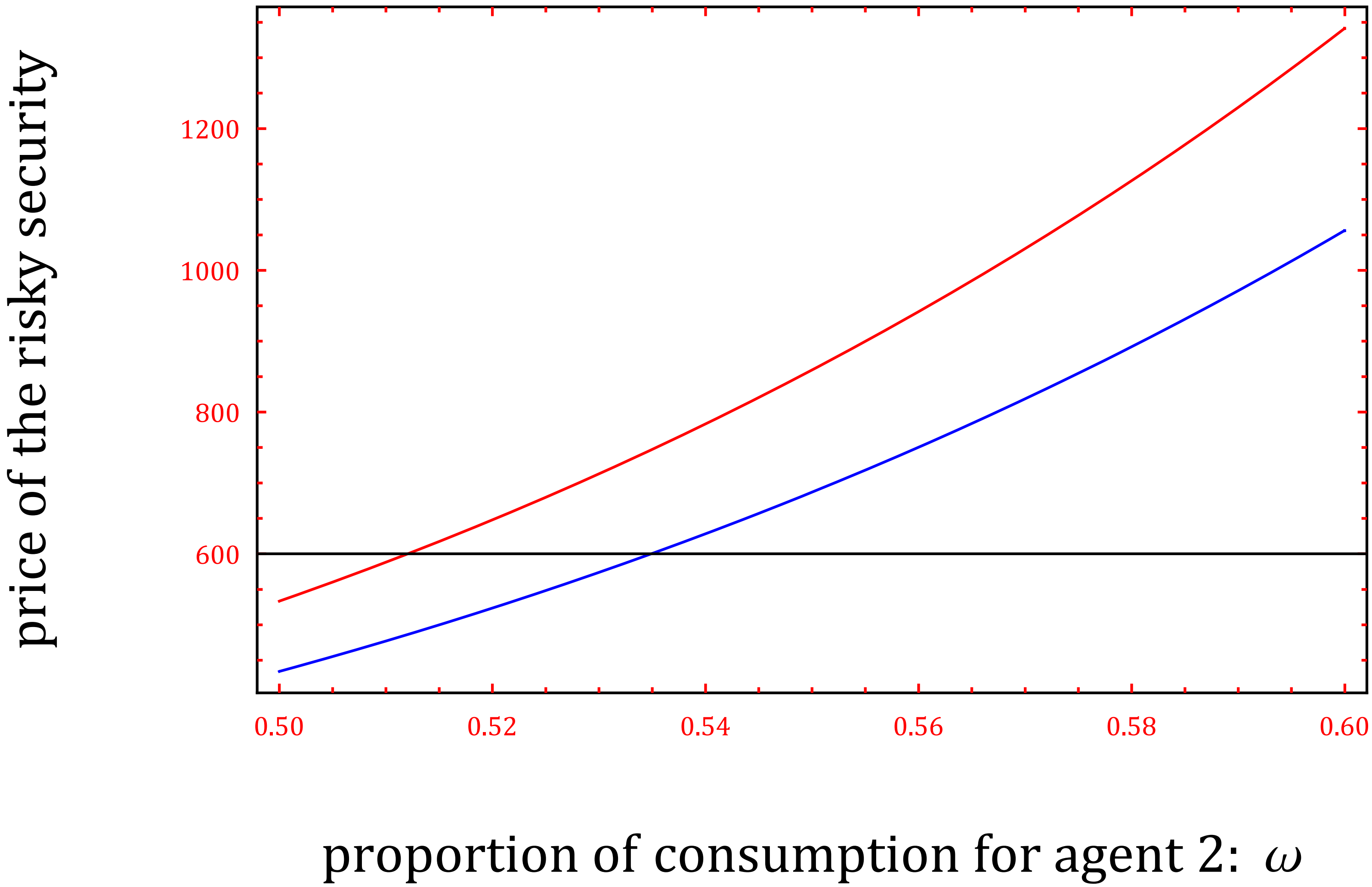
price of the risk−free security
8
6
4
2
0
0.0
0.2
0.4
0.6
0.8
1.0
proportion of consumption for agent 2: ω

(60) Examples



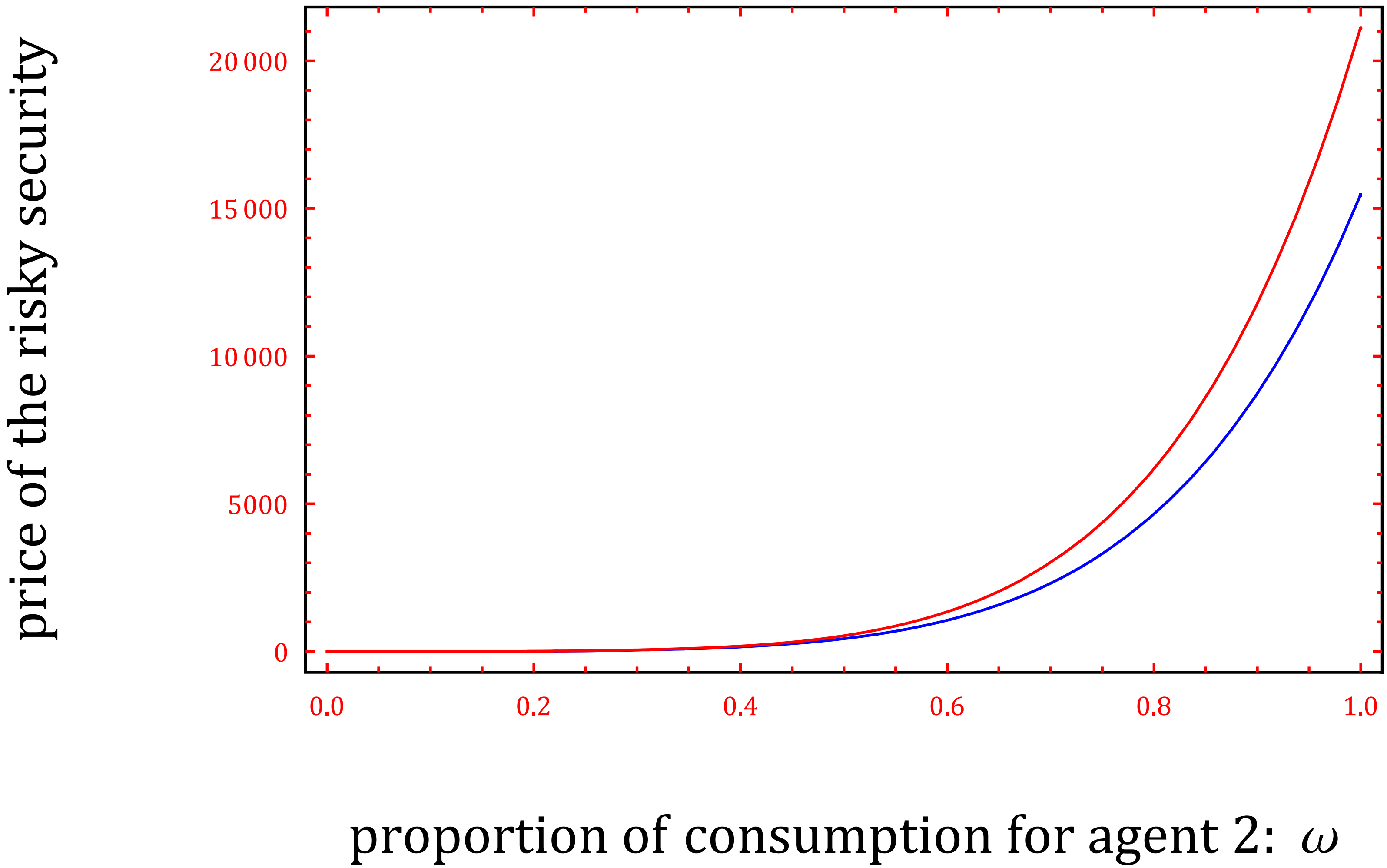
price of the risk−free security
1.34
1.32
1.30
1.28
1.26
1.24
1.22
0.540
0.545
0.550
0.555
0.560
proportion of consumption for agent 2: ω

(61) Examples



price of the risky security
1200
1000
800
600
0.50
0.52
0.54
0.56
0.58
0.60
proportion of consumption for agent 2: ω

(62) Examples

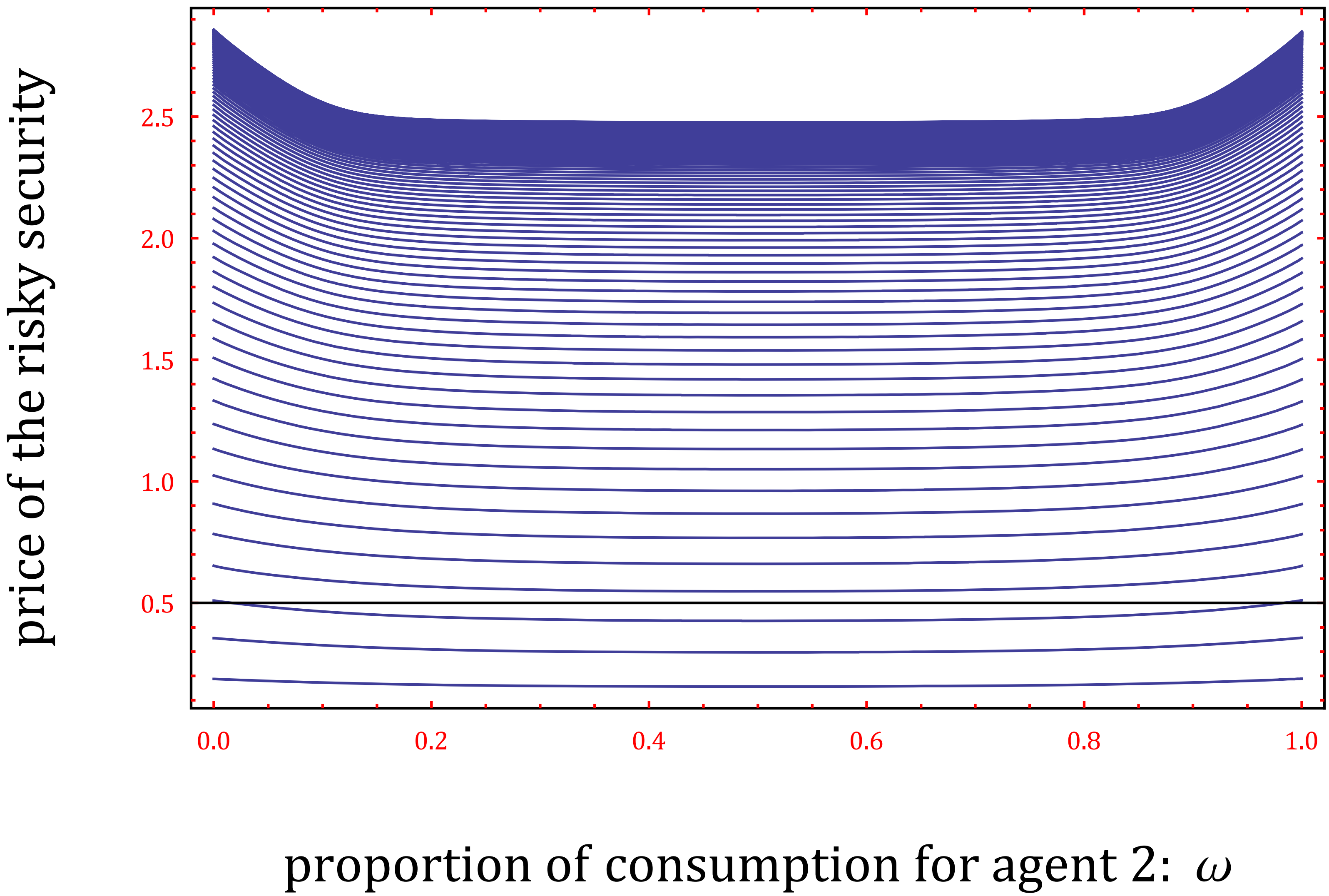

price of the risky security
20 000
15 000
10 000
5000
0
0.0
0.2
0.4
0.6
0.8
1.0
proportion of consumption for agent 2: ω

(63) Examples

Back to Heaton-Lucas: the same 8-state MC, same CRRA = 1.5, except that now the transition probabilities for each state are replaced by the steady-state probabilities. From the point of view of calculating the equilibrium the states are now indistinguishable. As a result one has only one system per iteration.

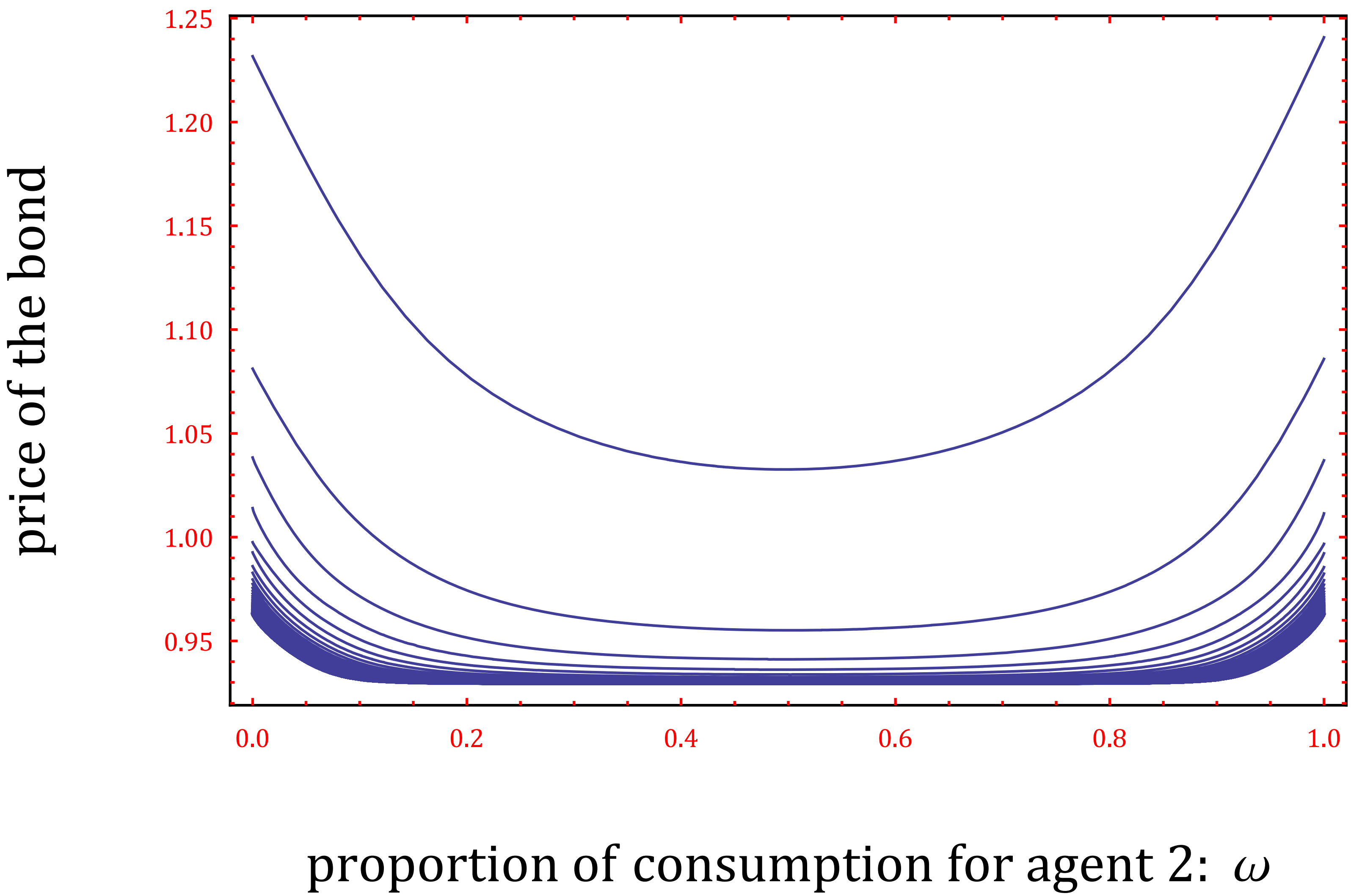
(64) Examples

$T = 100$



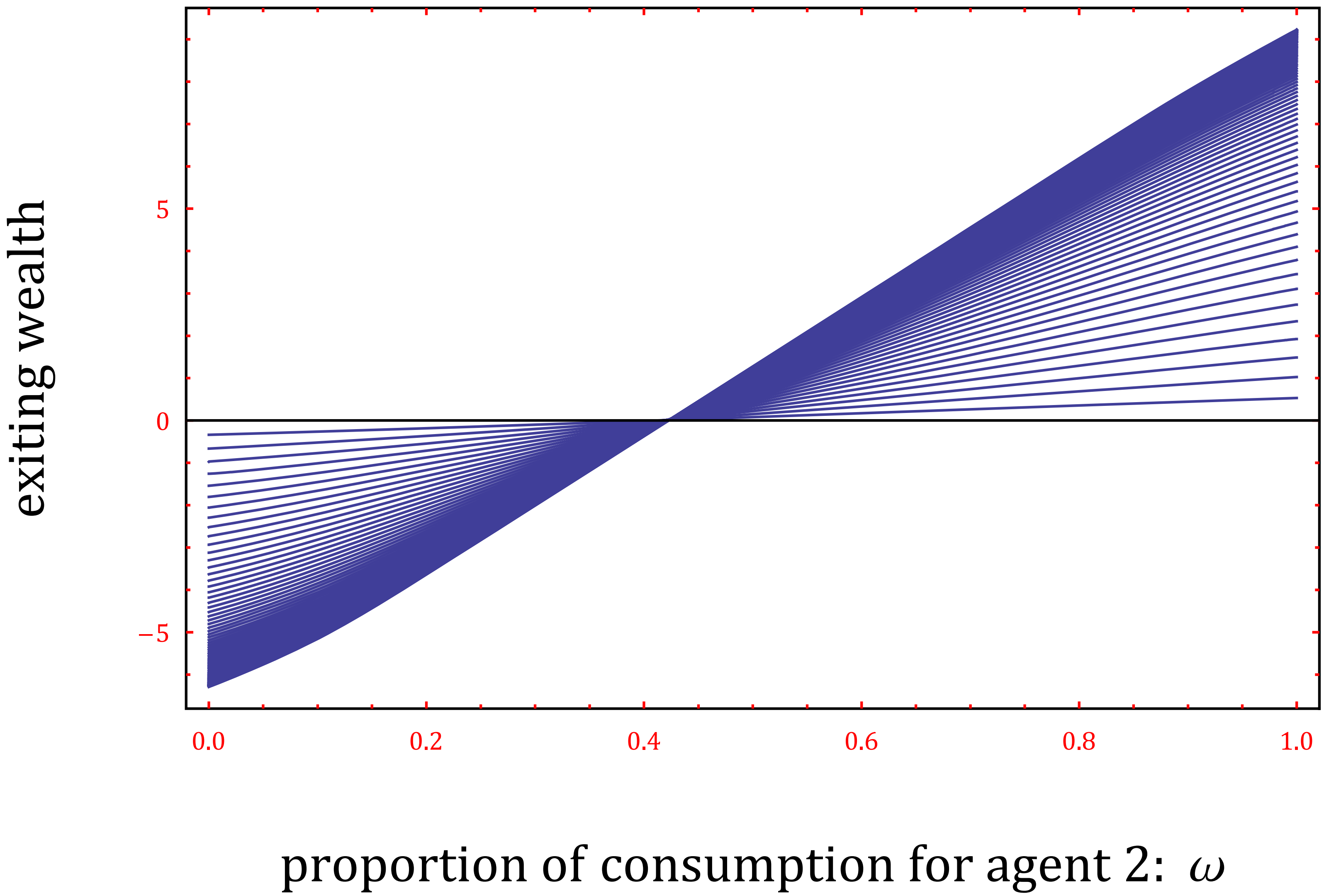
(65) Examples

$T = 100$



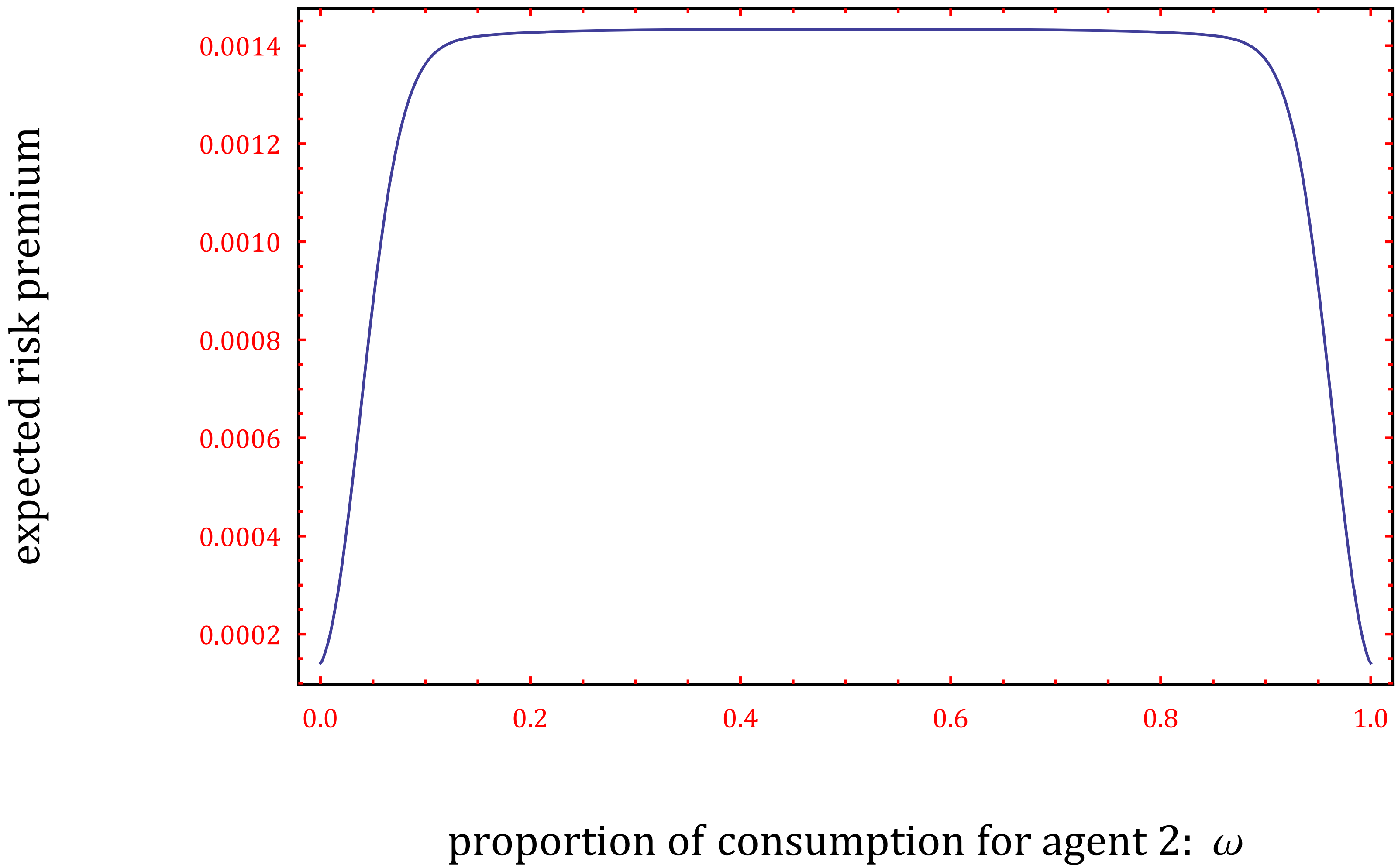
(66) Examples

$T = 100$



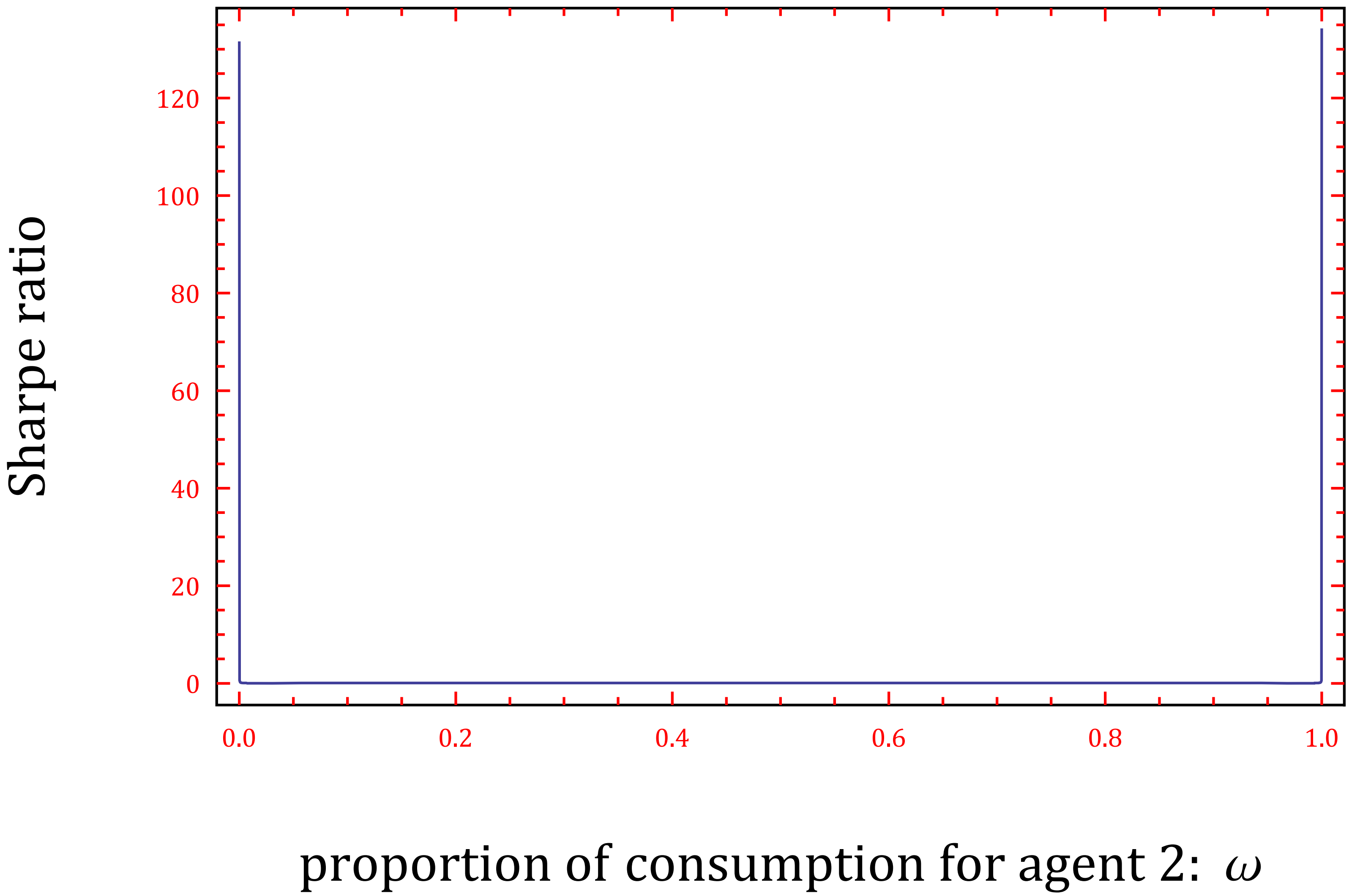
(67) Examples

$T = 100$



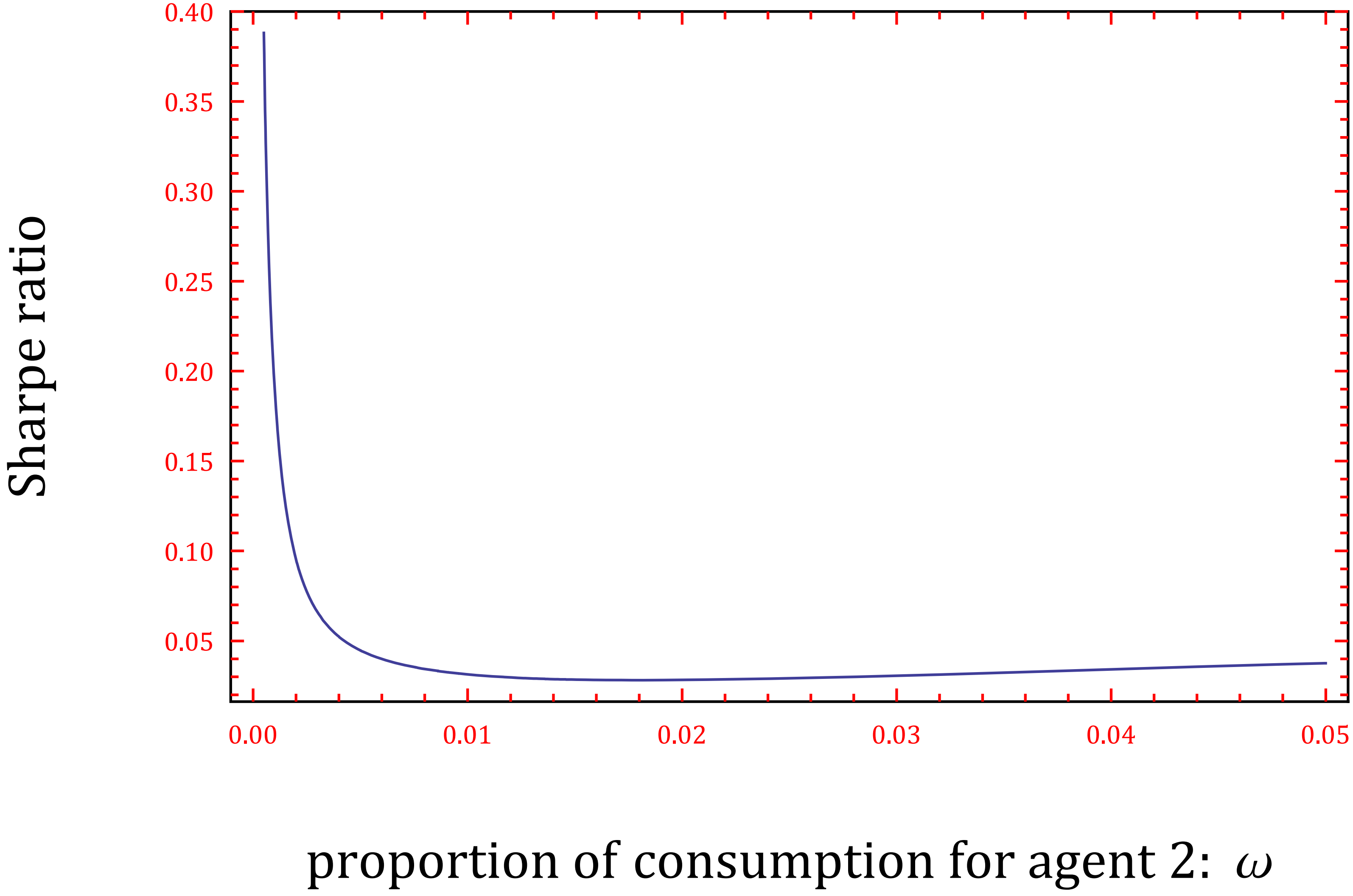
(68) Examples

$T = 100$



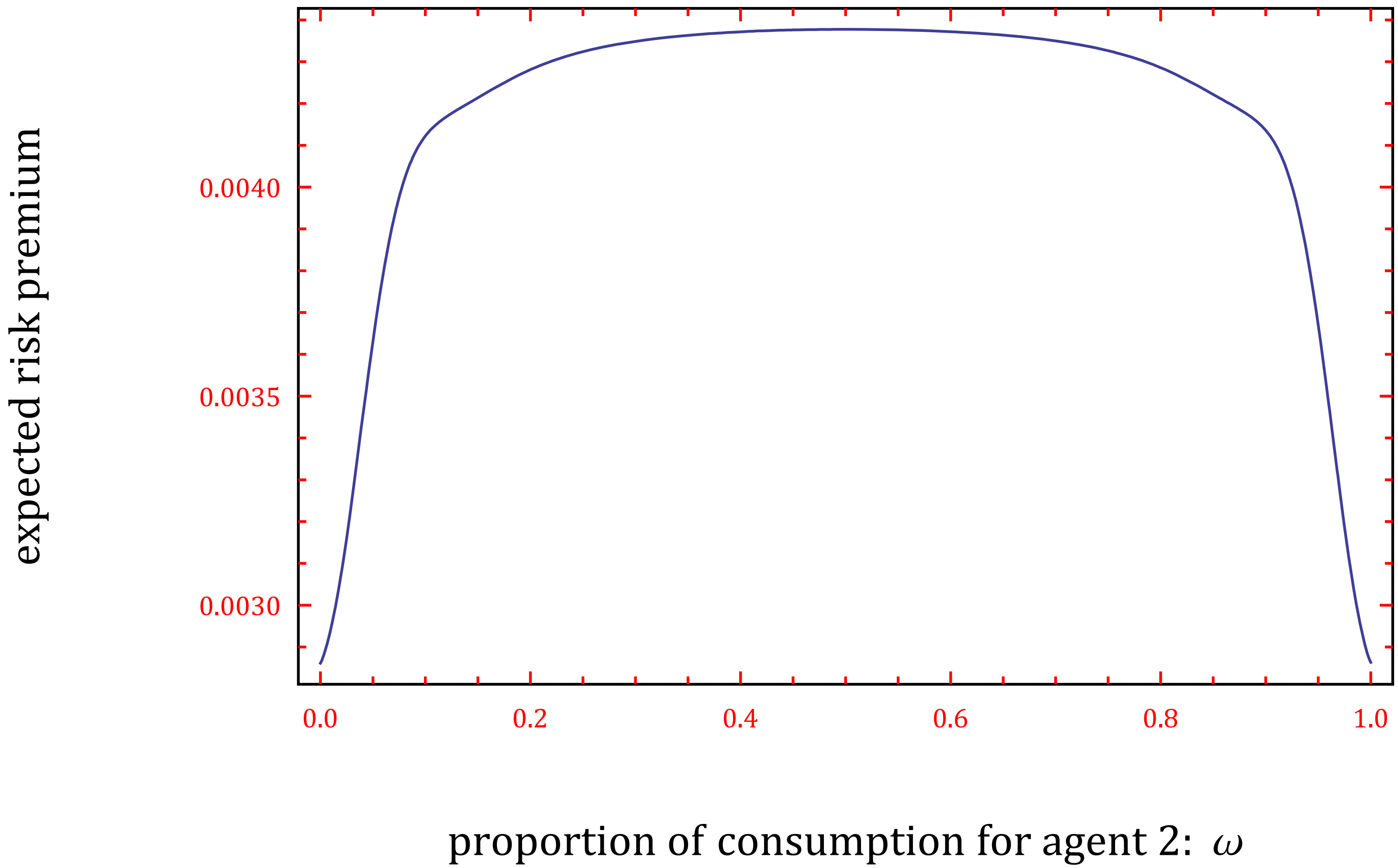
(69) Examples

$T = 100$



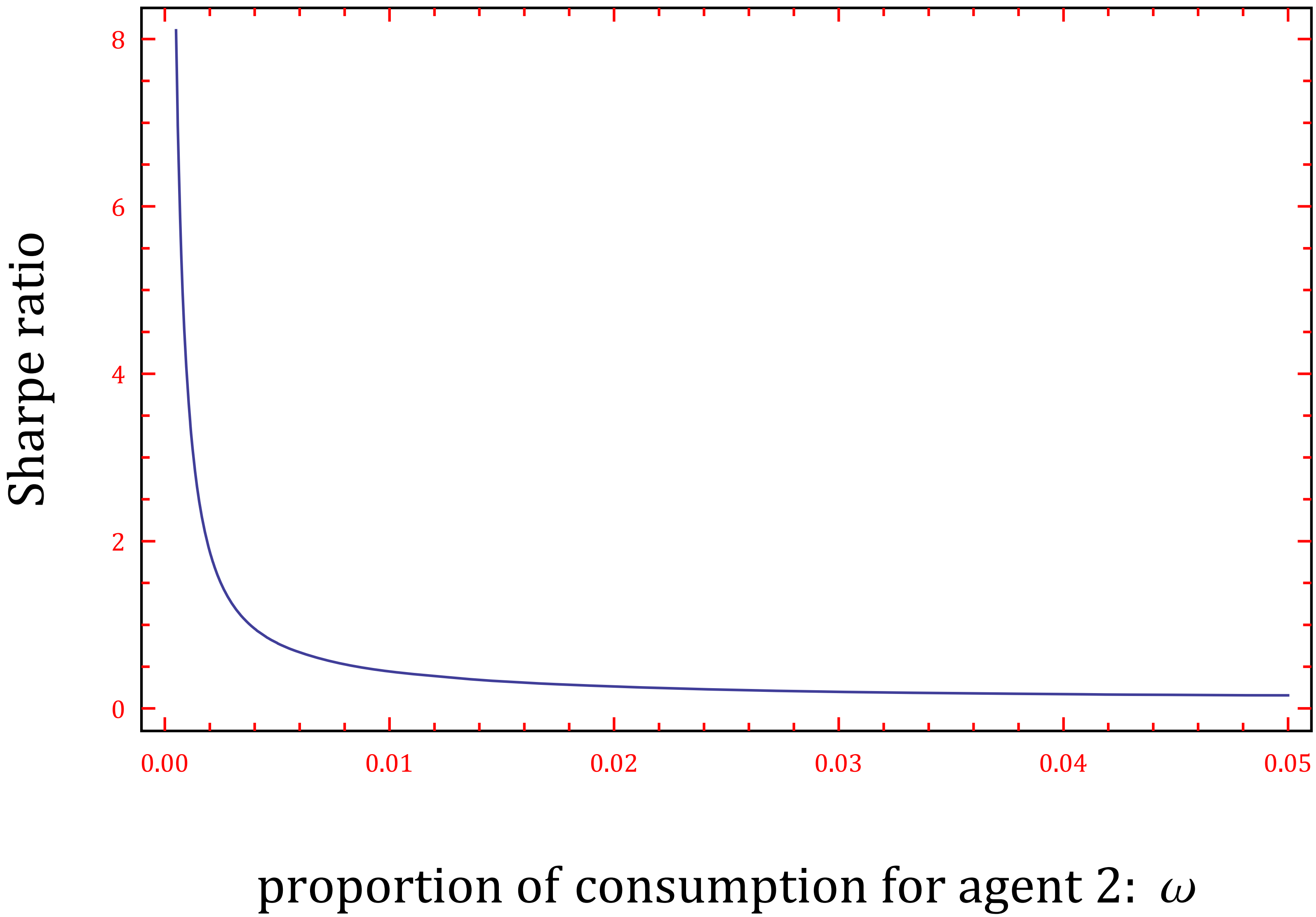
(70) Examples

$T = 50$



(71) Examples

$T = 50$



(72) Comments on the method

- when the tree is really small the global method, when it converges to a solution, provides a single-point solution much faster than does the recursive method
- It should be pointed out, however, that the recursive method delivers *a whole set of points* as in the figures above.
- In principle the recursive method can be used on a very large tree

(73) Comments on the method

- For the case in which the tree is binomial, we emphasize very strongly that, even when the exogenous state variables are Markovian, the *global approach does not permit the use of a recombining tree*.
  - This is because a recombining node would have a unique value of the exogenous state variables but would correspond to two different values of the endogenous state variables, depending on which node the process is coming from.
  - Avoiding the path-dependence aspect is of a great advantage for the recursive method.
- The complexity of the problem does not increase significantly by increasing the number of assets

(74) Future prospects

- Categorize cases in which incomplete markets can and cannot explain asset-pricing puzzles
- Transactions costs
- Recursive utility and default risk
- Large population
- Production economies
- Monetary policy
- International finance
- Continuous time